Entrants' reputation and industry dynamics*

by Bernardita Vial† and Felipe Zurita‡

December 22, 2015

forthcoming in the International Economic Review

Abstract

This paper introduces the analysis of entry-exit decisions in a market where reputation determines the price that firms may charge. It does so by developing a rational-expectations model of competition in a non-atomic market under heterogeneous reputations. A crucial distinction is made between a seller and its name. Names, that can be kept or discarded, are vehicles for information transmission. The analysis focuses on the class of name-switching reputational equilibria, in which a firm discards its name if and only if its reputation falls below the entrants' reputation. The main technical result is the existence of a unique steady-state equilibrium within this class. This equilibrium generates a rich steady-state industry dynamics, largely on agreement with the findings in the empirical literature. In addition, the paper studies the determinants of the entrants' reputation.

JEL Classification: C7, D8, L1
Keywords: reputation, industry dynamics, free entry, exit and entry rates

1 Introduction

It has long been recognized that consumer trust, or reputation, constitutes a valuable–even determinant–asset to a firm (Tadelis, 1999). Firms with good names are able to charge higher prices, expand more easily and live longer. Conversely, a bad name may be an obstacle to firm survival and expansion (Cabral and Hortacsu, 2010), and as such, it may be worth changing (McDevitt, 2011, Wu, 2010).

This paper presents a theoretical model of a competitive market in which reputation is the driving force behind entry, exit, name changing and pricing decisions. Firms with better reputations charge higher prices; entrants, in turn, are willing to sell below cost with the expectation that building a

*We are grateful to the editor Hanming Fang and the three anonymous referees for their careful revision and insightful comments; we are also grateful to Axel Anderson, Luís Cabral, Francesc Dilme, Juan Dubra, Mehmet Ekmekci, Juan Escobar, Hugo Hopenhayn, David K. Levine, César Martinelli, Steve Thompson, Rodrigo Wagner, and Federico Weinschelbaum, to conference participants at the Association for Public Economic Theory, the European and Latin American meetings of the Econometric Society, the European Association for Research in Industrial Economics annual conference, International Industrial Organization Conference, International Workshop of the Game Theory Society, *Jornadas de Economía del Banco Central del Uruguay, Jornadas Latinoamericanas de Teoría Económica*, Midwest Economic Theory Meetings, *Sociedad de Economía de Chile*, Spain-Italy-Netherlands Meeting on Game Theory, World Congress of the Game Theory Society, and seminar participants at *Universidad de Santiago*, *Universidad de Chile*, and *Pontificia Universidad Católica de Chile*. This paper shares content with an earlier manuscript entitled "On Reputational Rents as an Incentive Mechanism in Competitive Markets." Financial support from FONDECYT grant #1121096 is gratefully acknowledged.

†*Instituto de Economía*, *Pontificia Universidad Católica de Chile*. E-mail:bvial@uc.cl
‡*Instituto de Economía*, *Pontificia Universidad Católica de Chile*. E-mail:fzurita@uc.cl

reputation will allow them to recover those initial losses. When reputation matters, free entry has an additional consequence besides keeping prices low; namely, if the entrants' reputation is high enough some existing firms may wish to discard their names by exiting the market and re-enter under a new name. The explicit consideration of this option is the distinctive feature of our analysis.

The model predicts that older firms have stochastically larger reputations than new firms, that their exit rates are smaller and their prices higher as well. At the industry-wide level, it predicts that industries in which ability can be lost more easily will have higher entry-level reputations. Yet, higher entry-level reputations will not necessarily translate into higher turnover ratios.

The predominant explanation for firm dynamics in the literature is based on technological shocks. Firms exit either because others drove them out through product and process innovation, as in the creative destruction hypothesis (see the seminal paper by Hopenhayn, 1992), or because of adverse shocks that increased their production costs. While such shocks are certainly a key factor in the observed dynamics, some evidence suggests that reputation-driven exit is also at play. For instance, there is evidence that exit is more likely subsequent to poor consumer reviews or complaints in some industries (McDevitt, 2011; Cabral and Hortacsu, 2010). The literature also finds that a significant number of firms of different sizes change their names rather than leave the market. As changing names is a strategy aimed at affecting consumer beliefs rather than controlling costs, the interplay between reputation and firm decisions seems worth exploring. A reputational theory of industry dynamics must also confront the fact that there is considerable heterogeneity across industries. McDevitt (2011) finds that about 8% of residential plumbing services firms in Illinois changed names within one year, while Wu (2010) finds that this frequency is much smaller among CRSP-listed companies: about 0.5% per year on average in the 1925-2000 period.

To address these issues, we develop an adverse selection model with imperfect public monitoring along the lines of Mailath and Samuelson (2001). There are two types of firm: competent and inept. There is perfect competition in the sense of Gretsky et al. (1999): There is a continuum of price-taking firms, a continuum of consumers, and entry is free. The incumbents' reputation is the Bayesian update of a common prior given an observed history of (imperfect, public) signals; the entrants' reputation μ_E is also a consistent belief. The equilibrium price function is increasing in the seller reputation.

In equilibrium, firms exit when their reputations fall below μ_E. As competent firms obtain stochastically larger signals than inept firms, their expected present value is higher at each reputation level. As a result, competent firms always want to participate, either with their old names or new ones. In the steady state, there may be exit and entry flows even if the industry as a whole is stagnant. Exit probability is found to depend on firms' characteristics, like type and age. In particular, the reputation of competent firms stochastically dominates that of inept firms and the reputation of older firms dominates that of younger firms as well. Yet there is always heterogeneity both within and between cohorts; in fact, the reputation distributions for all cohorts have full support.

The entry-level reputation provides an endogenous lower bound on reputations, which is jointly determined with the steady-state reputation distribution. The support of the reputation distribution at the trading stage is affected by the entrant's reputation. Reciprocally, the fraction of competents among entrants depends on how many competent firms choose to change their names, which is determined by the reputation distribution. Using a fixed-point argument, we show that there is a unique pair consisting of a mutually consistent entry-level reputation and a steady-state reputation distribution. The relationship between the entry-level reputation and the equilibrium exit rate is non-trivial: A change in a parameter that increases μ_E may also shift the reputation distributions such that firms fall below μ_E less often, reducing the equilibrium turnover rate. Thus, treating μ_E and the reputation distribution as exogenous may be misleading. The endogenization of the entry-level reputation has proven to be important in other contexts; for instance, Atakan and Ekmekci (2014) and Jullien and Park (2014) find that entrants' reputation affects, respectively, the equilibrium assignment in a search market with bargaining, and the equilibrium announcement strategy in a reputation model with pre-trade communication. On the other hand, μ_E emerges as a determinant of the equilibrium price level, as potential entrants that would enjoy a reputation μ_E if they enter must be indifferent between entering or staying out.

Firms are subject to the possibility that their types change exogenously. Our analysis shows that a key determinant of entrants' reputation is the type-change probability. Indeed, a smaller probability that a competent firm becomes inept implies a lower entry-level reputation. The explanation is purely informational: If competence is more persistent, then the fraction of competents among entrants is reduced since the likelihood that their histories of signals are bad enough to motivate an exit from the market is lower, as they have probably been competent–obtaining larger signals than inept firms in a stochastic sense–for a long period of time.

Related literature

There is a growing empirical literature on the dynamics of firms within an industry. Among the most salient patterns that have consistently been found are[1]: (1) The presence of sizable entry and exit rates even in industries that are scarcely growing, with significant heterogeneity across industries; (2) younger firms are–*ceteris paribus*–more likely to exit and also (3) more likely to charge lower prices; (4) the probability that a given seller will exit the market increases as its reputation worsens; and (5) the firms that are more likely to change names or exit are those with worse or shorter track records.

The theoretical literature has investigated a number of possible explanations for these patterns. One strand asks whether such dynamics might be the result of individual productivity shocks in a perfectly competitive market for a homogeneous good (the seminal paper of Hopenhayn, 1992, stands out). A related strand looks at the combination of productivity shocks and financial frictions (Cooley and Quadrini, 2001, Albuquerque and Hopenhayn, 2004, Clementi and Hopenhayn, 2006) or labor market frictions (Hopenhayn and Rogerson, 1993). While (1) and (2) are consistent with this view, the law of one price is at odds with (3). Also, the empirical concepts of reputation and track records do not have a theoretical counterpart in this setting. The same is true in Fishman and Rob (2003), in which industry dynamics are driven by consumer inertia in a context of search costs and older firms sell more because they have a larger customer base.

On the other hand, there is a theoretical literature that looks at the creation and maintenance of firms' reputations in markets for experience goods (e.g., Klein and Leffler, 1981, Fudenberg and Levine, 1989, and Mailath and Samuelson, 2001, to name just a few; Mailath and Samuelson, 2006, and Bar-Isaac and Tadelis, 2008, present comprehensive expositions of the literature.) This literature discusses primarily the monopoly case. In spite of this, some papers still manage to look at entry and exit decisions. For instance, Bar-Isaac (2003) assumes that the firm has the option to leave the market. When the firm knows its own type, in equilibrium the high-quality firm never leaves, while the low-quality firm plays a strictly mixed strategy at low levels of reputation–i.e., below some threshold. The mixed strategy is such that the post-exit reputation of any firm that has crossed the threshold becomes the threshold. Having a strictly positive probability of exiting, the low-quality type eventually leaves; this implies that there is complete separation in the long run. Board and Meyer-ter Vehn (2010) extend this analysis by incorporating moral hazard and the possibility of entry, and focus their analysis on the investment and exit decisions over the life cycle of the firm. In this equilibrium, the entry-level reputation coincides with the threshold as well.

Within the literature that looks at reputation dynamics in competitive markets, some papers focus on markets in which the information flow to potential customers is quite limited and fundamentally different from that to existing customers–namely, private monitoring; Hörner (2002) and Rob and Fishman (2005) stand out. Instead, we want to examine markets where information–albeit imperfect–flows constantly to potential customers as well; for instance, the eBay feedback system (Cabral and

[1]See Section 7. The empirical literature has also given a great deal of attention to firm growth and firm size. We will abstract from this issue by assuming that all firms have a capacity constraint of one unit.

Hortacsu, 2010), or the complaint record of plumbing firms (McDevitt, 2011). Indeed, Internet-related technological progress moves an increasing number of markets into this category by providing means of communication among customers; one example of this is the role of TripAdvisor, Expedia, etc. in the travel industry.

Tadelis (1999) is one of the first papers to formally analyze competition under imperfect public monitoring. It presents an adverse-selection model with a continuum of firms. However, the author focuses on an equilibrium where firms leave the market after one bad outcome; this means that active firms either don't have any history (they are new), or they must have impeccable records. Tadelis (2002) develops a similar model, under moral hazard. While this kind of model can explain certain stylized facts of industry dynamics, like the differences in pricing and probability of exit between cohorts, it cannot explain the observed heterogeneity in these variables after controlling by age: All firms of the same age must have the same records and reputation. In particular, it cannot account for observations (4) and (5) beyond age.

Our paper contributes to recent literature on reputation under competition that features heterogeneous reputations. Some papers do not consider entry (e.g., Vial, 2010); some use entry-level reputation as an exogenous parameter (e.g., Ordoñez, 2013), while others obtain it independently from the reputation distributions because of their focus on mixed strategies (e.g., Atkeson et al., 2015). In our paper entry-level reputation is endogenously determined and depends only on informational variables; the price function fulfills the market-clearing role. In contrast, in Bar-Isaac (2003), Board and Meyer-ter Vehn (2010) and Atkeson et al. (2015), the price function is determined by consumer valuations and μ_E adjusts to clear the market, through a zero-profit condition.

The rest of the paper is organized as follows: Section 2 presents the model and Section 3 introduces the equilibrium concept. Section 4 discusses the existence and uniqueness of the consistent reputation distribution and entry-level reputation. Section 5 presents the comparative statics results while Section 6 analyzes the dynamics of the industry in the steady state. Section 7 discuses robustness and extensions, and Section 8 concludes. All proofs are contained in the Appendix.

2 The model

We consider an infinite-horizon model in which, at every stage or date $t \in \mathbb{N}$, a market for a given service opens. A continuum of long-run firms faces an infinite sequence of generations of non-atomic consumers. Each generation of consumers is of mass 1. In contrast, the mass of potential firms is $1+\kappa$, with $\kappa > 0$. Consumers choose whether to buy one unit of the service–and from which seller–or none at all. Firms may be *active at* t (i.e., produce one unit) and sell under their old name (action $a = O$) or sell under a new name ($a = N$), or be *inactive*, i.e., not produce at all ($a = I$). Thus, firms' action space is $A = \{O, N, I\}$.

The service is an experience good as per Nelson (1970): Its quality is ex ante unobservable to buyers. There are two types of firms, indexed by $\tau \in \{H, L\}$. Competent firms ($\tau = H$) can only produce a high-quality service, while inept firms ($\tau = L$) can only produce a low-quality one. Types are privately observed–hence, this is a pure adverse selection model.

There is no communication among consumers. Since they only live for one period, the information each one obtains as a result of consuming the service is not transferred to the next generation, but lost altogether; nevertheless, an imperfect signal s of the quality provided becomes publicly available. For an active firm (i.e., a firm that chooses $a_t \in \{O, N\}$) the signal is $r \in (0, 1)$; if $a_t = I$, however, the firm doesn't provide the service and therefore its signal is empty ($s = \emptyset$).[2]

[2]For instance, if the firm were an eBay seller, r could be the feedback score; if the firms were schools, r could be the score percentile on a standardized test; if the firms were health care providers, r could be their medical malpractice track records; if the firms were car makers, r could be consumer reports, and so on.

The cdf of signal r for a type-τ firm is denoted by F^τ. We assume that F^H and F^L are differentiable probability distributions with densities f^H and f^L, and that they are mutually absolutely continuous with common full support in the unit interval; hence, no level of r will ever be perfectly informative. The likelihood ratio $\frac{f^H(r)}{f^L(r)}$ is a monotonically increasing bijection from $(0,1)$ to $(0,\infty)$–hence, invertible.

Consumers can keep track of events through firms' names: there is perfect recall of signals and actions only while the firm produces under the same name. The *prior public history* of a firm at the beginning of stage t is denoted by $\overline{h}_t$. After choosing an action a_t the *interim public history* becomes

$$h_t = \begin{cases} (\overline{h}_t, a_t) & \text{if } a_t = O, \\ a_t & \text{if } a_t \in \{N, I\}. \end{cases} \tag{1}$$

Thus, when a firm keeps its name ($a_t = O$) this action is added to its public history. In contrast, if a firm either changes its name ($a_t = N$) or chooses inactivity ($a_t = I$), its prior public history is erased and the interim public history consists solely of this latter action. The addition of the realization of the signal s_t to the interim public history yields the next period's prior public history:[3]

$$\overline{h}_{t+1} = (h_t, s_t). \tag{2}$$

Types are subject to the possibility of changing exogenously: The probability that a firm of current type τ will be of type τ' in the next period is denoted by $\lambda^{\tau\tau'} \equiv \Pr(\tau_{t+1} = \tau' | \tau_t = \tau) \in (0,1)$. We denote by Λ the corresponding transition matrix:

$$\Lambda \equiv \begin{pmatrix} \lambda^{HH} & \lambda^{LH} \\ \lambda^{HL} & \lambda^{LL} \end{pmatrix}. \tag{3}$$

This transition matrix applies to all firms, active or inactive, after the signals become available. We assume that $\lambda^{LH} < \lambda^{HH}$ and also that $\kappa < \overline{\kappa}$, where

$$\overline{\kappa} \equiv \frac{\lambda^{HL}}{\lambda^{LH}}(\lambda^{HH} - \lambda^{LH}). \tag{4}$$

This upper bound on κ ensures that the mass of (active or inactive) firms that become competent remains small enough so that the adverse selection problem is non-trivial.

A firm's reputation is the consumers' belief about the firm's type, i.e., the probability that the firm is competent, conditional on all available information. Consumers have common priors and observe the same events, hence they have common beliefs. The prior reputation $\overline{\mu}_t$ and the interim reputation μ_t, respectively, are defined by

$$\begin{aligned} \overline{\mu}_t &\equiv \Pr(H|\overline{h}_t) \quad \text{and} \\ \mu_t &\equiv \Pr(H|h_t). \end{aligned}$$

The timeline for each stage is shown in Figure 1. At the beginning of the stage, each firm is endowed with a type τ_t and a public history $\overline{h}_t$ attached to its name. Firms choose actions $a_t \in A$, leading to updated histories h_t. The market opens. After trading, the signals r_t are added to the public histories of active firms, and the type-change process occurs. The stage ends.

Let $c \geq 0$ be the production cost. A firm that sells at a price p_t receives a profit of $(p_t - c)$ at t, and zero if it does not sell. Then, the flow payoff is

$$(p_t - c)\mathbb{1}_{\{a_t \neq I\}}, \tag{5}$$

[3]Note that the three actions are available after every history; for instance, a firm that was inactive at $t-1$ can choose to keep its name at t yielding $h_t = (I, \emptyset, O)$, or use a new one yielding $h_t = (N)$.

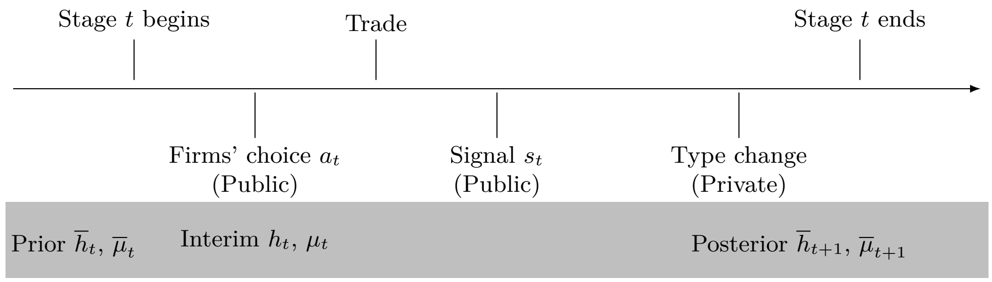


Figure 1: Timeline for the date-t stage

where $\mathbb{1}_{\{a_t \neq I\}}$ is the indicator function that takes the value 1 if and only if the firm chooses to be active at t. Firms maximize the sum of expected discounted payoffs, with discount factor $\delta \in (0,1)$.

A consumer who buys at a price p from a firm with reputation μ gets an expected payoff of

$$\mu u^H + (1-\mu)u^L - p, \tag{6}$$

where u^H is the utility of consuming a high-quality service, and u^L a low-quality service. A consumer who doesn't buy gets a payoff of zero. We assume that $u^H > u^L > c \geq 0$.

3 Equilibrium definition

We focus on steady state equilibria since our goal is to study the dynamics of industries that are neither growing nor shrinking.

The state variables for a firm include its prior reputation and its current type, as they may affect the current price and the expectation of future prices. We restrict attention to stationary symmetric Markov strategies, of which these are the *only* state variables. Therefore, we require that the strategy of all firms of a given type τ with histories that lead to the same prior reputation $\overline{\mu}$ are identical. The state-space is denoted by $\Omega \equiv \overline{U} \times \{H, L\}$, where $\overline{U} \subset [0,1]$ denotes the set of prior reputations.

We consider a competitive market in the sense of Gretsky et al. (1999): Entry is free, and all consumers and firms are price takers. A firm with interim reputation μ is unable to (favorably) influence the price, as it faces competition from other firms with the same or arbitrarily close reputations. The market price for a service provided by a firm with reputation μ is denoted by $p(\mu)$, thus defining a function $p : U \to \mathbb{R}$, where $U \subset [0,1]$ denotes the set of interim reputations. Since all consumers are identical, they must be indifferent among all (active) providers; from Equation (6), consumer indifference is obtained if

$$\frac{dp(\mu)}{d\mu} = u^H - u^L. \tag{7}$$

This defines the equilibrium price function $p(\mu)$, up to a constant p_0, i.e., $p(\mu) = p_0 + (u^H - u^L)\mu$. On the other hand, free entry implies a zero-profit condition for a mass κ of potential entrants, namely, that their sum of expected discounted payoffs is null. This condition determines the equilibrium p_0.

The firm's behavior strategy is a map $\boldsymbol{\sigma} : \Omega \to \Delta(A)$, where $\Delta(A)$ is the set of probability distributions over A. In turn, the maximum sum of expected discounted payoffs is the firm's value function $v : \Omega \to \mathbb{R}$.[4]

After the firm chooses an action $a \in A$, its prior public history will be updated according to Equation (1) and its prior reputation $\overline{\mu}_t$ will be updated to $\mu_t = \varphi\left(\overline{\mu}_t | a_t\right)$, where $\varphi : \overline{U} \times A \to U$. As consumers

[4]To save on notation, we will not explicitly refer to consumers' strategies, as consumers are homogeneous and will always buy in equilibrium.

do not distinguish among entrants with new names (i.e., those who chose $a_t = N$) they will have the same interim reputation μ_E, which is also constant over time. Given an interim reputation μ_t and a signal realization s_t, the posterior probability that the firm is competent is denoted by $\overline{\varphi}(\mu_t|s_t)$, where $\overline{\varphi} : U \times ((0,1) \cup \emptyset) \to \overline{U}$. Under these beliefs, the maximization of expected, discounted lifetime profits in recursive form is associated with the following Bellman equation:

$$
\begin{aligned}
v(\overline{\mu}, \tau) = \max_{\boldsymbol{\sigma} \in \Delta(A)} \Bigg\{ & (\sigma_O + \sigma_N)(p_0 - c) + (u^H - u^L)(\sigma_O \varphi(\overline{\mu}|O) + \sigma_N \varphi(\overline{\mu}|N)) \\
& + \delta \sum_{\tau' \in \{H,L\}} \lambda^{\tau\tau'} \Bigg(\int_0^1 \Big[\sigma_O v(\overline{\varphi}(\varphi(\overline{\mu}|O)|r), \tau') + \sigma_N v(\overline{\varphi}(\varphi(\overline{\mu}|N)|r), \tau') \Big] dF^{\tau} \\
& + \sigma_I v(\overline{\varphi}(\varphi(\overline{\mu}|I)|\emptyset), \tau') \Bigg) \Bigg\}, \quad (8)
\end{aligned}
$$

where σ_a denotes the probability that the firm chooses action a, with $\sum_a \sigma_a = 1$. The resulting policy function is denoted by $\sigma_a(\overline{\mu}, \tau) \equiv \Pr(a|\overline{\mu}, \tau)$.

While firms may have heterogeneous and ever-changing reputations, in the steady state the reputation distribution is constant over time. $\overline{G}$ denotes the cdf of prior reputations of active firms at the beginning of each stage; $\overline{m}^{\tau}$ denotes the mass of type-τ firms in that group. Similarly, G denotes the cdf of interim reputations of active firms, and m^{τ} the corresponding mass of type-τ firms. Thus, G and $\overline{G}$ differ because some incumbents choose to exit, some to re-enter, and some inactive firms decide to enter.

In equilibrium firms maximize their discounted expected profits, the market clears, consumers' expectations are correct and satisfy Bayes' rule whenever possible, and the population distributions of firms' reputations are constant over time:

Definition 1 (Equilibria)**.** *An equilibrium is a strategy $\boldsymbol{\sigma}^*$, a price function p, and a belief system $(\varphi, \overline{\varphi})$ with corresponding distributions $(G, \overline{G})$ such that:*

i. Optimality: $\boldsymbol{\sigma}^* : \Omega \to \Delta(A)$ is the policy function associated with Equation (8)*;*

ii. Market clearing: The mass of active firms equals the mass of consumers: $m^H + m^L = 1$;

iii. Consistency of beliefs: Firms' reputations are derived from $\boldsymbol{\sigma}^*$ using Bayes' rule whenever possible;

iv. Steady state: The reputation distributions of active firms G and $\overline{G}$ and the entrants' reputation are constant over time.

As is common in this kind of models, many equilibria may exist, supported by different beliefs. We focus on the class of reputational name-switching equilibria in which the entrants' reputation acts as a "reservation reputation": firms with reputations above μ_E keep their names, while any firm whose reputation falls below μ_E switches its name.

Definition 2 (Name-switching equilibria)**.** *An equilibrium where firms use the following strategy $\boldsymbol{\sigma}^*(\overline{\mu}, \tau)$ is called* name-switching equilibrium*:*

$$
(\sigma_O^*(\overline{\mu}, \tau) \;\; \sigma_N^*(\overline{\mu}, \tau) \;\; \sigma_I^*(\overline{\mu}, \tau)) \equiv \begin{cases} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} & \text{if } \overline{\mu} < \mu_E \;\wedge\; \tau = H, \\ \begin{pmatrix} 0 & \xi & 1-\xi \end{pmatrix} & \text{if } \overline{\mu} < \mu_E \;\wedge\; \tau = L, \\ \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} & \text{if } \overline{\mu} \geq \mu_E \;\wedge\; (\tau = H \;\vee\; \tau = L), \end{cases} \quad (9)
$$

for some $\xi \in (0,1)$ and for some $\mu_E \in [0,1]$.

In this equilibrium class competent firms always produce, as do inept firms with prior reputation above μ_E. Active firms sell under their old names if and only if their prior reputations are above μ_E. Inept firms with prior reputation below μ_E are indifferent between being active or inactive; they produce under a new name with probability ξ, and stay inactive with probability $1-\xi$. Observe that if μ_E is higher than λ^{LH} there will be entry and exit flows in equilibrium; if, on the other hand, μ_E is lower than λ^{HH}, a positive mass of firms will keep their names, giving thus rise to a non-trivial industry dynamics. The next section shows that this is indeed the case.

4 Equilibrium characterization

4.1 Belief updating

Bayesian updating conditioning on a and $\overline{\mu}$ yields

$$\varphi(\overline{\mu}|a) = \frac{\sigma_a\left(\overline{\mu}, H\right)\overline{\mu}}{\sigma_a\left(\overline{\mu}, H\right)\overline{\mu} + \sigma_a\left(\overline{\mu}, L\right)\left(1-\overline{\mu}\right)} \tag{10}$$

whenever $\sigma_a(\overline{\mu}, \tau) > 0$ for some $\tau \in \{H, L\}$. When $\overline{\mu} \geq \mu_E$, both types are expected to keep their names ($a = O$) so that the action is uninformative. On the other hand, $a = O$ is a zero-probability event when $\overline{\mu} < \mu_E$; in this case, we assume that such off-equilibrium move is also uninformative:

$$\varphi(\overline{\mu}|O) = \overline{\mu} \quad \forall \overline{\mu} \in \overline{U} \tag{11}$$

When $a \in \{N, I\}$, $\overline{h}$ becomes unobservable, so that conditioning on $\overline{\mu}$ is not possible. In this case, consistency of beliefs requires that the interim reputation for firms choosing each action be equal to the fraction of competent firms in that group (if non-empty), given $\boldsymbol{\sigma}^*$. Thus, the entrants' reputation $\varphi(\overline{\mu}|N)$ satisfies

$$\mu_E = \frac{\overline{m}^H \overline{G}\left(\mu_E|H\right) + \lambda^{LH}\kappa}{\overline{G}\left(\mu_E\right)} \tag{12}$$

whenever $\overline{G}(\mu_E) > 0$. In fact, in the steady state the mass of entrants (i.e., new names) is equal to the mass of firms that exit (i.e., the mass of lost names $\overline{G}(\mu_E)$); the competent "entrants", in turn, are the inactive firms that become competent (with mass $\lambda^{LH}\kappa$) plus the competents that exited in order to change their names (with mass $\overline{m}^H\overline{G}(\mu_E|H)$). Similarly, the interim reputation of inactive firms $\varphi(\overline{\mu}|I)$ is null as no competent chooses this action.

Hence, there are two cases in which the interim reputation would differ from the prior: either the name changed, or the action is perfectly revealing.

Summarizing, the interim reputation is given by

$$\varphi(\overline{\mu}|a) = \begin{cases} \overline{\mu} & \text{if } a = O, \\ \mu_E & \text{if } a = N, \\ 0 & \text{if } a = I. \end{cases} \tag{13}$$

After a signal $r \in (0, 1)$ is realized, the interim reputation of an active firm is updated to

$$\overline{\varphi}\left(\mu|r\right) = \lambda^{LH} + \left(\lambda^{HH} - \lambda^{LH}\right)\frac{f^H\left(r\right)\mu}{f^H\left(r\right)\mu + f^L\left(r\right)\left(1-\mu\right)}, \tag{14}$$

which amounts to Bayes' rule upon consideration of the possibility of type-change. Since f^H and f^L have full support, Equation (14) is well-defined for all $\mu \in [0, 1]$ and $r \in (0, 1)$. Notice that the assumption that $(\lambda^{HH} - \lambda^{LH}) > 0$ implies that the posterior reputation is strictly increasing in the interim reputation.

Regarding inactive firms, Bayes' rule requires their posterior reputation to be

$$\overline{\varphi}\left(\mu|\emptyset\right) = \lambda^{LH} \tag{15}$$

as they are subject to the same type change process.

As the likelihood ratio $\frac{f^H(r)}{f^L(r)}$ is a monotonically increasing bijection from $(0,1)$ to $(0,\infty)$, it follows that the range of prior reputations is $\overline{U} = [\lambda^{LH}, \lambda^{HH}]$, and the range of interim reputations is $U = [\lambda^{LH}, \lambda^{HH}] \cup \{0, \mu_E\}$.

Consistency of beliefs requires also that the mean prior and the mean interim reputation of active firms are $\overline{m}^H$ and m^H respectively. Moreover, for any $x \in \overline{U}$, the probability of being competent conditional on the firm's prior reputation being x is exactly x. If the reputation distributions are absolutely continuous, this translates into

$$\frac{\overline{m}^H \overline{g}\left(x|H\right)}{\overline{g}\left(x\right)} = x, \tag{16}$$

where $\overline{g}\left(\cdot|\tau\right)$ denotes the pdf of prior reputations conditional on current type τ, and $\overline{g}\left(\cdot\right)$ denotes the unconditional pdf.

4.2 Optimality and market clearing in steady state

We now turn to requirements *i* and *ii*. Given the equilibrium beliefs, the value function in Equation (8) is given by

$$\begin{aligned} v\left(\overline{\mu},\tau\right) = \max_{\boldsymbol{\sigma}\in\Delta(A)} \Bigg\{ &(\sigma_O+\sigma_N)(p_0-c) + (u^H-u^L)(\sigma_O\overline{\mu}+\sigma_N\mu_E) \\ &+ \delta \sum_{\tau'\in\{H,L\}} \lambda^{\tau\tau'} \left(\int_0^1 \left[\sigma_O v\left(\overline{\varphi}(\overline{\mu}|r),\tau'\right) + \sigma_N v(\overline{\varphi}(\mu_E|r),\tau')\right] dF^\tau + \sigma_I v(\lambda^{LH},\tau')) \right) \Bigg\} \end{aligned} \tag{17}$$

for a firm of prior reputation $\overline{\mu}$ and current type τ. The flow payoff $(\sigma_O+\sigma_N)(p_0-c)+(u^H-u^L)(\sigma_O\overline{\mu}+\sigma_N\mu_E)$ and the law of motion $\overline{\varphi}$ are continuous and non-decreasing functions in $\overline{\mu}$. On the other hand, the signal for a competent firm first-order stochastically dominates that of an inept one. It follows that the value function is non-decreasing in $\overline{\mu}$ for both types, and that the value for the competent type is larger than the value for an inept type at any reputation level.[5]

The fact that the value function is non-decreasing in $\overline{\mu}$ implies that any firm with prior reputation below μ_E that chooses to produce will sell under a new name; hence, $\sigma^*_O(\overline{\mu},\tau) = 0$ is indeed optimal if $\overline{\mu} < \mu_E$.

On the other hand, as $v\left(\overline{\mu},H\right) > v\left(\overline{\mu},L\right)$, it follows that market clearing (Requirement *ii*) amounts to a zero-profit condition that is binding only for inept entrants, namely,

$$v\left(\mu_E,L\right) = 0. \tag{18}$$

As the flow payoff is linear in p_0, for each cost level $c \in \mathbb{R}_+$ there will be one $p_0 \in \mathbb{R}$ such that Equation (18) holds. As a consequence, the policy function in Equation (9) is indeed optimal (Requirement *i*).

[5] Let $\mathcal{N}$ be the set of bounded, continuous, non-decreasing functions that map $\mathbb{R}^2$ into $\mathbb{R}$, endowed with the sup norm. Let us associate type $\tau = L$ to the number 0, and type $\tau = H$ to 1. Then, the Bellman operator maps $\mathcal{N}$ into $\mathcal{N}$, and as $\mathcal{N}$ is a closed subset of the set of continuous and bounded functions, its unique fixed point v also lies in this set. (See, for instance, Theorem 1.6, Lemma 1.8 and Lemma 1.10 in Chapter 12 of De la Fuente (2000)).

The levels of m^H and m^L are determined by the firms' strategies and the market clearing condition (Requirement *ii*), taking into account the type-change process. The type-change process applied to active firms yields

$$\begin{pmatrix} \overline{m}^H \\ \overline{m}^L \end{pmatrix} = \begin{pmatrix} \lambda^{HH} & \lambda^{LH} \\ \lambda^{HL} & \lambda^{LL} \end{pmatrix} \begin{pmatrix} m^H \\ m^L \end{pmatrix}. \tag{19}$$

The same transition matrix Λ applies to inactive firms; hence, after the mass of $\lambda^{LH}\kappa$ of inactive firms that become competent enters the market, with the corresponding exit of an equal mass of inept firms, we obtain

$$\begin{pmatrix} m^H \\ m^L \end{pmatrix} = \begin{pmatrix} \overline{m}^H \\ \overline{m}^L \end{pmatrix} + \begin{pmatrix} \lambda^{LH}\kappa \\ -\lambda^{LH}\kappa \end{pmatrix}. \tag{20}$$

Solving for m^H and taking into account that $m^H + m^L = 1$, we obtain

$$m^H = \frac{\lambda^{LH}}{\lambda^{LH} + \lambda^{HL}}(1+\kappa). \tag{21}$$

The upper bound on κ in Equation (4) implies that $m^H < \lambda^{HH}$. As consistency of beliefs implies that m^H is the mean interim reputation, this assumption rules out the case in which all firms change their names at all times.

After all inept firms with $\overline{\mu} < \mu_E$ leave the market, the mass of inept potential entrants is $\overline{m}^L\overline{G}\left(\mu_E|L\right)+\lambda^{LL}\kappa$. Since the mass of inept firms with $\overline{\mu} > \mu_E$ is $\overline{m}^L\left(1-\overline{G}\left(\mu_E|L\right)\right)$, then the equilibrium behavior strategy ξ for inept firms under the threshold μ_E must satisfy

$$\xi\left(\overline{m}^L\overline{G}\left(\mu_E|L\right)+\lambda^{LL}\kappa\right)+\overline{m}^L\left(1-\overline{G}\left(\mu_E|L\right)\right)=m^L.$$

Rearranging and using Equation (20), we get

$$\xi = \frac{\overline{m}^L\overline{G}\left(\mu_E|L\right)-\lambda^{LH}\kappa}{\overline{m}^L\overline{G}\left(\mu_E|L\right)+\lambda^{LL}\kappa}. \tag{22}$$

4.3 Distributions and entrants' reputation in steady state

Let $\tilde{r}\left(x,\mu\right)$ denote the signal that a firm with interim reputation μ needs to obtain a posterior reputation x, and $\tilde{\mu}\left(x,r\right)$ denote the interim reputation that a firm requires to get a posterior reputation x after a signal r. Both functions are defined implicitly by

$$x = \overline{\varphi}(\mu|\tilde{r}(x,\mu)) \quad \text{and} \tag{23}$$

$$x = \overline{\varphi}(\tilde{\mu}(x,r)|r), \tag{24}$$

where $\overline{\varphi}$ is defined by Equation (14). Let $\gamma^{\tau\tau'} \equiv \dfrac{m^\tau}{\overline{m}^{\tau'}}\lambda^{\tau\tau'}$ denote the fraction of type-τ' firms at the beginning of any particular period that were type-τ in the previous period. We know that $\sum_{\tau\in\{H,L\}}\gamma^{\tau\tau'} = 1$ and $\gamma^{HH} > \gamma^{HL}$. The prior reputation distribution for current type-τ' firms is a mixture of the posterior reputation distributions for previously type-H and type-L firms, where $\gamma^{H\tau'}$ and $\gamma^{L\tau'}$ are the corresponding mixture weights. Thus, the steady-state conditional distributions of prior and interim reputations for active firms satisfy

$$\begin{pmatrix} \overline{G}\left(x|H\right) \\ \overline{G}\left(x|L\right) \end{pmatrix} = \begin{pmatrix} \gamma^{HH} & \gamma^{LH} \\ \gamma^{HL} & \gamma^{LL} \end{pmatrix} \begin{pmatrix} \int_0^{\tilde{r}(x,\mu_E)} G\left(\tilde{\mu}\left(x,r\right)|H\right)dF^H \\ \int_0^{\tilde{r}(x,\mu_E)} G\left(\tilde{\mu}\left(x,r\right)|L\right)dF^L \end{pmatrix}. \tag{25}$$

Equation (25) shows the prior reputation distributions of competent and inept active firms, respectively, as a function of the previous period's interim reputation distributions. From each particular group of firms, the firms with reputation smaller than or equal to x are those that in the previous period had an interim reputation no greater than $\tilde{\mu}(x,r)$ whose signal realization was r; the measure of the group that originally was type-τ is $\int_0^{\tilde{r}(x,\mu_E)} G(\tilde{\mu}(x,r)|\tau)\,dF^\tau$. The integrals go from 0 to $\tilde{r}(x,\mu_E)$ because a signal higher than $\tilde{r}(x,\mu_E)$ is required to reach a reputation x today only if the interim reputation was smaller than μ_E.

In turn, the distributions of interim reputations conditional on type τ are given by

$$G(x|\tau) = \begin{cases} 0 & \text{if } x < \mu_E, \\ \frac{1}{m^\tau}\left((m^\tau - \overline{m}^\tau) + \overline{m}^\tau \overline{G}(x|\tau)\right) & \text{if } x \geq \mu_E. \end{cases} \tag{26}$$

The interim distributions in Equation (26) take into consideration the strategy $\boldsymbol{\sigma}^*$, and hence differ from the prior distributions in two respects: (1) the entry of a mass $\lambda^{LH}\kappa$ of inactive firms that become competent, which replaces an equal mass of inept firms; and (2) the changing of names by firms with priors lower than μ_E that remain active.

The next theorem establishes that there is a unique consistent tuple of steady-state conditional distributions for active firms and entry-level reputation:

Theorem 1. *There is a unique tuple $\left(\mu_E, \overline{G}(\cdot|H), \overline{G}(\cdot|L)\right)$ of entry-level reputation and steady-state reputation distributions for active firms that jointly satisfy equations* (25) *and* (26)*, and such that μ_E satisfies Equation* (12)*. The reputation distributions are absolutely continuous, with support $[\lambda^{LH}, \lambda^{HH}]$ for the prior reputation distributions $\overline{G}(\cdot|\tau)$ and $[\mu_E, \lambda^{HH}]$ for the interim reputation distributions $G(\cdot|\tau)$, and $\mu_E \in \left(\lambda^{LH}, m^H\right)$.*

Theorem 1 asserts that the steady-state reputation distributions and the entry-level reputation are uniquely determined. Uniqueness is not only computationally useful but economically significant, as it implies that the level of p_0 that solves the free-entry condition in Equation (18) is also uniquely determined. Therefore, the equilibrium is unique in its class.

A sketch of the proof is as follows: First, think of μ_E as a parameter. Replacing Equation (26) in Equation (25), we see that the pair of prior distributions $\left(\overline{G}(\cdot|H), \overline{G}(\cdot|L)\right)$ is the fixed point of an operator. Theorem 1 establishes the existence and uniqueness of this fixed point which depends continuously on μ_E, and also on the parameters of the transition matrix Λ and on κ. The limiting distributions are absolutely continuous; hence, Equation (16) applies.

Now, consider a fixed pair of reputation distributions for competent and inept firms. The fraction of competent firms among entrants is given by the right-hand side of Equation (12), which we define as the function $\psi : (\lambda^{LH}, \lambda^{HH}] \to [0, \infty)$:

$$\psi(x) \equiv \frac{\overline{m}^H \overline{G}(x|H) + \lambda^{LH}\kappa}{\overline{G}(x)}. \tag{27}$$

Thus, Equation (12) says that any consistent entry-level reputation μ_E is a fixed point of the function ψ defined by Equation (27). Such a fixed point exists: The sets $\{x : x < \psi(x)\}$ and $\{x : x > \psi(x)\}$ are non-empty and ψ is continuous.[6] On the other hand, observe that for any given cutoff value x, $\psi(x)$ is the average reputation among entrants while x is the reputation of the marginal entrants. Then, the average reputation is increasing (resp., decreasing) if and only if the marginal reputation is above (resp., below) it. It follows that $\psi' = 0$ if and only if $x = \psi(x)$ and therefore the fixed point is unique: There is a unique value of μ_E at which the cutoff point coincides with the expected fraction of competents in the group of entrants. This is illustrated with a numerical example in Figure 2.

[6] In fact, the function $\psi(x)$ is constructed under the assumption that firms change their names if and only if their reputation falls below the cutoff x and all competents enter. Consider the value $x^* \in (\lambda^{LH}, \lambda^{HH})$ that satisfies $\overline{m}^L \overline{G}(x^*|L) = \lambda^{LH}\kappa$; in that case, all entrants are competent and their reputation is the highest possible: $\psi(x^*) = 1$, and therefore $x^* \in \{x : x < \psi(x)\}$. On the other hand, at $x = \lambda^{HH}$ all firms exit and $\psi(x) = m^H < \lambda^{HH}$, so that $\{x : x > \psi(x)\}$ is also non-empty.

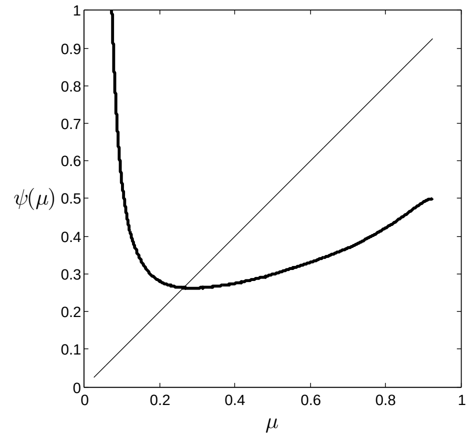


Figure 2: Consistent entry-level reputation as a fixed point

Notes: $f^H(r) = Be(r|3,2)$ and $f^L(r) = Be(r|2,3)$. The parameters are $\lambda^{HL} = 0.075$, $\lambda^{LH} = 0.025$ and $\kappa = 1$. The resulting μ_E is 0.26.

Notice that μ_E must be strictly larger than λ^{LH}, so that a positive mass of firms chooses to become inactive, making room for the entrants. At the same time, the mean interim reputation for active firms must be larger than μ_E, the minimum reputation in the support of the distribution; as consistency of beliefs implies that this mean reputation is m^H, we conclude that $\mu_E \in (\lambda^{LH}, m^H)$.

Moreover, Equation (16) implies that $\frac{\overline{g}(x|H)}{\overline{g}(x|L)}$ is increasing in x and thus the ratio $\frac{g(x|H)}{g(x|L)}$ is also increasing. In other words, consistency of beliefs implies that the monotone likelihood ratio property–which is assumed for the distributions of signals conditional on types–extends to the distributions of reputations conditional on types in the population of active firms. As likelihood ratio dominance implies first-order stochastic dominance, it follows that:

Corollary 1 (Reputation across types)**.** *The distribution of reputations (both prior and interim) of competent firms first-order stochastically dominates that of inept firms. Formally,* $(\forall x \in [\lambda^{LH}, \lambda^{HH}])$*,*

$$\overline{G}(x|H) \leq \overline{G}(x|L) \text{ and } G(x|H) \leq G(x|L)$$

with strict inequality in the interior.

The proof of Theorem (1) goes beyond our previous discussion, in that it considers simultaneously the consistency of the entry-level reputation μ_E and the distributions $(\overline{G}(\cdot|H), \overline{G}(\cdot|L))$. Clearly, μ_E affects the shape of the steady-state distributions: No firm would ever keep its name should its reputation fall below that threshold, and the interim distributions would have a point mass at μ_E every period (namely, the mass of firms that enter).

5 Comparative statics

The parameter λ^{LH}, the probability for an inept firm to become competent, measures the rate at which competency is acquired–a sort of technical progress or product innovation. In our model, this rate is exogenous and homogeneous among firms; Section 7 below contains a brief discussion of endogeneity and heterogeneity. The parameter λ^{HL}, on the other hand, measures the depreciation rate of this skill.

We are interested in how the rates of competency acquisition λ^{LH} and depreciation λ^{HL}, and outside competitive pressure κ shape the equilibrium. Not only the steady-state distributions depend continuously on them; the entry-level reputation does as well, as Proposition 1 asserts:

Proposition 1 (Continuous dependence on parameters)**.** *The tuple* $(\mu_E, \overline{G}(\cdot|H), \overline{G}(\cdot|L))$ *of entry-level reputation and steady-state reputation distributions depends continuously on the parameters of the transition matrix* Λ *and on* κ*.*

As an immediate corollary, the exit rate $\overline{G}(\mu_E)$ is also continuous on those parameters.

The level of the entrants' reputation is affected by κ and the transition parameters λ^{HL} and λ^{LH} in a non-trivial way, as they affect $\overline{m}^{\tau}$ and m^{τ} differently. In particular, the effects of changes in λ^{HL} and λ^{LH} over μ_E are not symmetric: While λ^{HL} only affects the transition matrix in Equation (19), λ^{LH} also affects the entry flow of new competent firms in Equation (20). What is symmetric, however, is that any change in parameters that results in higher values of γ^{HL} and γ^{LH} without reducing $\lambda^{LH}\kappa$ implies a higher μ_E:

Proposition 2 (Comparative statics)**.** *The entry-level reputation* μ_E *increases after any change in the parameters of the transition matrix* Λ *or* κ *that results in a increase in the mixture weights* γ^{HL} *and* γ^{LH} *without reducing the entry flow of new competent firms* $\lambda^{LH}\kappa$*.*

Increasing γ^{HL} and γ^{LH} means that the gap between γ^{HH} and γ^{HL} shortens; therefore, the prior reputation distributions for competent and inept firms move closer to each other. By this mechanism, the adverse selection that entrants face is reduced, as the fraction of competent firms among those below the threshold increases. Hence, provided that $\lambda^{LH}\kappa$ is not reduced, an increase in γ^{HL} and γ^{LH} causes μ_E to increase.

Consider first an increase in λ^{HL}, which affects only active firms in equilibrium. Intuitively, a higher λ^{HL} implies a higher "depreciation rate" of information, as the weight of older signals in predicting the current firm type decreases. Histories become less informative about types and the mixing weights γ^{HL} and γ^{LH} are increased; hence, the composition of the pool of firms changing their names improves. Therefore, the effect of an increase in λ^{HL} over μ_E is unambiguous: the entrants' reputation increases after an increase in λ^{HL}. The same result obtains if the increase in λ^{HL} is compensated, either by a proportional increase in λ^{LH} or by an increase in κ, so that m^H remains fixed.

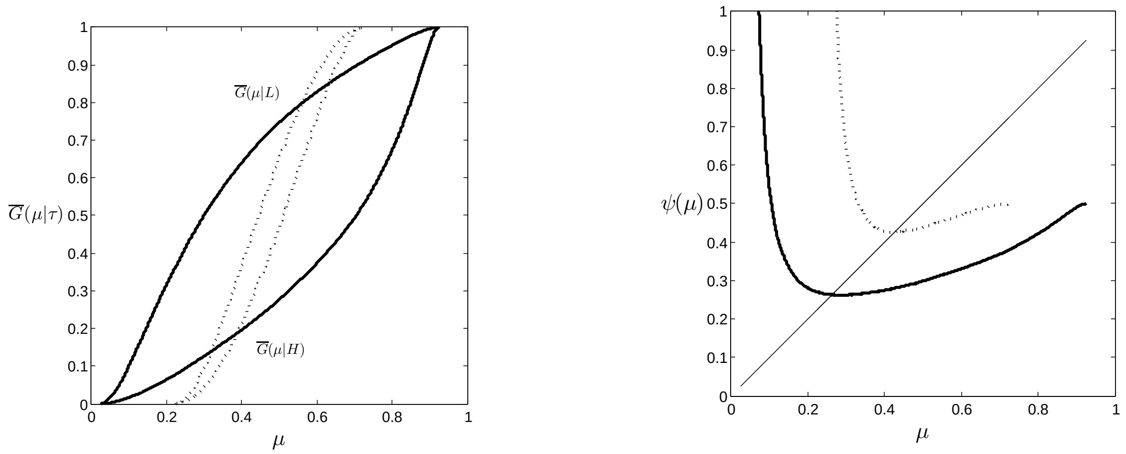
The case of an isolated increase in λ^{LH} is different, however, because this type-change probability not only affects active firms but also inactive ones. A higher λ^{LH} implies a higher $\overline{m}^H$ as active inept firms are more likely to become competent; the effect over m^H is proportionally larger, however, because a higher flow of new competent firms is added to $\overline{m}^H$ at each period. This implies that the mixing weight γ^{LH} decreases after an increase in λ^{LH}, so that the sufficient condition in Proposition 2 is not met.

Finally, μ_E increases if λ^{HL} and λ^{LH} jointly increase while κ decreases so that the flow of new competent firms $\lambda^{LH}\kappa$ does not change, and both m^H and $\overline{m}^H$ remain constant. Figure 3 illustrates this case.

Proposition 3 below summarizes these results:

Proposition 3. *The sufficient conditions for an increase in* μ_E *from Proposition 2 are met in all the following circumstances:*

1. *An increase in* λ^{HL}*;*
2. *A proportional increase in* λ^{HL} *and* λ^{LH}*;*
3. *An increase in* λ^{HL} *and* κ *so that* m^H *remains constant;*
4. *An increase in* λ^{HL} *and* λ^{LH}*, and a reduction in* κ *such that* m^H *and* $\overline{m}^H$ *remain constant.*



(a) Distributions $\overline{G}(\mu|H)$ and $\overline{G}(\mu|L)$

(b) Fraction of competents among entrants $\psi(\mu)$

Figure 3: Distributions and entry-level reputation for different combinations of λ^{LH}, λ^{HL} and κ.

Notes: $f^H(r) = Be(r|3,2)$ and $f^L(r) = Be(r|2,3)$. The parameters are $\lambda^{HL} = 0.075, \lambda^{LH} = 0.025$ and $\kappa = 1$ (solid line) and $\lambda^{HL} = 0.275, \lambda^{LH} = 0.225$ and $\kappa = 1/9$ (dotted line), so that $m^H = 0.5$ and $\overline{m}^H = 0.475$ in both cases. The resulting μ_E is 0.26 in the first case and 0.43 in the second one.

6 Industry dynamics

We are ready to describe the dynamics of entry, exit, and reputations within the name-switching reputational equilibrium. Exit occurs when a name was used at some date and not on the following one. Our focus is on a steady state where the net entry rate is zero. Still, there is a constant renewal (exit and entry), given by the turnover rate $\overline{G}(\mu_E)$, i.e., the fraction of incumbents that leave the market.

6.1 Exit rates and turnover

The exit probability is the probability that the firm's reputation will fall below the threshold μ_E. As competent firms' signals are stochastically larger than inept firms', among firms with the same prior reputation $\overline{\mu}$ the exit rate is lower for competent than for inept firms.

On the other hand, according to Bayes' rule the posterior probability of an event is increasing in its prior. Consequently, the lower the prior reputation $\overline{\mu}$, the higher the exit rate among firms with the same signal. Moreover, as the likelihood ratio is monotone, the exit rate is higher for firms with lower signals r among those with the same prior.

At the industry level, the exit probability is the turnover rate $\overline{G}(\mu_E)$. Because the turnover rate depends on μ_E and on the population-wide reputation distribution, a larger entry-level reputation is not necessarily associated to a larger exit probability: An increase in μ_E means that each firm will replace its name in a larger set of states, but it also will shift the reputation distribution, which may make the firm less likely to reach those states where its name is replaced. The result is ambiguous: The frequency of name changes may increase or decrease.

6.2 Age

A firm's age $n = 0, 1, 2, ...$ is an attribute of its current name, namely, the number of dates that it has been used in the market. In the steady state, the group of age n is identical to the group of age 0, n periods into the future. In this sense, studying the cross-sectional variation (across cohorts) is equivalent to studying the evolution over time of a given cohort.[7]

All names that were introduced n periods ago make up the cohort n; the prior mass (i.e., before exit) of cohort n is denoted by $\overline{m}_n$, and the interim mass (i.e., after exit) by m_n. Let $\overline{G}\left(.|\tau, n\right)$ denote the prior reputation distribution of the set of firms of type τ and cohort n, and $\overline{m}_n^\tau$ its mass; similarly, $G\left(.|\tau, n\right)$ and m_n^τ denote the interim reputation distributions and the cohort's mass. The corresponding probability density functions of prior and interim reputations conditional on type τ and age n will be denoted by $\overline{g}\left(.|\tau, n\right)$ and $g\left(.|\tau, n\right)$, respectively.

At any date, a new cohort of mass $\overline{G}\left(\mu_E\right)$ enters. A fraction of them, μ_E, is competent: $m_0^H = \mu_E \overline{G}\left(\mu_E\right)$. As all new names carry the same reputation μ_E, we have for $\tau \in \{H, L\}$:

$$G(x|\tau, 0) = \begin{cases} 0 & \text{if } x < \mu_E, \\ 1 & \text{if } x \geq \mu_E. \end{cases} \tag{28}$$

As time goes by, in each period two changes occur: (i) The type-change process shifts the masses of competent and inept firms within the cohort according to

$$\overline{m}_{n+1}^\tau = \lambda^{H\tau} m_n^H + \lambda^{L\tau} m_n^L, \tag{29}$$

and (ii) The mass of surviving names in each subpopulation τ shrinks by a factor of $\left(1 - \overline{G}\left(\mu_E|\tau, n+1\right)\right)$, as those firms that exit are not replaced by other firms from the same cohort. Hence,

$$m_{n+1}^\tau = \overline{m}_{n+1}^\tau \left(1 - \overline{G}\left(\mu_E|\tau, n+1\right)\right). \tag{30}$$

Let $\gamma_n^{\tau\tau'} \equiv \dfrac{m_n^\tau}{\overline{m}_{n+1}^{\tau'}} \lambda^{\tau\tau'}$. The evolution of the prior reputation distributions in a given cohort at different ages is given by

$$\begin{pmatrix} \overline{G}\left(x|H, n+1\right) \\ \overline{G}\left(x|L, n+1\right) \end{pmatrix} = \begin{pmatrix} \gamma_n^{HH} & \gamma_n^{LH} \\ \gamma_n^{HL} & \gamma_n^{LL} \end{pmatrix} \begin{pmatrix} \int_0^{\tilde{r}(x,\mu_E)} G\left(\tilde{\mu}\left(x, r\right)|H, n\right) dF^H \\ \int_0^{\tilde{r}(x,\mu_E)} G\left(\tilde{\mu}\left(x, r\right)|L, n\right) dF^L \end{pmatrix}. \tag{31}$$

For $n > 0$, the distributions of interim reputations relate to the priors' as follows:

$$G\left(x|\tau, n\right) = \begin{cases} 0 & \text{if } x < \mu_E, \\ \frac{\overline{G}(x|\tau,n) - \overline{G}(\mu_E|\tau,n)}{1 - \overline{G}(\mu_E|\tau,n)} & \text{if } x \geq \mu_E. \end{cases} \tag{32}$$

Equation (31) is analogous to Equation (25); the difference is in the mixing weights. In the population of active firms as a whole the total mass and the ratio of competent to inept are constant over time. In contrast, not only is each cohort losing mass over time, but also each type does so at different rates. Similarly, Equation (32) resembles Equation (26); they differ in that within each cohort there is only exiting and no entry.

[7] Although firms may remain inactive for some time, when they become active again they do so under a new name in equilibrium. This is why the age of active firms coincides with the number of consecutive dates of production under the current name.

6.2.1 Differences across types

Starting from the interim reputation distribution for new firms in Equation (28) and applying equations (31) and (32), it can be shown by induction that for all $n \in \mathbb{N}$,

$$\frac{\overline{m}_n^H \overline{g}(x|H,n)}{\overline{m}_n \overline{g}(x|n)} = x. \tag{33}$$

Equation (33) is the analogous to Equation (16) when conditioning on age. Consistency of beliefs thus implies the monotone likelihood ratio property of the distributions of reputations conditional on types within cohorts, as the likelihood ratios $\frac{\overline{g}(x|H,n)}{\overline{g}(x|L,n)}$ and $\frac{g(x|H,n)}{g(x|L,n)}$ are increasing in x. It follows that:

Proposition 4 (Reputation across types by cohort)**.** *Within each cohort $n \geq 1$ the (prior, interim) reputation of competent firms first-order stochastically dominates the (prior, interim) reputation of inept firms. Formally,* $(\forall n \in \mathbb{N})\,(\forall x \in [\lambda^{LH}, \lambda^{HH}])$,

$$\overline{G}(x|H,n) \leq \overline{G}(x|L,n) \text{ and } G(x|H,n) \leq G(x|L,n),$$

with strict inequality in the interior.

In view of the linear connection between prices and interim reputations and the connection between exit rates and prior distributions, Proposition 4 implies:

Corollary 2. *Competent firms charge higher prices (in a stochastic sense) and exit less often than inept firms, both within each cohort and throughout the population.*

6.2.2 Differences across cohorts

Clearly, the interim reputation of older cohorts first-order stochastically dominates that of the cohort of age 0. As the equilibrium price is a linear function of the firms' interim reputation, the price distributions inherit the properties of the interim reputation distributions. In particular, the price that older cohorts charge first-order stochastically dominates that of entrants.

On the other hand, the exit (or name-switching) decision is made in response to the prior reputation. The next proposition shows that the prior reputation of older cohorts also first-order stochastically dominates that of age 1, which implies that the exit probability of older firms is smaller.

Proposition 5 (Reputation across cohorts)**.** *The prior reputation of firms of age 1 is first-order stochastically dominated by the prior reputation of firms of any older cohort, both conditional and unconditional on types. Formally,* $(\forall n \in \mathbb{N})\,(\forall x \in [\lambda^{LH}, \lambda^{HH}])$*:*

$$\overline{G}(x|\tau,n) \leq \overline{G}(x|\tau,1) \text{ and } \overline{G}(x|n) \leq \overline{G}(x|1),$$

with strict inequality in the interior for $n > 1$.

Again, the connection between the prior distribution and the exit rate implies:

Corollary 3. *Older firms exit less often than firms of age 1, both conditioning and not conditioning on type.*

Figure 4 shows the family of prior reputation distributions by cohorts in a numerical example where older firms (i.e., names) have stochastically better reputations than younger ones. This means that exit rates are monotonically decreasing in age, and the price distributions are also ordered by first-order stochastic dominance.

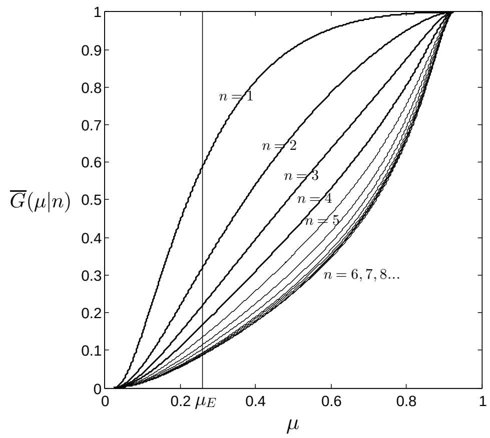


Figure 4: Reputation distributions for different ages, $\overline{G}\left(\mu|n\right)$

Notes: $f^H\left(r\right) = Be\left(r|3,2\right)$ and $f^L\left(r\right) = Be\left(r|2,3\right)$. The parameters are $\lambda^{HL} = 0.075$, $\lambda^{LH} = 0.025$ and $\kappa = 1$. The resulting μ_E is 0.26.

6.3 Relation to empirical literature

The empirical literature on industry dynamics finds considerable heterogeneity among industries in terms of entry and exit rates. Typically, gross entry and exit rates are similar within an industry (Dunne et al., 1988). The focus on the steady state is thus particularly suitable for analyzing those industries.

The analysis in Section 6.1 suggests that heterogeneity among firms within an industry should also be expected: The exit probability is higher for names with worse reputations and/or worse signals. The empirical literature is consistent with these predictions: Cabral and Hortacsu (2010), in a study of eBay auctions, find that the probability that a name exits the market increases as its reputation declines (as defined by eBay's reputation mechanism.) In turn, McDevitt (2011) studies the plumbing services market in Illinois and finds that, all else being equal, the firms more likely to change names or withdraw are those with worse track records–a variable that resembles the history of public signals in our model.

The analysis also shows that inept firms are more likely to leave the market than competent ones. Although types are unobservable to the econometrician, this proposition has testable implications: As the exit process is biased towards inept firms, older firms have (stochastically) larger reputations than younger firms, as Proposition 5 asserts. Moreover, since firms with better reputations charge higher prices, this implies that older firms charge (stochastically) larger prices. These predictions are also consistent with the empirical literature, which finds systematic differences between firms of different ages: In Foster et al. (2008) younger firms are more likely to exit, and charge lower prices, than older firms. In the same vein, in McDevitt (2011) the exit probability is monotonically decreasing with age. Moreover, while these studies find a strong positive relation between survival and age, quality may explain the apparent correlation between those variables (Thompson, 2005).

In view of these findings, one would expect a selected sample of old, highly reputable firms to change names at a low rate; this is precisely what Wu (2010) finds when examining CRSP-listed companies. Correspondingly, an unbiased sample of firms within an industry should exhibit a (perhaps considerably) higher incidence of name-switching, just as McDevitt (2011) finds in the Illinois plumbing-service industry.

The previous discussion suggests that a reputation-based model can be very helpful in improving our understanding of industry dynamics.

7 Discussion

The model developed in this paper is almost as simple as possible within the class of dynamic adverse selection models: There are two types, all long-run players use the same behavior strategy $\boldsymbol{\sigma}^*$–which depends on the simplest state space–, and the attention is centered on the steady state. Imperfect public monitoring coupled with a type-change process results in a market with a continuum of firms with different reputations charging different prices. Reputation heterogeneity is limited by strategic exit and entry: Firms have the option to reset their reputations by exiting the market and reentering under a new name.

Many features of the model may be amended without essential effects on the equilibrium dynamics:

Moral hazard. Probably the foremost real-world issue that was left aside is moral hazard. However, the dynamics studied here under the assumption that competent firms can only provide high quality are the same as those of a model where competent firms *choose* to do so. The conditions under which they do are studied in Vial and Zurita (2013) in a similar setting.

Long-lived consumers. In many real-world examples–some of which the cited literature describes–consumers purchase more than once. If long-lived consumers accumulate private information based on their experiences with suppliers, the belief homogeneity exploited in this paper breaks down, opening up the possibility of relationship-building. What is crucial to the results in a setting with a continuum of buyers and sellers, however, is the type of monitoring rather than the repeated interaction between them. For instance, in Mailath and Samuelson (2001) consumers are long-lived, but monitoring remains public.[8] Thus, our model can be interpreted as one with long-lived consumers and public monitoring as well.

Type change vs. replacement. The effect of the type change process in our model is to permanently "replenish" the uncertainty about types, i.e., to put tighter upper and lower bounds on reputations. We could instead consider an exogenous replacement process with the same effect.[9] By the replacement process, any name can pass from an old firm to a new one. Let's define λ and θ so that $\lambda^{LH} = \lambda\theta$ and $\lambda^{HL} = \lambda(1-\theta)$, while $\lambda^{LH} + \lambda^{HL} = \lambda$. Then λ could be interpreted as the probability that an active firm is replaced (i.e., dies); if this event occurs, the name of the old firm is (randomly) assigned to a new competent firm with probability $\theta \in (0,1)$ and to a new inept firm with probability $1-\theta$. As the consequence of replacements and type-changes are identical from the consumer's point of view, the interim reputation would still be updated according to Bayes' rule in Equation (14). So, if the prior is updated according to Equation (13), the value function for a firm of prior reputation $\overline{\mu}$ and type τ would instead be given by

$$\begin{aligned} v(\overline{\mu},\tau) = \max_{\boldsymbol{\sigma}\in\Delta(A)} \Bigg\{ & (\sigma_O+\sigma_N)(p_0-c) + (u^H-u^L)(\sigma_O\overline{\mu}+\sigma_N\mu_E) \\ & + \delta(1-\lambda)\left(\int_0^1 \Big[\sigma_O v(\overline{\varphi}(\overline{\mu}|r),\tau) + \sigma_N v(\overline{\varphi}(\mu_E|r),\tau)\Big] dF^\tau + \sigma_I v(\lambda^{LH},\tau))\right)\Bigg\}. \end{aligned}$$

The policy functions, however, would not be affected: The strategy in Equation (9) is still optimal given

[8] They assume that the ex post utility is the signal itself. If so, u^τ would be the expected value of the signal conditional on the quality that a type-τ seller provides, i.e., $u^H = \int_0^1 r dF^H$ and $u^L = \int_0^1 r dF^L$. The condition $u^H > u^L$ would follow from stochastic dominance.

[9] We consider exogenous replacements. In contrast, Tadelis (1999) and Mailath and Samuelson (2001) study the possibility of trading names when this trading is unobservable to consumers, while Wang (2011) and Hakenes and Peitz (2007) look at the observable trading case. Board and Meyer-ter Vehn (2013) and Dilmé (2012) consider the possibility of choosing types.

consumers' beliefs if p_0 is such that $v(\mu_E, L) = 0$. Moreover, as reputations and age are associated with a name and not with the identity of the firms that carry that name at different stages, the industry dynamics would not be affected. The only element of the name-switching equilibrium that would be affected is the price function: The (unique) level of p_0 that solves the free-entry condition $v(\mu_E, L) = 0$ would be lower under the type-changing process than under the replacement process, as firms have the option to continue operating after a type change.

Quality cost differentials. In our model, competent firms unambiguously profit more than inept firms of the same reputation. This follows from the monotone likelihood ratio property and the assumption that competent and inept firms have the same production cost. This assumption ensures that competent firms strictly prefer to enter when inept firms are merely indifferent, resulting in a pool of entrants that is better than the pool of exiting firms. Instead, we could assume that competent firms have higher costs than inept ones, as long as the advantage of having an easier road to higher reputations that the monotone likelihood ratio property entails outweighs the cost differential.

The model showed the importance of the entry-level reputation as an equilibrating variable of the market. This message is likely to extend to other environments. Further interesting extensions include:

Differences between active and inactive firms. In our model, both active and inactive firms are subject to the same type-change process. Let λ_a^{LH} denote the type-change probability induced by action a. There may be examples where the probability of becoming competent is different between active and inactive firms: $\lambda_{O,N}^{LH} \neq \lambda_I^{LH}$. An interesting case occurs when the competency-acquisition rate is positive only among active firms: $\lambda_{O,N}^{LH} > 0$ and $\lambda_I^{LH} = 0$. In that case, the adverse selection problem that entrants face is so acute that there are neither entry nor exit flows in equilibrium. This is due to the fact that the only competent entrants would be firms with bad histories that decided to discard their names because their prior reputation was lower than μ_E, but consistency requires that the entrants' reputation be the mean prior reputation in that group–a contradiction. Then, the group of entrants must be empty. In other words, names with a poor reputation are replaced by new names if and only if $\lambda_I^{LH} > 0$, as firms with poor reputation gain from pooling as new entrants with new competents. Although two industries with the same m^H would not differ in terms of the mean reputation of trading firms, their exit and entry flows would be completely different if $\lambda_I^{LH} > 0$ in the first industry and $\lambda_I^{LH} = 0$ in the second one.

Endogenous types. A natural extension is giving firms the possibility of investing, or paying a cost, to become competent prior to entry, as in Atkeson et al. (2015). The fact that competent firms have an advantage over inept firms would generate a willingness-to-pay for an increased probability of becoming competent. The (endogenized) λ_I^{LH} should adjust until the marginal firm is indifferent between investing or not. The adjustment would occur through the influence of λ_I^{LH} on the entry-level reputation and its effect on the payoff advantage of competent firms. Therefore, given equal production costs for both types of firms, $\lambda_I^{LH} > 0$ would be an equilibrium outcome instead of an assumption.

Furthermore, the type-change process may be endogenized by also enabling incumbent firms to invest, as in Board and Meyer-ter Vehn (2013): Firms may pay a cost to increase the probability of becoming competent in the next stage, $\lambda_{O,N}^{LH}$. If most of the information that fosters innovative activity comes from outside the market, new firms would have an advantage over incumbent firms in acquiring competence; in contrast, if the main source of this information is nontransferable experience in the market, the advantage would reverse.[10]

Entry or name-switching costs. Another interesting extension would be to consider an entry cost. If it were a sunk cost, it would create a wedge between the entry-level and exit-level reputations, as the value of entering would be smaller than the value of staying in the market, all else being equal. This would affect the turnover rate and the age distribution of firms, as the incentive for switching names would be reduced. The equilibrium price level would depend on the magnitude of the entry cost, as the free entry condition is binding for inept firms paying this cost. With moral hazard, this

[10] See Chapter 3 in Audretsch (1995).

would also have implications for efficiency, as discussed by Atkeson et al. (2015) and Garcia-Fontes and Hopenhayn (2000). A similar wedge would emerge if erasing the history (by changing names in our model) were costly (or risky); moreover, as the entry-level reputation is larger than the exit-level reputation all entrants choose a new name in equilibrium. Thus, an entry cost is equivalent to a name-switching cost.

8 Conclusions

We presented a reputation model where industry dynamics is driven by the stochastic movement of firms' reputations along with an option to change names. The constant renewal of names of disgraced firms is prevalent in markets where identities can be changed or concealed at low cost; for instance, we observe this with some Internet trading sites offering services such as home repair and maintenance and with brick-and-mortar industries such as restaurants. Through a simple name-switching policy the model rationalizes important features present in the data, like the positive correlation of reputation and price with age, and the negative correlation of reputation with exit.

Beyond matching empirical findings, the model yields insight into the role of the different forces shaping the equilibrium within a complex causality network. The preference parameters (consumers' valuations and firms' discount factor and production costs) directly affect players' choices, and through them, the equilibrium assignment and price function. For a given strategy profile, the informational variables (the distribution of signals conditional on types, the parameters of the type-change process and the outside competitive pressure) determine consumer beliefs. Still, the type-change process and the outside competitive pressure are not merely informational variables, as they determine the composition of firms in the population. Since the price function solves a zero-profit condition that is binding for marginal entrants, it depends on the informational variables: While the entrants' reputation affects their flow payoff, the expectation of future payoffs that stem from their subsequent reputations affect their continuation payoffs.

The entrants' reputation emerges as a key equilibrium variable that determines the name-switching rate and therefore, the whole industry dynamics. The effect of the parameters of the transition process and the outside competitive pressure over the entry-level reputation is shown to depend crucially on its effect over the fraction of firms of a given type that were of the same type in the previous period: The lower this fraction, the closer to each other are the distributions for competent and inept firms and the better the composition of the pool of name-switching firms is. Hence, industry dynamics are driven by the endogenous name-switching process, while this is in turn determined by the exogenous type-change process.

While the results are obtained under a variety of simplifying assumptions, they are likely to be robust to the consideration of important phenomena like moral hazard or differences in competency acquisition and depreciation rates between active and inactive firms. Still, the analysis points towards many interesting questions for future research.

Appendix

A.1 Proof of Theorem 1

We proceed in two steps. First, the entry-level reputation is assumed to be an exogenous parameter $y \in (0, \lambda^{HH})$. Under this assumption, Lemma 1 shows that there is a unique steady-state distribution pair for competent and inept firms.

Second, the entry-level reputation y is endogenized by requiring it to be consistent: $y = \mu_E$. Indeed, consistency implies that the fraction of competent firms among active firms with a given reputation μ is precisely μ, and similarly, that the fraction of competent firms among entrants (if any) is precisely μ_E. Lemma 2 shows that there is at least one consistent entry-level reputation μ_E, while Lemma 3 shows that this entry-level reputation is unique.

Consider the system of integral equations defined by

$$\begin{pmatrix} \overline{G}_{t+1}(x|H) \\ \overline{G}_{t+1}(x|L) \end{pmatrix} \equiv \begin{pmatrix} \gamma^{HH} & \gamma^{LH} \\ \gamma^{HL} & \gamma^{LL} \end{pmatrix} \begin{pmatrix} \int_0^{\tilde{r}(x,y)} G_t(\tilde{\mu}(x,r)|H)\, dF^H \\ \int_0^{\tilde{r}(x,y)} G_t(\tilde{\mu}(x,r)|L)\, dF^L \end{pmatrix} \tag{34}$$

and

$$G_t(x|\tau) = \begin{cases} 0 & \text{if } x < y, \\ \frac{1}{m^\tau}\left((m^\tau - \overline{m}^\tau) + \overline{m}^\tau \overline{G}_t(x|\tau)\right) & \text{if } x \geq y. \end{cases} \tag{35}$$

Replacing Equation (35) in Equation (34) and rearranging, we get:

$$\begin{aligned} \begin{pmatrix} \overline{G}_{t+1}(x|H) \\ \overline{G}_{t+1}(x|L) \end{pmatrix} &= \lambda^{LH}\kappa \begin{pmatrix} \frac{\lambda^{HH}}{\overline{m}^H} & -\frac{\lambda^{LH}}{\overline{m}^H} \\ \frac{\lambda^{HL}}{\overline{m}^L} & -\frac{\lambda^{LL}}{\overline{m}^L} \end{pmatrix} \begin{pmatrix} F^H(\tilde{r}(x,y)) \\ F^L(\tilde{r}(x,y)) \end{pmatrix} \\ &+ \begin{pmatrix} \lambda^{HH} & \lambda^{LH}\frac{\overline{m}^L}{\overline{m}^H} \\ \lambda^{HL}\frac{\overline{m}^H}{\overline{m}^L} & \lambda^{LL} \end{pmatrix} \begin{pmatrix} \int_0^{\tilde{r}(x,y)} \overline{G}_t(\tilde{\mu}(x,r)|H)\, dF^H \\ \int_0^{\tilde{r}(x,y)} \overline{G}_t(\tilde{\mu}(x,r)|L)\, dF^L \end{pmatrix}. \end{aligned} \tag{36}$$

Alternatively, the change of variables $\mu = \tilde{\mu}(x,r)$ and $r = \tilde{r}(x,\mu)$ (as defined in equations (24) and (23)) inside the integral in Equation (36) allows it to be written as:

$$\begin{aligned} \begin{pmatrix} \overline{G}_{t+1}(x|H) \\ \overline{G}_{t+1}(x|L) \end{pmatrix} &= \lambda^{LH}\kappa \begin{pmatrix} \frac{\lambda^{HH}}{\overline{m}^H} & -\frac{\lambda^{LH}}{\overline{m}^H} \\ \frac{\lambda^{HL}}{\overline{m}^L} & -\frac{\lambda^{LL}}{\overline{m}^L} \end{pmatrix} \begin{pmatrix} F^H(\tilde{r}(x,y)) \\ F^L(\tilde{r}(x,y)) \end{pmatrix} \\ &- \begin{pmatrix} \lambda^{HH} & \lambda^{LH}\frac{\overline{m}^L}{\overline{m}^H} \\ \lambda^{HL}\frac{\overline{m}^H}{\overline{m}^L} & \lambda^{LL} \end{pmatrix} \begin{pmatrix} \int_y^1 \overline{G}_t(\mu|H)\, f^H(\tilde{r}(x,\mu))\, \frac{\partial \tilde{r}(x,\mu)}{\partial \mu} d\mu \\ \int_y^1 \overline{G}_t(\mu|L)\, f^L(\tilde{r}(x,\mu))\, \frac{\partial \tilde{r}(x,\mu)}{\partial \mu} d\mu \end{pmatrix}. \end{aligned} \tag{37}$$

Define the right-hand side of Equation (36) (or alternatively, that of Equation (37)) as an operator T in the set of pairs of continuous and normalized functions $\left(\overline{G}(\cdot|H), \overline{G}(\cdot|L)\right)$ endowed with the following metric:[11]

$$\rho\left(\left(\overline{G}(\cdot|H), \overline{G}(\cdot|L)\right), \left(\overline{G}'(\cdot|H), \overline{G}'(\cdot|L)\right)\right) = \max\left\{\rho_\infty\left(\overline{G}(\cdot|H), \overline{G}'(\cdot|H)\right), \rho_\infty\left(\overline{G}(\cdot|L), \overline{G}'(\cdot|L)\right)\right\},$$

where:

$$\rho_\infty\left(\overline{G}(\cdot|\tau), \overline{G}'(\cdot|\tau)\right) = \sup_{x\in\left[\lambda^{LH}, \lambda^{HH}\right]} \left|\overline{G}^\tau(x|\tau) - \overline{G}'(x|\tau)\right|$$

for $\tau \in \{H, L\}$. The supremum is taken over $x \in [\lambda^{LH}, \lambda^{HH}]$ since the domains of $\overline{G}$ and $\overline{G}'$ are always contained in this interval.

Note that equations (34) and (35) coincide with (25) and (26), respectively, in the steady state. As a consequence, the steady-state reputation distributions $\overline{G}(\cdot|H)$ and $\overline{G}(\cdot|L)$ described in equations (25) and (26) are a fixed point of T. Since T depends parametrically on y, so do $\overline{G}(\cdot|H)$ and $\overline{G}(\cdot|L)$.

We start by establishing that:

Lemma 1. *The operator T has a unique fixed point.*

Proof. Notice that there are no firms with reputation either below y or above λ^{HH} after entry-exit decisions are made, and that:

1. $\tilde{\mu}(x, \tilde{r}(x,y)) = y$; this is to say, the previous reputation of a firm that obtained a signal $\tilde{r}(x,y)$ that changed its reputation from y to x was y;
2. $\tilde{\mu}(x,r) < y \Leftrightarrow r > \tilde{r}(x,y)$: Those firms with a reputation x today and had a reputation lower than y in the previous period are those that obtained signals of at least $\tilde{r}(x,y)$; and
3. $\tilde{\mu}(x,r) > \lambda^{HH} \Leftrightarrow r < \tilde{r}(x, \lambda^{HH})$: Those firms that had a higher reputation than λ^{HH} in the previous period and have a reputation x today are those with signals lower than $\tilde{r}(x, \lambda^{HH})$.

Using these facts, the distance between $\overline{G}_{t+1}(\cdot|\tau)$ and $\overline{G}'_{t+1}(\cdot|\tau)$ can be shown to be bounded as follows:

$$\rho_\infty\left(\overline{G}_{t+1}(\cdot|\tau), \overline{G}'_{t+1}(\cdot|\tau)\right) \leq \beta \cdot \rho\left(\left(\overline{G}_t(\cdot|H), \overline{G}_t(\cdot|L)\right), \left(\overline{G}'_t(\cdot|H), \overline{G}'_t(\cdot|L)\right)\right),$$

[11] As $\tilde{r}(\lambda^{LH}, y) = 0$, $\tilde{r}(\lambda^{HH}, y) = 1$, $\tilde{\mu}(\lambda^{LH}, r) = 0$ and $\tilde{\mu}(\lambda^{HH}, r) = 1$, if the functions $\overline{G}_t(\cdot|\tau)$ for $\tau \in \{H, L\}$ are normalized, then $\overline{G}_{t+1}(\cdot|\tau)$ are also normalized; i.e. $\overline{G}_{t+1}(\lambda^{LH}|\tau) = 0$ and $\overline{G}_{t+1}(\lambda^{HH}|\tau) = 1$.

where $\beta \in (0,1)$ is defined by

$$\beta = \max \left\{ \sup_{x \in \left[\lambda^{LH}, \lambda^{HH}\right]} \left(F^H \left(\tilde{r}\left(x,y\right)\right) - F^H \left(\tilde{r}\left(x, \lambda^{HH}\right)\right)\right), \right.$$

$$\left. \sup_{x \in \left[\lambda^{LH}, \lambda^{HH}\right]} \left(F^L \left(\tilde{r}\left(x,y\right)\right) - F^L \left(\tilde{r}\left(x, \lambda^{HH}\right)\right)\right) \right\}.$$

It follows that

$$\rho\left(\left(\overline{G}_{t+1}\left(\cdot|H\right), \overline{G}_{t+1}\left(\cdot|L\right)\right), \left(\overline{G}'_{t+1}\left(\cdot|H\right), \overline{G}'_{t+1}\left(\cdot|L\right)\right)\right) \leq \beta\rho\left(\left(\overline{G}_{t}\left(\cdot|H\right), \overline{G}_{t}\left(\cdot|L\right)\right), \left(\overline{G}'_{t}\left(\cdot|H\right), \overline{G}'_{t}\left(\cdot|L\right)\right)\right),$$

i.e., T is a contraction mapping with modulus β.

On the other hand, the set of continuous, bounded real functions endowed with the sup norm is complete. Moreover, the subset of normalized functions is closed, [12] and thereby complete. Then, by Banach's fixed point theorem, T has a unique fixed point, which is a pair of continuous and normalized functions.

If y is consistent, $\overline{G}\left(x|H\right)$ and $\overline{G}\left(x|L\right)$ are increasing functions because $G\left(x|H\right)$ and $G\left(x|L\right)$ are non-negative in the whole domain, while $\tilde{r}\left(x,y\right)$ is increasing in x. Thus, $\overline{G}\left(x|H\right)$ and $\overline{G}\left(x|L\right)$ are not only normalized and continuous, but also increasing. In other words, they are distribution functions. □

The prior reputation distributions $\overline{G}\left(\cdot|\tau\right)$ have support $[\lambda^{LH}, \lambda^{HH}]$ because the likelihood ratio is onto; consequently, the interim reputation distributions $G\left(\cdot|\tau\right)$ have support $[\mu_E, \lambda^{HH}]$. All reputation distributions are absolutely continuous because the signal distributions are. Hence, Equation (16) applies. The parameter y affects the operator T, and therefore affects both the contraction modulus and the limiting distributions. Moreover, the limiting distributions are continuous in y, as they are the fixed point of a contraction.[13] Similarly, the limiting distributions also depend continuously on the parameters κ and $\lambda^{\tau\tau'}$ for $\tau, \tau' \in \{H, L\}$. We write T^y and $\overline{G}^y\left(x|\tau\right)$ or $T^{y;\alpha}$ and $\overline{G}^{y;\alpha}\left(x|\tau\right)$ to emphasize the dependence of the operator and its fixed point in the cutoff level y or in any other parameter α when necessary.

We now endogenize the entry-level reputation y. Consider the function $\psi : (\lambda^{LH}, \lambda^{HH}] \to [0, \infty)$ defined in Equation (27). Any consistent entry-level reputation must be a fixed point of ψ (see Equation (12)). When taking into consideration the dependence of the distributions on y, ψ should be written as

$$\psi\left(x,y\right) = \frac{\overline{m}^H \overline{G}^y\left(x|H\right) + \lambda^{LH}\kappa}{\overline{m}^H \overline{G}^y\left(x|H\right) + \overline{m}^L \overline{G}^y\left(x|L\right)}. \tag{38}$$

Define the function

$$\phi\left(\mu\right) \equiv \psi\left(\mu, \mu\right) \tag{39}$$

for $\mu \in (\lambda^{LH}, \lambda^{HH}]$. A consistent entry-level reputation satisfies $\mu_E = \psi\left(\mu_E, y\right)$ for given distributions; now we need to verify that those distributions were generated by the same entry-level reputation: $y = \mu_E$. In other words, we need to prove that $\phi\left(\mu\right)$ has a unique fixed point. We begin by observing that:

Lemma 2. *ϕ has at least one fixed point.*

Proof. The function $f\left(\mu\right) \equiv \overline{G}^{\mu}\left(\mu|L\right)$ is continuous, with $f\left(\lambda^{LH}\right) = 0$ and $f\left(\lambda^{HH}\right) = 1$. Then, by the Intermediate Value Theorem there is at least one $x^* \in (\lambda^{LH}, \lambda^{HH})$ such that $f\left(x^*\right) = \frac{\lambda^{LH}\kappa}{\overline{m}^L}$, and so $\phi\left(x^*\right) = 1$. We also know that ϕ is continuous in its domain, and that $\phi\left(x^*\right) - x^* = 1 - x^* > 0$ and $\phi\left(\lambda^{HH}\right) - \lambda^{HH} = m^H - \lambda^{HH} < 0$ (as $\kappa < \overline{\kappa}$ as defined in Equation (4)). Also by the Intermediate Value Theorem, there is at least one $\mu \in (x^*, \lambda^{HH})$ such that $\phi\left(\mu\right) - \mu = 0$. As a consequence, there is at least one $\mu_E \in (\lambda^{LH}, \lambda^{HH})$ such that $\psi\left(\mu_E, \mu_E\right) = \mu_E$. □

The next step is to establish uniqueness.

Lemma 3. *$\phi'\left(\mu_E\right) = 0$ if μ_E is a fixed point. Hence, the fixed point is unique.*

Proof. We prove that $\frac{\partial\psi}{\partial x} = 0$ and $\frac{\partial\psi}{\partial y} = 0$ at $x = y = \mu_E$, from which we deduce that $\phi'\left(\mu_E\right) = 0$ since

$$\phi'\left(\mu_E\right) d\mu_E = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy.$$

[12] See Lemma 1 in Vial (2010) for a proof.
[13] See De la Fuente (2000), Chapter 2, Theorem 7.18.

Taking the derivative of Equation 38 we obtain

$$\frac{\partial \psi}{\partial x}(x,y)=\frac{\overline{g}^{y}(x)}{\overline{G}^{y}(x)}\left(\frac{\overline{m}^{H}\overline{g}^{y}(x|H)}{\overline{g}^{y}(x)}-\psi(x,y)\right).$$

By Equation (16), $\frac{\overline{m}^{H}\overline{g}^{y}(x|H)}{\overline{g}^{y}(x)}=x$. Moreover, at a fixed point $\psi(x,y)=x$. Hence, $\frac{\partial\psi}{\partial x}=0$ at $x=y=\mu_E$. In words, the entrants' reputation $\psi(x,y)$ increases when the exit reputation level increases if and only if the firms that leave and reenter after this change have a higher reputation than those that are already replacing their names. At the fixed point, however, those firms have exactly the same average reputation, so moving the cutoff point will have no effect on the entrants' reputation.

As for y, it affects ψ through the distributions $\overline{G}^{y}(\cdot|H)$ and $\overline{G}^{y}(\cdot|L)$. Since the pair of steady-state distributions is the fixed point of a contraction mapping in a complete metric space, it can be obtained as the limit of the sequence $\left\{\overline{G}^{y}_{t}(\cdot|H),\overline{G}^{y}_{t}(\cdot|L)\right\}$ defined by iterating T^{y} starting from any pair $\overline{G}_{0}(\cdot|H)$ and $\overline{G}_{0}(\cdot|L)$, where $\overline{G}^{y}_{t}(\cdot|H)$ and $\overline{G}^{y}_{t}(\cdot|L)$ denote the t-th iteration of T^{y}. Define ψ_t as $\psi_t(x,y)\equiv\frac{\overline{m}^{H}\overline{G}^{y}_{t}(x|H)+\lambda^{LH}\kappa}{\overline{G}^{y}_{t}(x)}$. We will show that $\{\psi_t(\mu_E,y)\}$ is a constant sequence when the starting point is $\overline{G}^{y}_{0}(\cdot|\tau)\equiv\overline{G}^{\mu_E}(\cdot|\tau)$ with the associated interim distribution $G^{y}_{0}(\cdot|\tau)\equiv G^{\mu_E}(\cdot|\tau)$ (i.e., the steady-state distributions under T^{μ_E}) and $y=\mu_E+dy$ is infinitesimally different from μ_E; hence $\frac{\partial\psi}{\partial y}=0$ when evaluated at $x=y=\mu_E$.

Let us define $T^{y}_{0}(x|\tau)$ as

$$T^{y}_{0}(x|\tau)\equiv\gamma^{H\tau}\int_{0}^{\tilde{r}(x,y)}G^{y}_{0}(\tilde{\mu}(x,r)|H)\,dF^{H}+\gamma^{L\tau}\int_{0}^{\tilde{r}(x,y)}G^{y}_{0}(\tilde{\mu}(x,r)|L)\,dF^{L}. \tag{40}$$

After one iteration of T^{y} we obtain

$$\overline{G}^{y}_{1}(x|\tau)=\overline{G}^{y}_{0}(x|\tau)+\left.\frac{\partial T^{y}_{0}(x|\tau)}{\partial y}\right|_{y=\mu_E}dy.$$

From direct computation of the derivative of the right-hand-side of Equation (40) (with fixed distributions),

$$\left.\frac{\partial T^{y}_{0}(x|\tau)}{\partial y}\right|_{y=\mu_E}=\frac{\partial\widetilde{r}(x,y)}{\partial y}\left(\gamma^{H\tau}G^{y}_{0}(\mu_E|H)\,f^{H}(\tilde{r}(x,\mu_E))+\gamma^{L\tau}G^{y}_{0}(\mu_E|L)\,f^{L}(\tilde{r}(x,\mu_E))\right). \tag{41}$$

Taking into account that $\frac{m^{H}G^{y}_{0}(\mu_E|H)}{\overline{G}^{y}_{0}(\mu_E)}=\mu_E$ and $\frac{m^{L}G^{y}_{0}(\mu_E|L)}{\overline{G}^{y}_{0}(\mu_E)}=1-\mu_E$ and rearranging, we obtain:

$$\left.\frac{\partial T^{y}_{0}(x|\tau)}{\partial y}\right|_{y=\mu_E}=\overline{G}^{y}_{0}(\mu_E)\,\frac{\partial\widetilde{r}(x,\mu_E)}{\partial y}\,\frac{\lambda^{H\tau}\mu_E f^{H}(\tilde{r}(x,\mu_E))+\lambda^{L\tau}(1-\mu_E)\,f^{L}(\tilde{r}(x,\mu_E))}{\overline{m}^{\tau}} \tag{42}$$

Moreover, using Equation (23) the pair $\left(\overline{G}^{y}_{1}(\cdot|H),\overline{G}^{y}_{1}(\cdot|L)\right)$ can be written as

$$\begin{pmatrix}\overline{G}^{y}_{1}(\cdot|H)\\ \overline{G}^{y}_{1}(\cdot|L)\end{pmatrix}(x)=\begin{pmatrix}\overline{G}^{y}_{0}(\cdot|H)\\ \overline{G}^{y}_{0}(\cdot|L)\end{pmatrix}(x)+\omega^{y}_{0}(x,y)\begin{pmatrix}\frac{x}{\overline{m}^{H}}\\ \frac{1-x}{\overline{m}^{L}}\end{pmatrix} \tag{43}$$

with $\omega^{y}_{0}(x,y)\equiv\frac{\partial\widetilde{r}(x,y)}{\partial y}\overline{G}^{y}_{0}(\mu_E)\left(\mu_E f^{H}(\tilde{r}(x,\mu_E))+(1-\mu_E)\,f^{L}(\tilde{r}(x,\mu_E))\right)dy$, while $\psi_1(\mu_E,y)$ can be written as

$$\psi_1(\mu_E,y)=\frac{\overline{m}^{H}\overline{G}^{y}_{0}(\mu_E|H)+\mu_E\omega^{y}_{0}(\mu_E,y)+\lambda^{LH}\kappa}{\overline{G}^{y}_{0}(\mu_E)+\omega^{y}_{0}(\mu_E,y)}.$$

As $\frac{\overline{m}^{H}\overline{G}^{y}_{0}(\mu_E|H)+\lambda^{LH}\kappa}{\overline{G}^{y}_{0}(\mu_E)}=\mu_E$, we conclude that

$$\psi_1(\mu_E,y)=\psi_0(\mu_E,y)=\mu_E.$$

We now look at higher iterations of T^{y}. Applying the operator T^{y} as defined from Equation (37) to $\begin{pmatrix}\overline{G}^{y}_{t}(\cdot|H)\\ \overline{G}^{y}_{t}(\cdot|L)\end{pmatrix}$ we obtain

$$\begin{pmatrix}\overline{G}^{y}_{t+1}(\cdot|H)\\ \overline{G}^{y}_{t+1}(\cdot|L)\end{pmatrix}(x)=\begin{pmatrix}\overline{G}^{y}_{t}(\cdot|H)\\ \overline{G}^{y}_{t}(\cdot|L)\end{pmatrix}(x)-\begin{pmatrix}\frac{\overline{m}^{H}\lambda^{HH}}{\overline{m}^{H}} & \frac{\overline{m}^{L}\lambda^{LH}}{\overline{m}^{H}}\\ \frac{\overline{m}^{H}\lambda^{HL}}{\overline{m}^{L}} & \frac{\overline{m}^{L}\lambda^{LL}}{\overline{m}^{L}}\end{pmatrix}$$
$$\times\begin{pmatrix}\frac{\int_{y}^{1}\omega^{y}_{t}(\mu,y)\mu f^{H}(\tilde{r}(x,\mu))\frac{\partial\tilde{r}(x,\mu)}{\partial\mu}d\mu}{\overline{m}^{H}}\\ \frac{\int_{y}^{1}\omega^{y}_{t}(\mu,y)(1-\mu)f^{L}(\tilde{r}(x,\mu))\frac{\partial\tilde{r}(x,\mu)}{\partial\mu}d\mu}{\overline{m}^{L}}\end{pmatrix}. \tag{44}$$

where $\omega_t^y(x,y) \equiv -\int_y^1 \omega_{t-1}^y(\mu,y)\left(\mu f^H(\tilde{r}(x,\mu)) + (1-\mu) f^L(\tilde{r}(x,\mu))\right)\frac{\partial \tilde{r}(x,\mu)}{\partial \mu}d\mu$. Rearranging and using Equation (23), we get

$$\begin{pmatrix} \overline{G}_{t+1}^y(\cdot|H) \\ \overline{G}_{t+1}^y(\cdot|L) \end{pmatrix}(x) = \begin{pmatrix} \overline{G}_t^y(\cdot|H) \\ \overline{G}_t^y(\cdot|L) \end{pmatrix}(x) + \omega_t^y(x,y)\begin{pmatrix} \frac{x}{\overline{m}^H} \\ \frac{1-x}{\overline{m}^L} \end{pmatrix}.$$

Accordingly,

$$\psi_{t+1}(\mu_E,y) = \frac{\overline{m}^H \overline{G}_t^y(\mu_E|H) + \mu_E\omega_t^y(\mu_E,y) + \lambda^{LH}\kappa}{\overline{G}_t^y(\mu_E) + \omega_t^y(\mu_E,y)}.$$

Hence, we can prove by induction on t that $\psi_t(\mu_E,y) = \mu_E$ for all t: By assuming that $\psi_t(\mu_E,y) = \mu_E$, we deduce that $\psi_{t+1}(\mu_E,y) = \mu_E$. Moreover, $\psi_1(\mu_E,y) = \mu_E$ had already been established. Thus, $\psi_t(\mu_E,y) = \psi_0(\mu_E,y) = \mu_E$ for all $t > 0$, and therefore $\frac{\partial \psi}{\partial y}(x,y) = 0$ when $x = y = \mu_E$.

□

Finally, we conclude that $\mu_E < m^H$:

Lemma 4. *The consistent entry-level reputation is lower than the fraction of competents among active firms after the exit-entry process takes place:* $\mu_E < m^H$.

Proof. Notice that $\psi(\mu_E,\mu_E) = \mu_E$ and $\psi(\lambda^{HH},\mu_E) = m^H$. Moreover,

$$\frac{\partial \psi}{\partial x}(x,\mu_E) = \left(\frac{\overline{m}^H \overline{g}^{\mu_E}(x|H) + \overline{m}^L \overline{g}^{\mu_E}(x|I)}{\overline{G}^{\mu_E}(x)}\right)(x - \psi(x,\mu_E)).$$

This is strictly positive in the interval (μ_E, λ^{HH}). Hence, $\mu_E < m^H$. □

A.2 Proof of Proposition 1

The tuple $\left(\mu_E, \overline{G}(\cdot|H), \overline{G}(\cdot|L)\right)$ contains the entry-level reputation, which is a fixed point of ϕ (as defined in Equation (39)) and a pair of steady-state reputation distributions that are a fixed point of T (as defined in Equation (36)). As $\overline{G}(\cdot|\tau)$ depends continuously on κ and the parameters of the transition matrix Λ for a given entry-level reputation, the dependence of ϕ on those parameters is also continuous. We will show that this implies that μ_E also depends continuously on on κ and the parameters of the transition matrix Λ.

Proof. Denote $\phi^\kappa(x)$ as the value of $\phi(x)$ when the mass of potential firms is $1+\kappa$. Define the function $\nu : (0,\overline{\kappa})) \to (\lambda^{LH}, m^H)$ as $\nu(\kappa) = \left\{x \in \left(\lambda^{LH}, m^H\right) : x = \phi^\kappa(x)\right\}$, which gives the fixed point μ_E as a function of κ. We know that $\phi^\kappa(x) > x$ when $x < \nu(\kappa)$ and $\phi^\kappa(x) < x$ when $x > \nu(\kappa)$.

We will prove that ν is continuous. Assume instead that ν is not continuous at some $\kappa_0 \in (0,\overline{\kappa})$. Then, there is some $\varepsilon^* > 0$ such that for all $\delta > 0$ we can find some $\kappa(\delta) \in (0,\overline{\kappa}))$ such that $|\kappa_0 - \kappa(\delta)| < \delta$ and $|\nu(\kappa_0) - \nu(\kappa(\delta))| \geq \varepsilon^*$. Notice that we can make ε^* arbitrarily small, in order that $\overline{x} \equiv \nu(\kappa_0) + \varepsilon^* < \lambda^{HH}$ and $\underline{x} \equiv \nu(\kappa_0) - \varepsilon^* > \lambda^{LH}$. Then, $|\nu(\kappa_0) - \nu(\kappa(\delta))| \geq \varepsilon^*$ implies that either $\phi^{\kappa_0}(\overline{x}) < \overline{x} < \phi^{\kappa(\delta)}(\overline{x})$ or $\phi^{\kappa_0}(\underline{x}) > \underline{x} > \phi^{\kappa(\delta)}(\underline{x})$. Hence, there is some $\varepsilon' \equiv \min\{\overline{x} - \phi^{\kappa_0}(\overline{x}), \phi^{\kappa_0}(\underline{x}) - \underline{x}\} > 0$ such that for all $\delta > 0$ there is some $\kappa(\delta) \in (0,\overline{\kappa}))$ that satisfies $|\kappa_0 - \kappa(\delta)| < \delta$ and $\left|\phi^{\kappa(\delta)}(\overline{x}) - \phi^{\kappa_0}(\overline{x})\right| \geq \varepsilon'$ or $\left|\phi^{\kappa(\delta)}(\underline{x}) - \phi^{\kappa_0}(\underline{x})\right| \geq \varepsilon'$. As this implies that $\phi^\kappa(x)$ is discontinuous in κ for some x, we arrive at a contradiction.

The proof of the continuity of μ_E in the parameters of the transition matrix Λ is analogous to that of the continuity in κ, except that the domain of ν is different since $\lambda^{LH} \in (0,\lambda^{HH})$ and $\lambda^{HH} \in (\lambda^{LH}, 1)$. □

A.3 Proof of Proposition 2

We will show that a sufficient condition for μ_E to increase when a given parameter α increases is that $\gamma^{HH} \equiv \frac{m^H \lambda^{HH}}{\overline{m}^H}$ and $\gamma^{LL} \equiv \frac{m^L \lambda^{LL}}{\overline{m}^L}$ are reduced while $\lambda^{LH}\kappa$ is not, where $m^H = \frac{\lambda^{LH}(1+\kappa)}{\lambda^{LH}+\lambda^{HL}} = 1 - m^L$, and $\overline{m}^H = m^H - \lambda^{LH}\kappa = 1 - \overline{m}^L$.

Proof. Let α denote any parameter affecting the entry-level reputation. We want to look at the effect of an increase in α on the fixed point of the function $\phi(\mu)$ defined in Equation 39. As $\phi'(\mu_E) = 0$, the change in α affects μ_E only through its potential direct effect on ψ and the indirect effect on the distributions $\overline{G}(\cdot|H)$ and $\overline{G}(\cdot|L)$.

Consider an initial value α_0, and the associated pair of conditional steady-state distributions $\overline{G}^{y_0;\alpha_0}(\cdot|\tau)$ for $\tau \in \{H, L\}$, where y_0 is the consistent entry-level reputation when $\alpha = \alpha_0$. As in the proof of Lemma 3, we analyze the sequence of distributions $\left\{\overline{G}_t^{y_0}(\cdot|H), \overline{G}_t^{y_0}(\cdot|L)\right\}$ defined by iterating the operator $T^{y_0;\alpha}$ starting from $\overline{G}_0^{y_0}(\cdot|\tau) \equiv \overline{G}^{y_0;\alpha_0}(\cdot|\tau)$ with the associated interim distribution $G_0^{y_0}(\cdot|\tau) \equiv G^{y_0;\alpha_0}(\cdot|\tau)$, and the associated sequence $\{\psi_t(y_0, y_0)\}$ defined by $\psi_t(x, y_0) \equiv \frac{\overline{m}^H \overline{G}_t^{y_0}(x|H)+\lambda^{LH}\kappa}{\overline{G}_t^{y_0}(x)}$ when $\alpha = \alpha_0 + d\alpha$ is infinitesimally larger than α_0. We will show that a sufficient condition for $\lim_{t\to\infty} \psi_t(y_0, y_0) > y_0$ is that $\frac{\partial\gamma^{HH}}{\partial\alpha} < 0$ and $\frac{\partial\gamma^{LL}}{\partial\alpha} < 0$ while $\frac{\partial(\lambda^{LH}\kappa)}{\partial\alpha} \geq 0$.

Let us define $T_0^{y_0;\alpha}(x|\tau)$ as:

$$T_0^{y_0;\alpha}(x|\tau) \equiv \gamma^{H\tau}(\alpha) \int_0^{\tilde{r}(x,y_0;\alpha)} G_0^{y_0}(\tilde{\mu}(x, r; \alpha)|H)\, dF^H + \gamma^{L\tau}(\alpha) \int_0^{\tilde{r}(x,y_0;\alpha)} G_0^{y_0}(\tilde{\mu}(x, r; \alpha)|L)\, dF^L, \tag{45}$$

where $\gamma^{H\tau} + \gamma^{L\tau} = 1$. After one iteration of the operator $T^{y_0;\alpha}$ we obtain

$$\overline{G}_1^{y_0}(x|\tau) = \overline{G}_0^{y_0}(x|\tau) + \left.\frac{\partial T_0^{y_0;\alpha}(x|\tau)}{\partial\alpha}\right|_{\alpha=\alpha_0} d\alpha. \tag{46}$$

Taking the derivative of the right-hand side of Equation (45) (with fixed distributions) and rearranging, we get

$$\begin{aligned}
\left.\frac{\partial T_0^{y_0;\alpha}(x|\tau)}{\partial\alpha}\right|_{\alpha=\alpha_0} &= \frac{\partial\gamma^{H\tau}}{\partial\alpha}\left(\int_0^{\tilde{r}(x,y_0)} G_0^{y_0}(\tilde{\mu}(x,r)|H)\, dF^H - \int_0^{\tilde{r}(x,y_0)} G_0^{y_0}(\tilde{\mu}(x,r)|L)\, dF^L\right) \\
&+ \frac{\partial\tilde{r}(x,y_0)}{\partial\alpha}\left(\gamma^{H\tau}\int_0^{\tilde{r}(x,y_0)} G_0^{y_0}(y_0|H)\, dF^H + \gamma^{L\tau}\int_0^{\tilde{r}(x,y_0)} G_0^{y_0}(y_0|L)\, dF^L\right) \\
&+ \gamma^{H\tau}\int_0^{\tilde{r}(x,y_0)} \frac{\partial\tilde{\mu}(x,r)}{\partial\alpha} g_0^{y_0}(\tilde{\mu}(x,r)|H)\, dF^H + \gamma^{L\tau}\int_0^{\tilde{r}(x,y_0)} \frac{\partial\tilde{\mu}(x,r)}{\partial\alpha} g_0^{y_0}(\tilde{\mu}(x,r)|L)\, dF^L
\end{aligned}$$

As the distributions $\overline{G}_0^{y_0}(\cdot|\tau)$ for $\tau \in \{H, L\}$ are steady-state distributions, then $\overline{G}_0^{y_0}(x|\tau) = T_0^{y_0;\alpha_0}(x|\tau)$. Hence:

$$\overline{G}_0^{y_0}(x|L) - \overline{G}_0^{y_0}(x|H) = \left(\gamma^{HH} - \gamma^{HL}\right)\left(\int_0^{\tilde{r}(x,y_0)} G_0^{y_0}(\tilde{\mu}(x,r)|L)\, dF^L - \int_0^{\tilde{r}(x,y_0)} G_0^{y_0}(\tilde{\mu}(x,r)|H)\, dF^H\right)$$

and also

$$\begin{aligned}
\overline{g}_0^{y_0}(x|\tau) &= \frac{\partial\tilde{r}(x,y_0)}{\partial x}\left(\gamma^{H\tau}\int_0^{\tilde{r}(x,y_0)} G_0^{y_0}(y_0|H)\, dF^H + \gamma^{L\tau}\int_0^{\tilde{r}(x,y_0)} G_0^{y_0}(y_0|L)\, dF^L\right) \\
&+ \gamma^{H\tau}\int_0^{\tilde{r}(x,y_0)} \frac{\partial\tilde{\mu}(x,r)}{\partial x} g_0^{y_0}(\tilde{\mu}(x,r)|H)\, dF^H + \gamma^{L\tau}\int_0^{\tilde{r}(x,y_0)} \frac{\partial\tilde{\mu}(x,r)}{\partial x} g_0^{y_0}(\tilde{\mu}(x,r)|L)\, dF^L.
\end{aligned}$$

From the definitions of $\tilde{r}(x, y)$ and $\tilde{\mu}(x, r)$, by taking derivatives and rearranging we obtain

$$\begin{aligned}
\frac{\partial\tilde{r}(x,y)}{\partial\alpha}\left(\frac{\partial\tilde{r}(x,y)}{\partial x}\right)^{-1} &= -\left(\frac{\partial\lambda^{LH}}{\partial\alpha}\left(\frac{\lambda^{HH} - x}{\lambda^{HH} - \lambda^{LH}}\right) + \frac{\partial\lambda^{HH}}{\partial\alpha}\left(\frac{x - \lambda^{LH}}{\lambda^{HH} - \lambda^{LH}}\right)\right) \\
&= \frac{\partial\tilde{\mu}(x,r)}{\partial\alpha}\left(\frac{\partial\tilde{\mu}(x,r)}{\partial x}\right)^{-1}.
\end{aligned}$$

Using these results and noticing that $\overline{g}_0^{y_0}(x|H) = \overline{g}_0^{y_0}(x)\frac{x}{\overline{m}^H}$ and $\overline{g}_0^{y_0}(x|L) = \overline{g}_0^{y_0}(x)\frac{1-x}{\overline{m}^L}$ we finally obtain

$$\begin{pmatrix} \overline{G}_1^{y_0}(\cdot|H) \\ \overline{G}_1^{y_0}(\cdot|L) \end{pmatrix}(x) = \begin{pmatrix} \overline{G}_0^{y_0}(\cdot|H) \\ \overline{G}_0^{y_0}(\cdot|L) \end{pmatrix}(x) + \upsilon_0^\alpha(x, y_0)\begin{pmatrix} -\frac{\partial\gamma^{HH}}{\partial\alpha} \\ -\frac{\partial\gamma^{HL}}{\partial\alpha} \end{pmatrix} + \omega_0^\alpha(x, y_0)\begin{pmatrix} \frac{x}{\overline{m}^H} \\ \frac{1-x}{\overline{m}^L} \end{pmatrix},$$

where $\upsilon_0^\alpha(x, y_0) \equiv \frac{\overline{G}_0^{y_0}(x|L) - \overline{G}_0^{y_0}(x|H)}{\gamma^{HH} - \gamma^{HL}} > 0$ and $\omega_0^\alpha(x, y_0) \equiv -\left(\frac{\partial\lambda^{LH}}{\partial\alpha}\left(\frac{\lambda^{HH}-x}{\lambda^{HH}-\lambda^{LH}}\right) + \frac{\partial\lambda^{HH}}{\partial\alpha}\left(\frac{x-\lambda^{LH}}{\lambda^{HH}-\lambda^{LH}}\right)\right)\overline{g}_0^{y_0}(x)$.

Repeatedly iterating the operator $T^{y_0;\lambda}$ as defined in Equation (37), noticing that

$$\begin{aligned}
\frac{\lambda^{HH}\mu f^H(\tilde{r}(x,\mu)) + \lambda^{LH}(1-\mu) f^L(\tilde{r}(x,\mu))}{x} &= \mu f^H(\tilde{r}(x,\mu)) + (1-\mu) f^L(\tilde{r}(x,\mu)) \\
&= \frac{\lambda^{HL}\mu f^H(\tilde{r}(x,\mu)) + \lambda^{LL}(1-\mu) f^L(\tilde{r}(x,\mu))}{1-x},
\end{aligned}$$

and after rearranging, we obtain

$$\begin{pmatrix} \overline{G}_{t+1}^{y_0;\lambda}(\cdot|H) \\ \overline{G}_{t+1}^{y_0;\lambda}(\cdot|L) \end{pmatrix}(x) = \begin{pmatrix} \overline{G}_t^{y_0;\lambda}(\cdot|H) \\ \overline{G}_t^{y_0;\lambda}(\cdot|L) \end{pmatrix}(x) + \upsilon_t^\alpha(x, y_0)\begin{pmatrix} \frac{1}{\overline{m}^H} \\ -\frac{1}{\overline{m}^L} \end{pmatrix} + \omega_t^\alpha(x, y_0)\begin{pmatrix} \frac{x}{\overline{m}^H} \\ \frac{1-x}{\overline{m}^L} \end{pmatrix}$$

for all $t > 0$, where υ_t^α and ω_t^α are defined as:

$$\begin{aligned}\upsilon_1^\alpha (x, y_0) \equiv& \int_{y_0}^{1} \upsilon_0^\alpha (\mu, y_0) \left(\frac{\partial \gamma^{HH}}{\partial \alpha} \left(\lambda^{HH} - x \right) \overline{m}^H f^H (\tilde{r} (x, \mu)) \right. \\ & \left. + \frac{\partial \gamma^{HL}}{\partial \alpha} \left(\lambda^{LH} - x \right) \overline{m}^L f^L (\tilde{r} (x, \mu)) \right) \frac{\partial \tilde{r} (\mu, y_0)}{\partial \mu} d\mu, \\ \omega_1^\alpha (x, y_0) \equiv& \int_{y_0}^{1} \left(\upsilon_0^\alpha (\mu, y_0) \left(\frac{\partial \gamma^{HH}}{\partial \alpha} \overline{m}^H f^H (\tilde{r} (x, \mu)) + \frac{\partial \gamma^{HL}}{\partial \alpha} \overline{m}^L f^L (\tilde{r} (x, \mu)) \right) \right. \\ & \left. - \omega_0^\alpha (\mu, y_0) \left(\mu f^H (\tilde{r} (x, \mu)) + (1 - \mu) f^L (\tilde{r} (x, \mu)) \right) \right) \frac{\partial \tilde{r} (x, \mu)}{\partial \mu} d\mu,\end{aligned}$$

and

$$\begin{aligned}\upsilon_t^\alpha (x, y_0) \equiv& - \int_{y_0}^{1} \upsilon_{t-1}^\alpha (\mu, y_0) \left(\left(\lambda^{HH} - x \right) f^H (\tilde{r} (x, \mu)) \right. \\ & \left. + \left(x - \lambda^{LH} \right) f^L (\tilde{r} (x, \mu)) \right) \frac{\partial \tilde{r} (x, \mu)}{\partial \mu} d\mu, \\ \omega_t^\alpha (x, y_0) \equiv& - \int_{y_0}^{1} \left(\upsilon_{t-1}^\alpha (\mu, y_0) \left(f^L (\tilde{r} (x, \mu)) - f^H (\tilde{r} (x, \mu)) \right) \right. \\ & \left. + \omega_{t-1}^\alpha (\mu, y_0) \left(\mu f^H (\tilde{r} (x, \mu)) + (1 - \mu) f^L (\tilde{r} (x, \mu)) \right) \right) \frac{\partial \tilde{r} (x, \mu)}{\partial \mu} d\mu.\end{aligned}$$

As $x \in (\lambda^{LH}, \lambda^{HH})$ and $\frac{\partial \tilde{r}(\mu, y_0)}{\partial \mu} < 0$, a sufficient condition for $\upsilon_t^\alpha (x, y_0) > 0 \; \forall t$ is that $\frac{\partial \gamma^{HH}}{\partial \alpha} < 0$ and $\frac{\partial \gamma^{HL}}{\partial \alpha} > 0$.

For all $t > 0$, $\psi_t (y_0, y_0; \lambda)$ can be written as

$$\psi_t (y_0, y_0) = \frac{\psi_{t-1} (y_0, y_0) \overline{G}_{t-1}^{y_0;\alpha} (y_0) + \upsilon_{t-1} (y_0, y_0) + y_0 \omega_{t-1} (y_0, y_0) + d (\lambda^{LH} \kappa)}{\overline{G}_{t-1}^{y_0;\alpha} (y_0) + \omega_{t-1} (y_0, y_0) .}$$

Since $\psi_0 (y_0, y_0) = y_0$, recursive substitution yields

$$\psi_t (y_0, y_0) = y_0 + \upsilon_0^\alpha (y_0, y_0) \left(\frac{- (1 - y_0) \frac{\partial \gamma^{HH}}{\partial \alpha} \overline{m}^H + y_0 \frac{\partial \gamma^{HL}}{\partial \alpha} \overline{m}^L}{\overline{G}_t^{y_0;\alpha} (y_0)} \right) + \frac{\sum_{j=1}^{t} \upsilon_j (y_0, y_0)}{\overline{G}_t^{y_0;\alpha} (y_0)} + d \left(\lambda^{LH} \kappa \right).$$

Hence, $\psi_t (y_0, y_0) > y_0 + \upsilon_0^\alpha (y_0, y_0) \left(- (1 - y_0) \frac{\partial \gamma^{HH}}{\partial \alpha} \overline{m}^H + y_0 \frac{\partial \gamma^{HL}}{\partial \alpha} \overline{m}^L \right)$ for all $t > 0$ if $\frac{\partial \gamma^{HH}}{\partial \alpha} < 0$ and $\frac{\partial \gamma^{HL}}{\partial \alpha} > 0$; this implies that $\lim_{t \to \infty} \psi_t (y_0, y_0) > y_0$. Therefore, $\phi (y_0) > y_0$ when any parameter α increases infinitesimally if $\frac{\partial \gamma^{HH}}{\partial \alpha} < 0$ and $\frac{\partial \gamma^{HL}}{\partial \alpha} > 0$, while $\lambda^{LH} \kappa$ is not reduced. We conclude that the consistent entry-level reputation obtained under this new value of α is strictly larger than y_0 if these sufficient conditions are met. □

A.4 Proof of Proposition 3

We will prove that the sufficient conditions for an increase in μ_E from Proposition 2 are met after: (1) an increase in λ^{HL}; (2) a proportional increase in λ^{HL} and λ^{LH}; (3) an increase in λ^{HL} and κ so that m^H remains constant; or (4) an increase in λ^{HL} and λ^{LH} with a reduction in κ so that m^H and $\overline{m}^H$ remain constant.

Proof. (1) The effect of an increase in λ^{HL} is directly obtained by taking derivatives and rearranging:

$$\frac{\partial \gamma^{HH}}{\partial \lambda^{HL}} = - \left(\frac{m^H}{\overline{m}^H} \right)^2 \frac{1 - \lambda^{LH} \kappa}{1 + \kappa} \text{ and } \frac{\partial \gamma^{LL}}{\partial \lambda^{HL}} = - \left(\frac{m^H}{\overline{m}^H} \right)^2 \frac{1 + \lambda^{LH} \kappa}{1 + \kappa}.$$

(2) If both λ^{HL} and λ^{LH} increase in the same proportion, then the mean interim reputation m^H remains fixed. Hence, λ^{LH} can be written as $\lambda^{LH} = \frac{\lambda^{HL} m^H}{1 - m^H + \kappa}$. After replacing on γ^{HH} and γ^{LL}, by taking the derivatives with respect to λ^{HL} and rearranging we obtain in this case:

$$\frac{\partial \gamma^{HH}}{\partial \lambda^{HL}} = - \left(\frac{m^H}{\overline{m}^H} \right)^2 \frac{m^L}{m^L + \kappa} \text{ and } \frac{\partial \gamma^{LL}}{\partial \lambda^{HL}} = - \left(\frac{m^H}{\overline{m}^L} \right)^2 \frac{m^L}{m^H}.$$

(3) If the increase in λ^{HL} is compensated by an increase in κ so that m^H remains constant, then κ can be written as: $\kappa = \frac{m^H(\lambda^{LH}+\lambda^{HL})-\lambda^{LH}}{\lambda^{LH}}$. After replacing on γ^{HH} and γ^{LL}, by taking the derivatives with respect to λ^{HL} and rearranging we obtain in this case:

$$\frac{\partial \gamma^{HH}}{\partial \lambda^{HL}} = -\left(\frac{m^H}{\overline{m}^H}\right)^2 \frac{\lambda^{LH} m^L}{m^H} \text{ and } \frac{\partial \gamma^{LL}}{\partial \lambda^{HL}} = -\left(\frac{m^H}{\overline{m}^L}\right)^2 \frac{\lambda^{LL} m^L}{m^H}.$$

(4) If λ^{HL} and λ^{LH} increase but κ is reduced so that both the mean interim reputation m^H and the mean prior reputation $\overline{m}^H$ remain fixed, then $\lambda^{LH} = \frac{\overline{m}^H - m^H(1-\lambda^{HL})}{m^L}$ and $\kappa = \frac{(m^H - \overline{m}^H)m^L}{\overline{m}^H - m^H(1-\lambda^{HL})}$; hence, $\lambda^{LH}\kappa$ does not change. By taking the derivatives with respect to λ^{HL} we obtain in this case:

$$\frac{\partial \gamma^{HH}}{\partial \lambda^{HL}} = -\frac{m^H}{\overline{m}^H} \text{ and } \frac{\partial \gamma^{LL}}{\partial \lambda^{HL}} = -\frac{m^H}{\overline{m}^L}.$$

□

A.5 Proof of Proposition 5

As $G(\tilde{\mu}(x,r)|\tau,n) \le G(\tilde{\mu}(x,r)|\tau,0)$, we obtain an upper bound for $\overline{G}(x|\tau,n+1)$ in Equation (31) as follows:

$$\begin{aligned}\overline{G}(x|\tau,n+1) &\le \gamma_n^{H\tau}\int_0^{\tilde{r}(x,\mu_E)} G(\tilde{\mu}(x,r)|H,0)\,dF^H + \gamma_n^{L\tau}\int_0^{\tilde{r}(x,\mu_E)} G(\tilde{\mu}(x,r)|L,0)\,dF^L \\ &= \overline{G}(x|\tau,1) + \left(\gamma_n^{H\tau} - \gamma_0^{H\tau}\right)\left(\int_0^{\tilde{r}(x,\mu_E)} G(\tilde{\mu}(x,r)|H,0)\,d\left(F^H - F^L\right)\right).\end{aligned}$$

Evaluating the integral, we find that

$$\overline{G}(x|\tau,n+1) \le \overline{G}(x|\tau,1) + \left(\gamma_n^{H\tau} - \gamma_0^{H\tau}\right)\left(F^H(\tilde{r}(x,\mu_E)) - F^L(\tilde{r}(x,\mu_E))\right).$$

However, $\left(\gamma_n^{H\tau} - \gamma_0^{H\tau}\right) > 0 \Leftrightarrow \frac{m_n^H}{m_n} > \mu_E$; but as $\frac{m_n^H}{m_n} = E[\mu|n]$, then $\frac{m_n^H}{m_n} > \mu_E$. As $F^H(\tilde{r}(x,\mu_E)) - F^L(\tilde{r}(x,\mu_E)) < 0$ we conclude that $\overline{G}(x|\tau,n+1) \le \overline{G}(x|\tau,1)$. □

References

Albuquerque, R. and H. A. Hopenhayn, "Optimal Lending Contracts and Firm Dynamics," *The Review of Economic Studies* 71 (2004), pp. 285–315.

Atakan, A. E. and M. Ekmekci, "Bargaining and reputation in search markets," *The Review of Economic Studies* 81 (2014), 1–29.

Atkeson, A., C. Hellwig and G. Ordoñez, "Optimal Regulation in the Presence of Reputation Concerns," *The Quarterly Journal of Economics* 130 (2015), 415–464.

Audretsch, D. B., *Innovation and industry evolution* (MIT Press, 1995).

Bar-Isaac, H., "Reputation and Survival: Learning in a Dynamic Signalling Model," *The Review of Economic Studies* 70 (2003), pp. 231–251.

Bar-Isaac, H. and S. Tadelis, "Seller reputation," *Foundations and Trends® in Microeconomics* 4 (2008), 273–351.

Board, S. and M. Meyer-ter Vehn, "A Reputational Theory of Firm Dynamics," Technical Report, UCLA, 2010.

———, "Reputation for Quality," *Econometrica* 81 (November 2013), 2381–2462.

Cabral, L. and A. Hortacsu, "The Dynamics of Seller Reputation: Evidence from eBay," *The Journal of Industrial Economics* 58 (2010), 54–78.

Clementi, G. L. and H. A. Hopenhayn, “A Theory of Financing Constraints and Firm Dynamics,” *The Quarterly Journal of Economics* 121 (2006), pp. 229–265.

Cooley, T. F. and V. Quadrini, “Financial Markets and Firm Dynamics,” *The American Economic Review* 91 (2001), pp. 1286–1310.

De la Fuente, A., *Mathematical methods and models for economists* (Cambridge University Press, 2000).

Dilmé, F., “Building (and Milking) Trust: Reputation as a Moral Hazard Phenomenon,” University of Pennsylvania, 2012.

Dunne, T., M. J. Roberts and L. Samuelson, “Patterns of Firm Entry and Exit in U.S. Manufacturing Industries,” *The RAND Journal of Economics* 19 (1988), pp. 495–515.

Fishman, A. and R. Rob, “Consumer inertia, firm growth and industry dynamics,” *Journal of Economic Theory* 109 (2003), 24 – 38.

Foster, L., J. Haltiwanger and C. Syverson, “Reallocation, Firm Turnover, and Efficiency: Selection on Productivity or Profitability?,” *The American Economic Review* 98 (2008), pp. 394–425.

Fudenberg, D. and D. K. Levine, “Reputation and Equilibrium Selection in Games with a Patient Player,” *Econometrica* 57 (1989), 759–778.

Garcia-Fontes, W. and H. Hopenhayn, “Entry restrictions and the determination of quality,” *Spanish Economic Review* 2 (2000), 105–127.

Gretsky, N., J. Ostroy and W. Zame, “Perfect Competition in the Continuous Assignment Model,” *Journal of Economic Theory* 88 (1999), 60–118.

Hakenes, H. and M. Peitz, “Observable Reputation Trading,” *International Economic Review* 48 (2007), 693–730.

Hopenhayn, H. and R. Rogerson, “Job Turnover and Policy Evaluation: A General Equilibrium Analysis,” *Journal of Political Economy* 101 (1993), pp. 915–938.

Hopenhayn, H. A., “Entry, Exit, and firm Dynamics in Long Run Equilibrium,” *Econometrica* 60 (1992), pp. 1127–1150.

Hörner, J., “Reputation and Competition,” *The American Economic Review* 92 (2002), 644–663.

Jullien, B. and I.-U. Park, “New, Like New, or Very Good? Reputation and Credibility,” *The Review of Economic Studies* 81 (2014), 1543–1574.

Klein, B. and K. B. Leffler, “The Role of Market Forces in Assuring Contractual Performance,” *The Journal of Political Economy* 89 (1981), 615–641.

Mailath, G. and L. Samuelson, *Repeated games and reputations: long-run relationships* (Oxford University Press, USA, 2006).

Mailath, G. J. and L. Samuelson, “Who Wants a Good Reputation?,” *The Review of Economic Studies* 68 (2001), 415–441.

McDevitt, R. C., “Names and Reputations: An Empirical Analysis,” *AEJ: Microeconomics* 3 (2011), 193–209.

Nelson, P., “Information and Consumer Behavior,” *The Journal of Political Economy* 78 (1970), 311–329.

Ordoñez, G. L., “Fragility of reputation and clustering of risk-taking,” *Theoretical Economics* 8 (2013), 653–700.

Rob, R. and A. Fishman, “Is Bigger Better? Customer Base Expansion through Word-of-Mouth Reputation,” *Journal of Political Economy* 113 (2005), pp. 1146–1162.

Tadelis, S., “What’s in a Name? Reputation as a Tradeable Asset,” *The American Economic Review* 89 (1999), 548–563.

———, “The Market for Reputations as an Incentive Mechanism,” *The Journal of Political Economy* 110 (2002), 854–882.

Thompson, P., “Selection and firm survival: evidence from the shipbuilding industry, 1825–1914,” *Review of Economics and Statistics* 87 (2005), 26–36.

Vial, B., “Walrasian Equilibrium and Reputation under Imperfect Public Monitoring,” *The BE Journal of Theoretical Economics, Advances* 10 (2010).

Vial, B. and F. Zurita, “Incentives and Reputation when Names Can Be Replaced: Valjean Reinvented as Monsieur Madeleine,” *Instituto de Economía, Working Paper 447* (2013).

Wang, T., “The Dynamics of Names: A Model of Reputation,” *International Economic Review* 52 (November 2011), 1039–1058.

Wu, Y., “What’s in a Name? What Leads a Firm to Change its Name and What the New Name Foreshadows,” *Journal of Banking & Finance* 34 (2010), 1344 – 1359.