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DEVELOPING COUNTRIES**

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Jaime Casassus\***

**Documento de Trabajo N° 376**

Santiago, Mayo 2010

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# Consumption and Hedging in Oil-Importing Developing Countries\*

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# Consumption and Hedging in Oil-Importing Developing Countries

## Abstract

We study the consumption and hedging strategy of an oil-importing developing country that faces multiple crude oil shocks. In our model, developing countries have two particular characteristics: their economies are mainly driven by natural resources and their technologies are less efficient in energy usage. The natural resource exports can be correlated with the crude oil shocks. The country can hedge against the crude oil uncertainty by taking long/short positions in existing crude oil futures contracts. We find that both, inefficiencies in energy usage and shocks to the crude oil price, lower the productivity of capital. This generates a negative income effect and a positive substitution effect, because today's consumption is relatively cheaper than tomorrow's consumption. Optimal consumption of the country depends on the magnitudes of these effects and on its risk-aversion degree. Shocks to other crude oil factors, such as the convenience yield, are also studied. We find that the persistence of the shocks magnifies the income and substitution effects on consumption, thus affecting also the hedging strategy of the country. The demand for futures contracts is decomposed in a myopic demand, a pure hedging term and productive hedging demands. These hedging demands arise to hedge against changes in the productivity of capital due to changes in crude oil spot prices. We calibrate the model for Chile and study up to what extent the country's copper exports can be used to hedge the crude oil risk.

*Keywords:* Crude oil prices, convenience yields, risk management, emerging markets, government policy, two-sector economies.

*JEL Classification:* G11, Q43, Q48, D92, O41, C60

# 1 Introduction

The recent steeply rise in crude oil prices is comparable to the hike observed in prices in the 70's and early 80's. The fact that nine out of the previous ten US recessions were preceded by an increase in oil prices has brought back the interest of researchers and policymakers in understanding the effect of energy shocks in the economy.<sup>1</sup> But the impact of oil-price shocks in developing economies is different than the one in more developed countries. In general, developing countries have higher energy-intensive manufacturing as a fraction of their GDP and use energy less efficiently (see International Energy Agency 2004). Also, many of these economies are less diversified than developed economies and rely on the export of a few primary commodities that flow from their natural resources.<sup>2</sup> These exports are sometimes called the *natural exports*. Interestingly, changes in the natural exports due to variations in the domestic commodity prices are sometimes correlated with crude oil shocks.<sup>3</sup> This correlation added to the higher energy usage make these economies different from more developed country. Surprisingly, there has not been enough attention to the risk-management policy that the countries can implement to confront these fluctuations. Nowadays crude oil futures are the most actively traded contracts and can significantly reduce the exposure of an economy to crude oil risk.

In this paper we study the consumption and hedging strategies of an oil-importing developing country that faces exogenous multiple crude oil shocks. To capture the relation between oil and the developing economy, we consider that the country has two productive sectors: a capital sector and the exports. The country combines oil and capital to produce more capital. There are some particular parameters in this technology that regulate the efficiency of oil usage. The second technology sector produces the natural exports of the country that can be correlated with the oil price shocks. Other types of exports are included in the capital's production technology. Under this setting, a less developed country has more natural exports relative to its capital than more developed economies. The country chooses how much capital to consume, how much oil to import at the prevailing market prices and also chooses the hedging strategy with financial instruments.

Recent financial studies have developed multi-factor Gaussian models that correctly captures the dynamics of crude oil prices (see for example Schwartz 1997, and Casassus and Collin-Dufresne 2005). We consider a generalization of these models. A multi-factor model is important because the risk management techniques involve trading in oil futures contracts that can be subject to numerous sources of uncertainty.<sup>4</sup> The optimal hedging strategy imply long/short positions in the existing crude oil futures contracts. There are at least as many futures contracts available as crude oil risk

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<sup>1</sup>See the reviews of Jones, Leiby, and Paik (2004) and Kilian (2007) for the current state of this literature.

<sup>2</sup>For example, using the data from Table 1 of Cashin, Cespedes, and Sahay (2004) we find that between 1991 and 1999, copper accounted for 85% of the exports of Zambia and 41% of the exports of Chile. In the same period, gold corresponded to 34%, 18% and 17% of Burundi, South Africa and Ghana exports, respectively.

<sup>3</sup>Using monthly average prices from Sep-1995 to Aug-2007 we find a correlation between oil and copper returns of 28.9% and between oil and gold returns of 16.4%.

<sup>4</sup>For example, futures prices in a 3-factor model depend on three sources of uncertainty that can be interpreted as the level, slope and curvature of the futures curve.

factors, so that the developing country can fully hedge the oil risk if it's optimal to do so. These financial instruments also enhances the investment opportunity set of the country. We assume that the country only chooses to hedge against the crude oil risk factors. The country decides not to hedge its own exports because there are no financial contracts available, or simply because it has a comparative advantage over other commodity producers.<sup>5</sup>

It is important to have close-form expressions to understand the economics behind the country's consumption and futures contract holdings. We use an asymptotic expansion technique to find approximate analytical expressions for the country's decisions. This technique expands the solution of our problem around the closed-form solution of a particular case (see Kogan and Uppal 2003). Indeed, as the input share of oil in the economy and the natural exports of the country goes to zero, the solution converges to the portfolio selection model of Merton (1969, 1971).

We find that the country's consumption increases with its natural exports, because they increase the country's wealth. The relative risk-aversion degree plays a crucial role in the country's decisions through the well-known income and substitutions effects with respect to the different variables of the model (see Kim and Omberg 1996, Campbell and Viceira 2002). In terms of oil usage, less efficient countries consume a lower fraction of their wealth if they are mainly worried about consumption smoothing (i.e. they have a risk-aversion degree greater than 1). Countries with lower risk-aversion degrees consume a higher fraction of their wealth than developed countries because of the substitution effect. Indeed, in this case the consumption good is more scarce in the future, implying that today's consumption good is relatively cheaper than tomorrow's consumption. The crude oil price has a negative effect for oil-importing economies, because it implies a decrease in the productivity of capital. Oil shocks affect the current state of the economy, but also the state in the future, specially if they are persistent. Highly risk-averse countries decrease their consumption if a price shock occurs, because of a negative income effect. Interestingly, countries with lower risk-aversion degree may increase today's consumption due to a positive substitution effect of crude oil prices. Shocks to other variables related to the crude oil dynamics, such as the convenience yield, alter consumption through their effect on the expected change in the crude oil price. A positive shock to the convenience yield has a positive effect for oil-importing economies because it decreases the expected oil price. The convenience yield creates a positive income effect and a negative substitution effect.

The country's hedging strategy is determined by the effect of the different variables in consumption. The strategy can be decomposed in three components. First, we obtain the standard myopic demand related to the risk-return trade-off of the financial instruments. Second, we find that the country takes positions in contracts for pure hedging purposes in order to minimize the variance of the country's wealth. The natural exports and their correlation with the oil shocks have a crucial role in determining the size of this component. A higher correlation implies short

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<sup>5</sup>Studying the hedging policies of some emerging countries in our sample shows that our assumption is quiet reasonable. For example Codelco, Chile's public copper mining company, that owns one of the largest copper mines in the world, has only 9% of its future production hedged for the period between 2006 and 2012.

positions in the futures that can potentially offset long positions due to other demands. Finally, the country has hedging demands with respect to each one of the crude oil risk factors. These demands arise because the oil factors affect the future productivity of the country. The persistence of the crude oil shocks have a significant impact in the magnitude of the positions in the futures contracts. We consider Chile as the benchmark developing economy and study its decisions in the case that the crude oil price is driven by a one-factor model. We find that a positive correlation between the Chilean natural exports and the crude oil price reduces considerably the positions in the crude oil futures contracts. The natural exports can potentially work as a *natural hedge* against crude oil risk. If we concentrate only on the hedging characteristics of the futures contracts and assume a high risk aversion degree for Chile, we obtain that the country hedges between -30% and 10% of the annual crude oil imports depending on the natural exports and their correlation with the crude oil shocks.

An extensive literature studies the link between oil prices and economic activity. Darby (1982), Hamilton (1983, 1988) and Mork (1989) report evidence supporting the hypothesis that oil prices have a significant effect on output. More recently, Hamilton (2003) propose a non-linear specifications for an oil shock considering the smaller effect of price shocks on real economic activity detected since the mid-1980s. The mechanism by which oil affects the economy remains unclear, specially because on average oil accounts only for a small part of the total marginal cost of production. Kim and Loungani (1992) explicitly include energy as an input in a real business cycle (RBC) model and find that oil price shocks should account only for a minor part of the output volatility. Rotemberg and Woodford (1996) consider the effects of imperfect competition and find that a model involving implicit collusion in the product market can significantly increase the effect of an energy price shocks on output. Finn (2000) proposes an explanation based on the relation between the capital utilization rate and energy prices.

Several papers study the connection between the economic performance of developing countries and the price of the commodities that these countries export (Deaton and Miller (1995), Deaton (1999), among others). Only few papers deal with the management of oil price risk in developing countries. Daniel (2001) and Devlin and Titman (2004) study the effectiveness of oil stabilization funds compared to managing risk with financial instruments when the country is a net exporter of oil. Both papers find that in theory the usage of derivatives dominates the stabilization fund approach, but in practice governments have favored the latter alternative. The authorities fear the political cost of ending up worse off and also lack of know how to implement these financial strategies. Devlin and Titman (2004) also argues that stabilization funds solution is even less efficient if oil price shocks are persistent. Claessens and Varangis (1991) studies a historical simulation of different hedging strategies of a state oil-importing company for the period 1986-1990. They show that the the company would have benefited substantially with the usage of futures contracts even if it were subject to basis risk.

Our paper relates to a large literature about hedging using commodity derivatives. The classical papers in this area focus on the hedging strategy of a producer that faces output price uncertainty

in a static framework (see Rolfo 1980, Anderson and Danthine 1980, Feder, Just, and Schmitz 1980, Newbery and Stiglitz 1981). Ho (1984) extends this problem to a dynamic setting. Other papers solve the hedging strategy from an investors point of view using futures contracts (see Adler and Detemple 1988a, Adler and Detemple 1988b, Duffie and Jackson 1990, Briys and de Varenne 1998, Lioui and Poncet 1996, Lioui and Poncet 2005). A couple of recent papers consider a stochastic convenience yield in an explicit way. Hong (2001) explains how the persistence of the convenience yield shocks affects the distribution of the open interest among contracts of different maturities. Mellios and Six (2008) studies the hedging problem when the commodity follows a multi-factor Gaussian process with a stochastic convenience yield. This paper and ours use the same machinery, however, the focus of the studies are different. Mellios and Six (2008) solves the general hedging problem and considers stochastic interest rates and time-varying risk premia which may play an important role for some commodities (i.e. for silver and gold). We are concentrated on the effect of the production side and the natural exports on the hedging strategy.

The paper is organized as follows. Section 2 presents the models for the developing country and for the crude oil price. Section 3 provides an analytical solution for the Hamilton-Jacobi-Bellman equation and discusses the resulting hedging and consumption strategies. Section 4 presents the empirical estimation and analyzes the economic implications for a one-factor crude oil pricing model. Finally, Section 5 concludes.

## 2 The Model

### 2.1 Production Technologies in the Developing Country

We assume that the oil-importing emerging economy has two productive sectors: a capital sector and a natural resource sector that exports the production of the domestic commodity (i.e. the natural exports).

The capital sector  $K(t)$  has a Cobb-Douglas production technology that uses capital and crude oil as inputs. We consider the following dynamics for the developing country's capital stock:

$$dK(t) = (\alpha K(t)^{1-\eta}(\omega Q(t))^\eta - S(t)Q(t) + X(t) - C(t)) dt, \quad (1)$$

where  $K(t)$  is the stock of capital,  $\alpha$  is the total factor productivity,  $\eta$  denotes the oil share of input in the production of capital,  $Q(t)$  is the demand for crude oil,  $S(t)$  is the price of a barrel of crude oil,  $X(t)$  are the natural exports and  $C(t)$  is consumption. The parameter  $\omega$  regulates the efficiency of oil usage. It is higher for countries with more efficient technologies, because oil is a more productive input. The country chooses how many barrels of oil to import and how much capital to consume at any given time  $t$ . The demand for oil is relatively small compared to the global aggregate demand, thus the country is assumed to be a crude oil price taker.

Rather than assuming a process for the natural resource stock, we directly model the exports from this sector. Other types of exports from alternative sources are included in the capital's production technology. We consider the natural exports,  $X(t)$ , to follow a geometric Brownian motion:

$$dX(t) = -\phi X(t)dt + \sigma_X X(t)d\widehat{Z}(t), \quad (2)$$

where  $\phi$  and  $\sigma_X$  are the 'depreciation rate' and the volatility of the export changes, respectively. The natural exports decreases over time because the natural resource is assumed to be exhaustible. Another interpretation for a decrease in the natural exports is that the economy develops over time, meaning that more developed countries have lower natural exports to capital ratios. Finally,  $\widehat{Z}(t)$  is a standard Brownian motion, that can be correlated with the crude oil shocks described in the next section.

There is an infinitely-lived emerging country that maximizes the expected utility of consumption given by

$$U(t, C) = e^{-\beta t} \frac{C^{1-\gamma}}{1-\gamma} \quad \text{for } \gamma > 0, \gamma \neq 1 \quad (3)$$

The effect of crude oil in the developing country is twofold. First, it has a direct impact in the economy's marginal productivity of capital, since the crude oil is as input to the economy. The higher the price of the oil, the lower the country's output. The second effect, is through a possible correlation between the crude oil shocks and the natural exports of the country. If these are positively correlated, then an increase in the oil price can generate an increase in the exports. In this case, the two oil effects have opposite directions, implying that the exports can potentially act as a natural hedge against crude oil shocks. In a dynamic economy like ours, crude oil shocks can also have a substantial effect in the economy's productivity in subsequent periods.

## 2.2 General Gaussian Crude Oil Price Process

For the crude oil price process we extend the approach of Casassus and Collin-Dufresne (2005) (CCD) to multiple sources of uncertainty. We introduce a canonical representation of an  $n$ -factor Gaussian model for crude oil (log) prices similar to the standard affine models from the term structure literature.<sup>6</sup> The model is in the  $A_0(n)$  family using the terminology of Dai and Singleton (2000).

We assume that the spot crude oil (log) price,  $u(t) = \log S(t)$ , follows the standard no-arbitrage dynamics under the equivalent martingale measure  $\mathbb{Q}$ :

$$du(t) = \left( r - \delta(t) - \frac{1}{2}\sigma_u^2 \right) dt + \sigma_u \left( \sqrt{1 - \varsigma^\top \varsigma} dZ_u^\mathbb{Q}(t) + \varsigma^\top dZ_v^\mathbb{Q}(t) \right) \quad (4)$$

---

<sup>6</sup>See Duffie and Kan (1996), Duffie, Pan, and Singleton (2000) and Dai and Singleton (2000).

where  $r$  is the interest rate,  $\delta(t)$  is the convenience yield,  $\sigma_u$  is the volatility of oil returns and  $Z_u^{\mathbb{Q}}(t)$  and  $Z_v^{\mathbb{Q}}(t)^{\top} = \{Z_{v_1}^{\mathbb{Q}}(t), \dots, Z_{v_{n-1}}^{\mathbb{Q}}(t)\}$  are  $n$  independent standard Brownian motions. The  $n \times 1$  vector  $\varsigma$ , defines the instantaneous correlation structure of the (log) price with other factors affecting the oil price dynamics.

The proposed Gaussian model considers time-varying expected crude oil returns. Its flexibility to fit the data is given by a stochastic specification for the convenience yield  $\delta(t)$ .<sup>7</sup> Empirical studies (Schwartz (1997), CCD among others) suggest that the variability of crude oil returns are mostly explained by changes in the convenience yield, rather than by changes in interest rates. For this reason and to keep the model simple, we assume a constant interest rate. We generalize the model in CCD and assume that the convenience yield is a linear function of the (log) price and  $n - 1$  other factors represented by  $v(t)^{\top} = \{v_1(t), v_2(t), \dots, v_{n-1}(t)\}$ :

$$\delta(t) = \psi_0 + \psi_u u(t) + \psi_v^{\top} v(t) \quad (5)$$

The vector  $v(t)$  follows a Gaussian diffusion process under the equivalent martingale measure  $\mathbb{Q}$ :

$$dv(t) = -\kappa_v v(t)dt + dZ_v^{\mathbb{Q}}(t), \quad (6)$$

where  $\kappa_v$  is an  $n \times n$  upper triangular matrix.<sup>8</sup>

The parameter  $\psi_u$  in equation (5) plays a crucial role in the dynamics of the oil price. This relation between convenience yields and oil prices allows the model to generate both, contango and backwardation in the futures curve. Indeed, if  $\psi_u$  is positive, the expected change in oil prices (under the  $\mathbb{Q}$  measure) is lower for high prices because the convenience yield is high, implying higher degrees of backwardation. The opposite effect occurs for low oil prices.

We have chosen a slightly different canonical representation than in CCD where the (log) spot price,  $u(t)$ , is a function of the latent factors. We want to explicitly have the oil price as a factor in the crude oil dynamics in order to understand the direct effect of this variable in the consumption and hedging strategies. Under the CCD representation, the hedging strategy would be in terms of  $n$  latent factors, rather than in terms of the spot price  $u(t)$  and  $n - 1$  latent factors.

To simplify the notation we define  $Y(t)$  as the stacked vector of the  $n$  crude oil factors,  $Y(t)^{\top} = \{u(t), v_1(t), \dots, v_{n-1}(t)\}$ . Using equations (4)-(6) we obtain the dynamics of  $Y(t)$ :

$$dY(t) = (\kappa_0 - \kappa_Y Y(t))dt + \sigma_Y dZ_Y^{\mathbb{Q}}(t), \quad (7)$$

---

<sup>7</sup>The convenience yield is defined as the implied benefit associated with holding the underlying physical good, in this case, a barrel of oil.

<sup>8</sup> From an empirical point of view, it is worth noting the parameters  $r$  and  $\psi_0$  cannot be separately identified. If we replace equation (5) in (4) we can see that the constant in the expected oil return is  $r - \psi_0$ . As we will see later, we estimate the model only with futures prices data, and since the convenience yield is an unobservable variable, it is impossible to identify  $\psi_0$  from the estimate of  $r - \psi_0$ . To circumvent this empirical issue we assume a value for  $r$  and estimate  $\psi_0$  from the data. We prefer this overidentified representation to isolate the convenience yield effect from the interest rates.

where  $Z_Y^{\mathbb{Q}}(t)^\top = \{Z_u^{\mathbb{Q}}(t), Z_{v_1}^{\mathbb{Q}}(t), \dots, Z_{v_{n-1}}^{\mathbb{Q}}(t)\}$ . The  $n \times 1$  vector  $\kappa_0$ , and the  $n \times n$  matrices  $\kappa_Y$  and  $\sigma_Y$ , collect the parameters from the dynamics of the crude oil factors  $u(t)$  and  $v(t)$ .

We assume the existence of crude oil futures contract in the financial market. It is well known (e.g., Duffie 2001) that when interests rates are constant, the futures price  $F_i(t)$  with maturity  $\tau_i$  is:

$$F_i(t) = \mathbb{E}_t^{\mathbb{Q}} \left[ e^{u(t+\tau_i)} \right] = e^{B_{0,i}(t) + B_{Y,i}(t)^\top Y(t)}, \quad (8)$$

where  $B_{0,i}(t)$  and  $B_{Y,i}(t) \equiv \{B_{u,i}(t), B_{v_1,i}(t), \dots, B_{v_{n-1},i}(t)\}$  are the solution to the following system of ordinary differential equations:

$$\frac{dB_{0,i}(t)}{dt} = -\frac{1}{2} B_{Y,i}(t)^\top \sigma_Y \sigma_Y^\top B_{Y,i}(t) - \kappa_0^\top B_{Y,i}(t) \quad (9)$$

$$\frac{dB_{Y,i}(t)}{dt} = \kappa_Y^\top B_{Y,i}(t) \quad (10)$$

with boundary conditions  $B_{0,i}(t+\tau_i) = 0$ ,  $B_{u,i}(t+\tau_i) = 1$  and  $B_{v_1,i}(t+\tau_i) = \dots = B_{v_{n-1},i}(t+\tau_i) = 0$ . These conditions ensure that at maturity  $F_i(t+\tau_i) = S(t+\tau_i)$ .

To complete the model we assume a constant risk-premia specification:

$$dZ_Y^{\mathbb{Q}}(t) = dZ_Y(t) + \lambda_Y dt \quad (11)$$

where  $Z_Y(t)$  is a  $n \times 1$  vector of Brownian motions on a standard filtered probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and  $\lambda_Y^\top = \{\lambda_u, \lambda_{v_1}, \dots, \lambda_{v_{n-1}}\}$  is the risk-premia vector.

Under the physical measure  $\mathbb{P}$ , the processes for the futures prices maturing  $\tau_i$  periods from now are given by<sup>9</sup>

$$dF_i(t) = F_i(t) B_{Y,i}(t)^\top \sigma_Y \lambda_Y dt + F_i(t) B_{Y,i}(t)^\top \sigma_Y dZ_Y(t). \quad (12)$$

The processes under  $\mathbb{P}$  are relevant for risk management decisions (rather than those under the risk-neutral measure  $\mathbb{Q}$ ), because the implementation of these strategies implies holding futures contracts over time. The country takes positions in futures contracts and demands a compensation (here  $B_{Y,i}(t)^\top \sigma_Y \lambda_Y$ ) for bearing the risk embedded in those contracts.

Finally, we define the process  $dF(t)$  as the vector of stacked processes  $dF_i(t)$ :

$$dF(t) = I_F(t) \sigma_F(t) \lambda_Y dt + I_F(t) \sigma_F(t) dZ_Y(t) \quad (13)$$

where  $I_F(t)$  is a matrix with the futures prices  $F_i(t)$  in the diagonal, and  $\sigma_F(t)$  stacks the  $n$  row vectors  $B_{Y,i}(t)^\top \sigma_Y$ .

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<sup>9</sup>Note that in the Gaussian model with constant risk premia, the futures returns  $dF_i(t)/F_i(t)$  are not affected by the level of  $Y(t)$ . These state variables enter only through the futures price  $F_i(t)$ .

### 3 Optimal Controls in the Developing Country

In this section we study the problem that faces the developing economy. At any given time, the country chooses: (i) how much crude oil to demand for its production technology, (ii) how much capital to consume, and (iii) the positions in the futures contracts in the economy.

The primary purpose of the futures contracts is hedging, however, they also enhance the investment opportunity set of the developing country because of their risk-return trade-off (i.e. Sharpe ratios). This creates additional incentives to take positions in the financial instruments. At any time  $t$  we allow the country to take long/short positions in  $n$  available crude oil futures contracts with maturities  $\tau_i = 1, \dots, n$ . A multi-factor specification for the crude oil dynamics implies that the economy wants to hedge not only against the crude oil price shocks, but also against changes in the convenience yield factors. The country continuously rebalances its position in a different set of contracts every time such that the maturities of the contracts remain constants. This assumption avoids the expiration of the futures that would otherwise face the infinitely-lived country. Trading in  $n$  contracts is enough to span the whole futures curve when an  $n$ -factor model for crude oil prices is considered. This also means that the country can fully hedge against the crude oil risks including the risk of rolling over the futures contracts.<sup>10</sup>

Let us define the  $n \times 1$  vector  $p(t)$  as the *number of crude oil futures contracts* held by the developing country at time  $t$  for each one of the  $n$  available contracts. A positive (negative) element  $i$  of  $p(t)$  means that the country takes a long (short) position in the futures contract maturing in  $\tau_i$  periods from time  $t$ . We restrict  $p(t)$  to be in the set of admissible strategies that lead to a strictly positive capital process ( $K(t) > 0$  a.s.). We only consider non-negative consumption and crude oil demand strategies.

The optimal consumption-demand-hedging strategy of the developing country is the solution to the following problem:<sup>11</sup>

$$\mathcal{J}(K(0), X(0), Y(0), 0) \equiv \sup_{\{C, Q, p\} \in \Psi} \mathbb{E}_0 \left[ \int_0^\infty U(s, C(s)) ds \right] \quad (14)$$

subject to:

$$dK(t) = (\alpha K(t)^{1-\eta} (\omega Q(t))^\eta - S(t)Q(t) + X(t) - C(t)) dt + p(t)^\top dF(t) \quad (15)$$

$$dX(t) = -\phi X(t) dt + \sigma_X X(t) (\rho_Y^\top dZ_Y(t) + \sqrt{1 - \rho_Y^\top \rho_Y} dZ_X(t)) \quad (16)$$

$$dY(t) = (\sigma_Y \lambda_Y + \kappa_0 - \kappa_Y Y(t)) dt + \sigma_Y dZ_Y(t) \quad (17)$$

where  $\mathcal{J}(K(t), X(t), Y(t), t)$  is the value function associated to the country's problem,  $\Psi$  is

<sup>10</sup>Neuberger (1996) presents an alternative way of solving the rolling over problem.

<sup>11</sup>For the moment we assume that the optimal controls exist and are admissible. In the next section, we obtain an approximated solution for the value function and determine the restrictions in the parameter space in order to have admissible controls.

the admissible set of strategies and  $dF(t)$  are the changes in the futures prices as defined in equation (13). The futures contracts are marked to market, which implies an instantaneous flow  $p(t)^\top dF(t)$  to the capital stock. Also, the country's natural exports can be correlated with both, the oil price and convenience yield shocks. To see this we rewrite the Brownian motion of the natural exports,  $\widehat{Z}(t)$ , as a linear combination of independent Brownian motions (compare equation (16) to (2)). We define  $Z_X(t) \equiv \frac{\widehat{Z}(t) - \rho_Y^\top Z_Y(t)}{\sqrt{1 - \rho_Y^\top \rho_Y}}$  as a Brownian motion that captures the unhedgeable risk of the country's natural exports (i.e.  $Z_X(t)$  is independent from the vector  $Z_Y(t)$ ) and  $\rho_Y^\top = \{\rho_u, \rho_{v_1}, \dots, \rho_{v_{n-1}}\}$  as the correlation vector. Here,  $\rho_u$  stands for the correlation between the exports and the crude oil shocks, and  $\rho_v$  defines the correlation between the exports and each one of the latent factors. In the rest of the paper we drop the time argument from the variables to simplify the notation.

Let us define the 'current' value function  $J(K, X, Y)$  of the country's problem, such that  $\mathcal{J}(K, X, Y, t) = e^{-\beta t} J(K, X, Y)$ . The function  $J(K, X, Y)$  satisfies the standard Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{aligned}
0 = \max_{\{C, Q, p\}} & \left\{ \frac{C^{1-\gamma}}{1-\gamma} - \beta J + (\alpha K^{1-\eta} (\omega Q)^\eta - S Q + X - C + p^\top I_F \sigma_F \lambda_Y) J_K - \phi X J_X \right. \\
& + (\sigma_Y \lambda_Y + \kappa_0 - \kappa_Y Y)^\top J_Y + \frac{1}{2} p^\top I_F \sigma_F \sigma_F^\top I_F p J_{KK} + \frac{1}{2} \sigma_X^2 X^2 J_{XX} + \frac{1}{2} \text{Tr}[\sigma_Y \sigma_Y^\top J_{YY}] \\
& \left. + \sigma_X X p^\top I_F \sigma_F \rho_Y J_{KX} + p^\top I_F \sigma_F \sigma_Y^\top J_{KY} + \sigma_X X \rho_Y^\top \sigma_Y^\top J_{XY} \right\}. \tag{18}
\end{aligned}$$

where  $J_i$  is the partial derivative of  $J$  with respect to the state variable  $i$  and  $J_{ij}$  are the second order derivatives.

The next proposition presents the optimal decisions for the country's problem in equations (14) to (17).

**PROPOSITION 1:** *The optimal consumption for the developing country is  $C^* = J_K^{-1/\gamma}$  and the optimal demand for crude oil is given by*

$$Q^* = \frac{K}{\omega} \left( \frac{\alpha \eta \omega}{S} \right)^{\frac{1}{1-\eta}}. \tag{19}$$

*The hedging strategy is determined by the  $n \times 1$  vector of contract holdings*

$$p^* = (I_F)^{-1} (\sigma_Y \sigma_F^{-1})^\top \left( \sigma_Y^{-1} \lambda_Y \frac{-J_K}{J_{KK}} + \sigma_X \sigma_Y^{-1} \rho_Y \frac{-X J_{KX}}{J_{KK}} + \frac{-J_{KY}}{J_{KK}} \right). \tag{20}$$

**Proof** See Appendix A.1.  $\square$

Proposition 1 shows that  $Q^*$  is the maximizer of the expected change in the capital stock. Indeed, let us define  $\mu$  as:

$$\mu = \frac{\alpha K^{1-\eta} (\omega Q)^\eta - S Q + X}{K}. \tag{21}$$

This variable is the expected change in the capital stock before consumption and measures the productivity per unit of capital in equation (1). The optimal demand for crude oil,  $Q^*$ , maximizes  $\mu$ . This simple result occurs because the capital technology is fully flexibly, i.e., there are no adjustment costs.

Equation (19) shows that  $Q^*$  is increasing in  $\omega$ , because oil is more productive for countries with higher  $\omega$ . The optimal crude oil demand is decreasing in the price of the crude oil  $S$ . It turns out that  $Q^*$  equates the marginal benefit and the marginal cost of an extra barrel of oil, thus it is independent from the exports and other variables in the economy. If we replace  $Q^*$  in equation (21) we obtain the optimal productivity of capital  $\mu^*$ . It is straightforward to show that the productivity of capital is decreasing in the price of the crude oil  $S$ , i.e.  $\frac{\partial \mu^*}{\partial S} < 0$ . This last point is central in what follows, because a higher crude oil price in the future will undoubtedly imply a decrease in the future productivity of capital.

Proposition 1 also shows that what matters for the hedging strategy is the product of quantities and prices (i.e.  $(p^*)^\top I_F$ ) which is in units of the numeraire good. Also, we find that the holdings in equation (20) are amplified by  $(\sigma_Y \sigma_F^{-1})^\top$ . If the futures returns volatilities are low, the country will take a larger position in the futures contracts to have the same hedging effect. This is the only place where the futures returns volatilities matter.<sup>12</sup>

The optimal holdings  $p^*$  in (20) result from the summation of three components. The first term is the standard myopic demand present in the classical Merton model and captures the risk-return trade-off of the positions in futures contracts. It is proportional to the Sharpe ratio of each risk factor,  $\sigma_Y^{-1} \lambda_Y$ . Its main purpose is to take advantage of the enhanced investment opportunity set rather than hedging against changes in oil prices. If there are no risk premia embedded in the futures contracts (i.e.  $\lambda_Y = 0$ ), there are no incentives for bearing crude oil risk and the myopic demand fade away. This demand is also present in standard static models of portfolio selection.

The second component in equation (20) is a pure hedging term that is also myopic in the sense that it appears even in a static version of the model. The risk-averse country is worried about the variance of the natural exports, because it affects the volatility of consumption. This type of hedging is sometimes called *statistical hedging*, because the coefficients  $\sigma_X \sigma_Y^{-1} \rho_Y$  are the  $\beta$ 's of  $n$  regressions where each crude oil factor is regressed on the natural exports. The correlations between the exports and the crude oil shocks,  $\rho_Y$ , play an important role, because they affect the hedging capacity of the futures contracts against shocks in the natural export. If  $\rho_Y^\top \rho_Y = 1$ , then the natural exports can be fully hedged with the futures contracts. If the country has no natural exports, this type of demand disappears. To see this, note that without exports the value function  $J(t)$  is independent of  $X(t)$  implying that  $J_{KX} = 0$ . The exports are a *natural hedge* against crude oil shocks as long as this term decreases the absolute holdings of futures contracts.

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<sup>12</sup>From equation (12) we find that the maturities of the futures contracts enter only through the volatility of the futures returns. This implies that the maturities are only relevant in our analysis to determine the amplifying factor  $(\sigma_Y \sigma_F^{-1})^\top$  in the hedging strategy.

The last term includes the productive hedging demands due to changes in each crude oil factor in  $Y(t)$ . The interpretation of this term is similar to the hedging demands in Merton (1973) and Breeden (1979), with the exception that here they hedge against future changes in the productivity of capital rather than against changes in the investment opportunity set.<sup>13</sup> These demands arise because the country worries about changes in the crude oil price since it affects the productivity of capital. For this reason we label these terms as *productive hedging demands*. Recall that the crude oil factors  $Y(t)$  can be decomposed in the (log) spot price,  $u(t)$ , and other latent factors,  $v(t)$ , associated to the convenience yield. A shock to the spot price can have a disparate effect in the economy depending on whether it is a permanent or a temporal shock. The country is more concerned about crude oil shocks if they persist in the economy for a longer period of time. If this is the case, the productive hedging demand with respect to  $u(t)$  is more significant. The latent factors  $v(t)$  influence crude oil prices in the future through the convenience yield, thus affecting the future productivity of capital. In the case that oil is useless for the economy (i.e.  $\eta = 0$  in equation (1)), the crude oil shocks have no effect in future production, thus these hedging demands disappear.

To the best of our knowledge, the problem that the developing country faces has no closed-form solution. In the next section, we present an approximated solution that is asymptotically exact. This will help us to get a better economic intuition about the decisions that the country takes.

### 3.1 An Approximated Solution

In this section we present the steps to obtain closed-form approximations for the consumption and hedging strategies of the developing country in Proposition 1. First, we use the homogeneity of the problem to reduce the number of state variables and then, we apply an asymptotic expansion technique to get an approximated solution around the standard Merton problem. The approximations deliver various economic insights that are helpful to understand the decisions of the country.

We note that consumption is homogeneous of degree one in  $K(t)$  and  $X(t)$  and that the CRRA utility function is homogeneous of degree  $(1 - \gamma)$ . These two properties imply that the value function  $J(t)$  is also homogeneous of degree  $(1 - \gamma)$ . We can use this feature to reduce the state space from  $n + 2$  to  $n + 1$  variables. The homogeneity feature implies that a country that doubles another one in natural exports and capital stock will consume twice the consumption, demand twice the number of crude oil barrels, and take twice the positions in futures contracts than the smaller country. For this reason we discuss the results in terms of the consumption-wealth ratio and the market value of the hedging positions to capital ratio. These variables are homogeneous of degree zero in  $K(t)$  and  $X(t)$ , meaning that the normalized natural exports to capital ratio,  $z(t)$ , and the crude oil factors,  $Y(t)$ , are enough to characterize the economy. Here we have defined  $z(t) = \frac{K(t)^{-1}X(t)}{x_0}$  where  $x_0$  is the initial natural exports to capital ratio, i.e.,  $x_0 \equiv K(0)^{-1}X(0)$ .<sup>14</sup>

<sup>13</sup>The investment opportunities in our model are given by the positions in the futures contracts, but the futures return are independent of the state variables  $Y(t)$ .

<sup>14</sup>The definition of  $x_0$  implies directly that  $z(0) = 1$ .

We write the current value function as

$$J(K, X, Y) = A_1^{-\gamma} \frac{(K e^{h(z, Y)})^{1-\gamma}}{1-\gamma} \quad (22)$$

where we have defined the constant

$$A_1 = \frac{1}{\gamma} \left( \beta - \alpha(1-\gamma) - \frac{1-\gamma}{\gamma} \frac{\lambda_Y^\top \lambda_Y}{2} \right) > 0. \quad (23)$$

Replacing equation (22) and the optimal controls  $\{C^*, Q^*, p^*\}$  from Proposition 1 in equation (18), yields a non-linear second-order PDE for  $h(z, Y)$  that we write as<sup>15</sup>

$$0 = G(z, Y) \quad (24)$$

It is hard to solve this equation numerically because it's a second-order equation in  $n + 1$  state variables. However, it is possible to obtain an approximation by doing an asymptotic expansion of the solution. This approximation method has become popular in finance lately (see for example, Kogan 2001, Kogan and Uppal 2003 and Janecek and Shreve 2004). This technique is exact in the limit and its main advantage is that it provides informative explicit expressions for the optimal consumption and hedging strategies.

The idea behind the asymptotic expansion technique is to do a Taylor expansion of the solution of equation (24) around a particular set of parameters under which this PDE has an exact solution. We note that the problem simplifies considerably if the oil is useless in the economy (i.e.  $\eta = 0$ ) and the country has no natural exports (i.e.  $x_0 = 0$ ).<sup>16</sup> In this case, the solution converges to the well-known closed-form solution of the infinite-horizon model of Merton (1969). Indeed, under this scenario the production technology  $K(t)$  has constant returns to scale, because: (i)  $Q^*(t)$  and  $X(t)$  are zero, and (ii) the investment opportunity set given by the futures returns is independent of the state variables. The value function is independent of  $X(t)$  and  $Y(t)$  and the problem reduces to the Merton solution. In this case  $h(z, Y) = 0$  which implies that the country consumes a constant fraction  $A_1$  of its capital and the positions in the futures contracts are proportional to  $\gamma^{-1} \sigma_Y^{-1} \lambda_Y$ .

The approximated solution is valid as long  $\eta$  and  $x_0$  stay relatively close to zero. As we will see later, even for small values of  $\eta$  and  $x_0$ , there is a lot of action in our model and the consumption and hedging strategies differ significantly from the Merton solutions. Moreover, these assumptions have reasonable economic foundations. We expect the ratio between the natural exports and capital to be a small figure even for less developed countries. For example, for Chile whose economy depends heavily on its copper exports, we estimate that the copper exports to capital ratio is less than 1%.

<sup>15</sup>Equation (A1) in Appendix A.2 shows the resulting differential equation. Here, we prefer to omit the details, because the equation is messy and uninformative.

<sup>16</sup>We could have expressed equation (22) in terms of the natural exports to capital ratio instead of  $z(t)$ , but using  $z(t)$  as a state variable clarifies the idea that we are expanding with respect to the *initial* natural exports to capital ratio,  $x_0$ .

The same happens with  $\eta$ . Recent RBC studies that include energy as a production factor use values around 4% for the oil share of income,  $\eta$  (see Finn (2000) and Wei (2003)).

We show the approximation technique for a first-order expansion, but this methodology can be implemented for higher-order expansions. We assume the following structure for the solution of equation (24)

$$h(z, Y) = h^{(\eta)}(z, Y) \eta + h^{(\epsilon)}(z, Y) \epsilon + h^{(x_0)}(z, Y) x_0 + O(2^{nd}\text{-order terms}) \quad (25)$$

where  $\epsilon = \frac{\eta}{1-\eta} \log(\eta)$ .<sup>17</sup> We replace this solution in the PDE and pursue a first-order Taylor expansion of  $G(z, Y)$  around  $\eta = 0$ ,  $\epsilon = 0$  and  $x_0 = 0$  to get

$$G(z, Y; \Theta) = G(z, Y; 0) + \Theta^\top \nabla G_\Theta(z, Y; 0) + O(2^{nd}\text{-order terms}) \quad (26)$$

where  $\Theta^\top = (\eta, \epsilon, x_0)$  and  $\nabla G_\Theta(\cdot)$  is the gradient vector with the partial derivatives of  $G(\cdot)$  with respect to  $\Theta$ . We seek for the functions  $h^{(\eta)}(z, Y)$ ,  $h^{(\epsilon)}(z, Y)$  and  $h^{(x_0)}(z, Y)$  such that the approximated PDE is satisfied to a first-order degree. The  $h^{(j)}(\cdot)$  functions need to be independent of  $\eta$  and  $x_0$ . Interestingly, we find that these are affine functions in the state variables  $z(t)$  and  $Y(t)$ . The next proposition shows the results after the first-order expansion has been performed.<sup>18</sup>

**PROPOSITION 2:** *Suppose that*

$$A_2 = \alpha + \phi + \sigma_x \rho_Y^\top \lambda_Y > 0. \quad (27)$$

*The approximated solution of equation (24) using a first-order asymptotic expansion in  $(\eta, \epsilon, x_0)$  around the origin is given by equation (25) where*

$$h^{(\eta)}(z, Y) = M_0^{(\eta)} - M_Y^{(\eta)\top} Y \quad (28)$$

$$h^{(\epsilon)}(z, Y) = A_1^{-1} \alpha$$

$$h^{(x_0)}(z, Y) = A_2^{-1} z \quad (29)$$

*and the  $M^{(\eta)}$ 's are constants depending on the fundamental parameters of the model.*

**Proof** See Appendix A.2.  $\square$

For the following we shall assume that conditions (23) and (27) are satisfied.

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<sup>17</sup>Appendix A.2 shows that equation (24) contains the hyper-power term  $\eta^{\frac{\eta}{1-\eta}}$ . A standard Taylor expansion around  $\eta = 0$  is undefined, because the first derivative of the hyper-power term has a singularity at the origin. To circumvent this inconvenience we define  $\epsilon$  such that  $e^\epsilon = \eta^{\frac{\eta}{1-\eta}}$  and expand the term around  $\epsilon = 0$ . We treat  $\epsilon$  as a different parameter for the expansion, however, we only require  $\eta$  and  $x_0$  to be small, because  $\lim_{\eta \rightarrow 0} \epsilon = 0$ .

<sup>18</sup>Technically speaking, taking advantage of the homogeneity of the problem is not necessary to get the approximated solution. It was useful, though, to understand that the solution in Proposition 2 was a function of  $K^{-1}X$ .

### 3.2 Characterizing the Optimal Controls

Now that we have an approximation for the value function  $J(t)$  we are ready to revisit the optimal controls from Proposition 1. We present approximated solutions that converge to Merton's solutions in the limit.

We need a measure of the total wealth of the country to better contrast our results with those from Merton's model. Indeed, in Merton's model the agent consumes a constant fraction of its wealth, so a fair comparison is to analyze the consumption-wealth ratio in our country. Here, the developing country's wealth is composed by its capital and the present value of future natural exports. We use utility indifference pricing to obtain the value an extra unit of natural exports in terms of the numeraire  $E(t)$ , thus

$$E(t) = \frac{J_X(t)}{J_K(t)} \quad (30)$$

Note that the price  $E(t)$  already considers the present value of future increments in the natural exports due to the extra unit today.<sup>19</sup> Let us define the total wealth of the country as

$$W(t) \equiv K(t) + E(t)X(t) \quad (31)$$

The definition of  $W(t)$  is correct as long as the marginal price of the natural exports  $E(t)$  corresponds to the average price. This is valid if  $E(t)$  is independent from  $X(t)$ , which is true at least to a first-order degree, because

$$\frac{W(t)}{K(t)} = 1 + A_2^{-1}x(t) + O(2^{nd}\text{-order terms}) \quad (32)$$

implying that  $E(t) \approx A_2^{-1}$ .<sup>20</sup>

The next proposition shows the asymptotically equivalent expansions for the consumption-wealth ratio and the market value of the hedging positions to capital ratio.

**PROPOSITION 3:** *Let us define  $c^*$  as the consumption-wealth ratio (i.e.  $c^* \equiv W^{-1}C^*$ ), and  $\pi^*$  as the ratio of the dollar amount invested in the futures contracts to the capital stock (i.e.  $\pi^* \equiv I_F K^{-1}p^*$ ).*

*Asymptotically equivalent expressions for optimal consumption and hedging strategies in the developing country are given by*

$$c^* = A_1 \left( 1 + \frac{1-\gamma}{\gamma} \left( \eta \left( -M_0^{(\eta)} + M_Y^{(\eta)\top} Y \right) - \epsilon \frac{\alpha}{A_1} \right) \right) + O(2^{nd}\text{-order terms}) \quad (33)$$

<sup>19</sup>The geometric Brownian motion specification for the natural exports in equation (2) means that changes in the exports are permanent. This implies that an increase of 1% in today's exports generates an increase of 1% in future exports as well.

<sup>20</sup>Note that  $A_2^{-1}$  acts as a discount factor for the perpetual flow of natural exports, which is why we restrict  $A_2$  to be positive in equation (27).

and

$$\begin{aligned} \pi^* = (\sigma_Y \sigma_F^{-1})^\top \times & \left( (1 + A_2^{-1}x) \frac{\sigma_Y^{-1} \lambda_Y}{\gamma} \right. \\ & \left. - \sigma_X \sigma_Y^{-1} \rho_Y A_2^{-1}x - \frac{1-\gamma}{\gamma} \eta M_Y^{(\eta)} + O(2^{nd}\text{-order terms}) \right). \end{aligned} \quad (34)$$

**Proof** See Appendix A.3.  $\square$

The analysis that follows is based on the approximated solutions from this proposition, therefore, the results are valid only to a first order degree.

### 3.2.1 Consumption Strategy

Equation (33) shows that the consumption-wealth ratio is independent from  $X(t)$ , which means that the main effect of the natural exports in consumption is through the wealth of the country. A positive shock to the exports increases the total wealth, and consumption increases proportionally to the wealth.

Crude oil impacts consumption because it is an input to the production technology. The effect of crude oil shocks  $Y(t)$  in consumption depends on the risk aversion parameter  $\gamma$ . This is related to the standard income and substitution effects with respect to each one the crude oil factors. For the analysis it is convenient to separate the crude oil price from the other factors, because the oil price is observable and directly affects the productivity of capital. The partial derivative of the consumption-wealth ratio with respect to the crude oil (log) price  $u(t)$  is:<sup>21</sup>

$$c_u^* \approx A_1 \frac{1-\gamma}{\gamma} \eta M_u^{(\eta)} \quad \text{where} \quad M_u^{(\eta)} = \frac{\alpha}{A_1 + \psi_u} > 0 \quad (35)$$

The crude oil price has two opposite effects in today's consumption-wealth ratio. The income effect in consumption is negative, because an increase in today's crude oil price has a negative impact in the capital accumulation process of the economy. On the other hand, the substitution effect in today's consumption is positive. The intuition is that the negative impact of crude oil in the economy decreases the *expected capital stock* even further because there is less capital to invest in every period. This shortage of expected capital increases the relative price of tomorrow's consumption, thus affecting today's consumption positively. Equation (35) shows that if  $\gamma > 1$ , the consumption-wealth ratio decreases with an increase in the crude oil price. Indeed, if the country is too worried about consumption smoothing (high  $\gamma$ ), it will consume less, even if today's consumption becomes relatively cheaper. In this case, the negative income effect dominates the

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<sup>21</sup>For the moment we assume that  $\psi_u \geq 0$ . CCD shows in a three-factor model that for crude oil prices this parameter is positive and highly significant. We obtain the same result in the next section for a one-factor model.

substitution effect. If  $\gamma < 1$ , the consumption-wealth ratio increases with crude oil shocks. The country is less concerned about the variability of consumption and takes advantage of the relatively lower price of today's consumption. Here, the positive substitution effect dominates the income effect. Both effects cancel out if risk aversion is unity which corresponds to the logarithmic utility case. In this case, the consumption-wealth ratio is constant.

The mean-reverting parameter  $\psi_u$  in (35) relates the spot price and the convenience yield, but also determines the persistence of the crude oil price shocks and the unconditional volatility of crude oil returns. The price shocks have a half-life of  $\psi_u^{-1} \log(2)$ . For values of  $\psi_u$  close to zero, the shocks are permanent and the impact in the economy is higher. An increase in the oil price persists for a long time in the economy and it affects the productivity of capital in every subsequent period of time. For high values of  $\psi_u$ , the price shocks are temporal, thus they only affect the short-term dynamics of crude oil prices. In this case, the effect in consumption is less important.

The general impact of the convenience yield factors  $v(t)$  in consumption is less intuitive. The reason is that in the maximal model these factors not only affect the current convenience yield through  $\psi_v$ , but also their own dynamics (see equations (5) and (6)). For example, a positive shock to  $v_j(t)$  modifies the expected change of the variables  $\{v_1(t), \dots, v_j(t)\}$ , because  $\kappa_v$  is an upper triangular matrix. The overall effect of this shock in the expected crude oil price depends on  $\psi_v$  and on the elements of column  $j$  of  $\kappa_v$ . Fortunately, there is one simple case to analyze. Shocks to  $v_1(t)$  affect the convenience yield and its own dynamics while leaving the other  $v$ 's unaltered. The derivative of the consumption-wealth ratio with respect  $v_1(t)$  is:<sup>22</sup>

$$c_{v_1}^* \approx A_1 \frac{1-\gamma}{\gamma} \eta M_{v_1}^{(\eta)} \quad \text{where} \quad M_{v_1}^{(\eta)} = -\frac{\psi_{v_1}}{A_1 + \kappa_{v_{11}}} M_u^{(\eta)} < 0 \quad (36)$$

Here,  $\psi_{v_1}$  is the effect of  $v_1(t)$  in the convenience yield and  $\kappa_{v_{11}}$  is the (1,1) element of  $\kappa_v$ .  $\kappa_{v_{11}}$  determines the persistence of the shocks to  $v_1(t)$ . The derivative  $c_{v_1}^*$  has the opposite sign than  $c_u^*$  in equation (35). The reason is simple. A positive shock to  $v_1(t)$  decreases the expected crude oil spot price, because the convenience yield has a negative effect in crude oil returns. It turn out also that the income effect of this variable is positive while its substitution effect is negative. These effects are the antithesis to the income and substitution effects with respect to price shocks. For  $\gamma > 1$  the consumption-wealth ratio increases because an increase in the convenience yield has a negative effect on prices, thus an overall positive effect in the economy (income effect). If  $\gamma < 1$  today's consumption decreases, because it becomes relatively more expensive with respect to tomorrow's consumption (substitution effect).

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<sup>22</sup>Without loss of generality, we assume that  $\psi_{v_1} > 0$ . Again, we use the results of CCD to consider that  $\kappa_{v_{11}} > 0$ .

### 3.2.2 Hedging Strategy

For the hedging strategy we use the dollar amount invested in the futures contracts to the capital stock,  $\pi^*$ , instead of the number of contracts. This measure is better for the analysis because it controls for the size of the country. The hedging strategy in (34) has exactly the same three components as  $p^*$  in Proposition 1. The myopic demand is positive as long as the Sharpe ratio is positive. As expected, it is decreasing in the degree of risk-aversion  $\gamma$ , which implies that more risk-averse countries seek less exposure to the crude oil risk factors. Also, the myopic demand is proportional to the total wealth of the country.<sup>23</sup> The natural exports increases the total wealth and allows the country to increase its investment in futures contracts. The second term of the hedging strategy is the statistical hedging demand. This demand is negative for those crude oil factors that have a positive correlation with the natural exports and viceversa. Indeed, a higher correlation of the exports with a particular factor, means that a portfolio of futures that is perfectly correlated with this factor works better as a hedge against shocks in the exports. This implies that fewer units of this portfolio are necessary for the hedge.

The third term in equation (34) has the productive hedging demands. It is not surprising that these demands have a similar structure than the sensitivity of consumption with respect to the crude oil shocks (i.e.  $c_u^*$  and  $c_{v_1}^*$ ). These demands are proportional to  $M_Y^{(\eta)}$ , because the country hedges against those crude oil shocks that impact consumption. Crude oil shocks are transferred to consumption through the productivity of capital. Again the sign depends on the risk-aversion of the country. Consider a portfolio of futures contracts,  $f_u$ , that is perfectly correlated with the shocks to the crude oil (log) price,  $u(t)$ . An increase in the crude oil price, has a negative effect on today's consumption if  $\gamma > 1$  and a positive effect if  $\gamma < 1$  (see equation (35)). Clearly, the country chooses a strategy that minimizes the effect of these shocks in consumption by taking a long position in  $f_u$  if  $\gamma > 1$  or a short position in  $f_u$  if  $\gamma < 1$ . The effects on consumption are compensated by the payoff from the marking-to-market of  $f_u$ . If these shocks are persistent (low  $\psi_u$ , high  $M_u^{(\eta)}$ ), the country is more worried about this type of uncertainty and takes a larger position in  $f_u$ . The converse occurs with the convenience yield shocks through the factor  $v_1(t)$ . An increase in the convenience yield has a positive effect on the capital accumulation process, because decreases expected oil prices. It has a positive effect on today's consumption if  $\gamma > 1$  and a negative effect if  $\gamma < 1$  (see equation (36)). If  $\gamma > 1$ , the country chooses a short position in a portfolio of futures  $f_{v_1}$ , that is perfectly correlated with  $v_1(t)$ . If  $\gamma < 1$ , the country takes a long position in this portfolio.

In the next section we take our model to the data. We study the decision of a developing country assuming that crude oil prices are driven by a one-factor Gaussian model. This simple framework will help us quantify the aggregate effect of the crude oil shocks and the natural exports on the country's decisions.

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<sup>23</sup>Recall that the first-order approximation to the total wealth to capital ratio is  $1 + A_2^{-1}x$ .

## 4 Empirical Results for a One-Factor Model

In this section we estimate a one-factor model for crude oil prices and analyze the consumption and hedging strategies of the developing country. One futures contract is enough to span the whole futures curve in a one-factor model. In this model it is also easier to connect the empirical results with the theoretical results discussed in the previous section. Despite its simplicity, the model has a time-varying convenience yield and is able to generate both, contango and backwardation, in the futures curves. The relation between the spot price and the convenience yield,  $\psi_u$ , also regulates the persistence of the crude oil shocks.

We first estimate the crude oil pricing model and calibrate the parameters for the country technologies and utility functions. Then we discuss the effect of the natural exports and its correlation with crude oil prices in consumption and the hedging strategy. Finally, we look at the particular effect of risk-aversion of the country and mean-reversion and volatility of crude oil prices in the optimal controls.

### 4.1 Crude Oil Estimates and Developing Country Parameters

We estimate a single factor model for the crude oil spot (log) price  $u(t)$ . The model is equivalent to the one in equations (4) and (5) with the extra restrictions that  $\varsigma = \psi_v = 0$ . Under the historical measure, the dynamics of the futures price on a contract that matures  $\tau_1$  periods from now is:

$$dF_1 = F_1 B_{u,1} \sigma_u \lambda_u dt + F_1 B_{u,1} \sigma_u dZ_u \quad (37)$$

with  $B_{u,1} = e^{-\psi_u \tau_1}$ . The convenience yield parameter  $\psi_u$  affects the volatility of the futures returns, and therefore, the futures risk premia.

The dataset consist on weekly crude oil futures prices from 1/2/1990 to 12/31/2005 from NYMEX with maturities of 1, 3, 6, 9, 12, 15, 18, 24, 30 and 36 months. We use maximum-likelihood estimation using both time-series and cross-sectional data. As in Chen and Scott (1993) and Pearson and Sun (1994), we assume that some of the data is observed with no error. In particular, we follow Collin-Dufresne, Goldstein, and Jones (2008) and CCD and choose to perfectly fit the first principal component of the futures curve. Since the principal component remains affine in the state variable, it can easily be inverted to obtain the state variable  $u(t)$ . The remaining principal components of futures prices are then over-identified and observed with “measurement errors,” which we assume follow AR(1) processes with the same autocorrelation degree.

Table 1 presents four groups of maximum-likelihood estimates for the single factor model. We set the interest rate to  $r = 0.03$  without loosing any generality, because we estimate the convenience yield parameter  $\psi_0$  (see footnote 8). The first group (Set 0) has the unconstrained parameter estimates. We obtain interesting results that are consistent with the findings of CCD.

First, we detect that the convenience yield parameters and the volatility of crude oil returns are all significantly different from zero. Moreover, the mean-reverting parameter that plays a particular role in the consumption and hedging strategies, is positive and highly significant ( $\psi_u = 0.076$ ). We also find that the risk premium parameter  $\lambda_u$  is not significant. Interestingly, CCD found that even for a richer structure, the crude oil risk-premia parameters are less significant than those for copper, silver and gold. The most important effect of  $\lambda_u$  is the existence of the myopic demand, a result that is well-known and broadly documented in the portfolio selection literature. Therefore, we decide to drop this parameter. The second group of parameters (Set 1) has the estimates assuming that  $\lambda_u = 0$ . We find that the other estimates are not affected by this assumption and that the change in the log-likelihood is not significant.<sup>24</sup> We use this parameter set for the analysis of the strategies below. The third and fourth parameter groups in Table 1 (Sets 2 & 3) have the estimates assuming a particular value for the mean-reversion parameter  $\psi_u$ . To analyze the effect of the parameter  $\psi_u$ , it is important to consider a realistic pricing model, thus the other parameters need to adjust to changes in  $\psi_u$ . For example, note that the parameter  $\psi_0$  changes radically for the different assumptions for  $\psi_u$ . This occurs, because the parameters  $\psi_0$  and  $\psi_u$  jointly regulate the slope of the futures curve. It doesn't make sense to change  $\psi_u$  while keeping  $\psi_0$  constant.

We choose Chile as the benchmark developing country, because it's economy has similar characteristics to the ones considered for our representative country. Chile's exports are mostly from the mining industry and it's the world's largest copper producer. According to the U.S. Geological Survey, Chile accounted for 35% of the world's copper production in 2006 followed by the U.S. with 8% of the global production (see USGS 2008). Chile has also more than 31% of the world's known copper reserves. In the last decades, Chile has had a stable economy absent from major governability problems, which are sometimes common in emerging economies. This means that its economic data is more related to the productivity parameters in our model, than to other political factors that are not considered in this study.

Table 2 shows the parameters that we use for the benchmark developing economy. We first calibrate the marginal productivity of capital (MPR) which in our case is  $\alpha(1-\eta)$ . We follow Caselli and Feyrer (2007) and use 9% for Chile. This paper estimates the MPR for various countries using a measure that accounts for natural capital adjustments and differences in prices of capital and consumption goods. Natural capital adjustments result in that the natural capital accounts, such as land and natural resources, are deducted from the national wealth, because only the payments to reproducible capital are relevant to estimate the MPR. For the oil share of income in Chile,  $\eta$ , we use 3%. Recent studies such as Finn (2000) and Wei (2003) use an energy share of 4%, but oil consumption accounts only for fraction of the total energy consumption of the country. Considering the MPR and the oil share of income, we calibrate a total factor productivity,  $\alpha$ , of  $\frac{9\%}{1-3\%} = 9.3\%$ . We normalize  $\omega$  to 1 for the efficiency of oil parameter in Chile. The International Energy Agency (2004) report documents that on average, oil-importing developing countries use more than twice of the oil than OECD countries to produce a unit of economic output. This means

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<sup>24</sup>The chi-squared statistic with one degree of freedom for the LR test is 2.8, while the critical value for a 5% significance level is 3.84 (i.e.  $\text{Prob}\{\chi_1^2 \geq 3.84\} = 0.05$ ).

that we should consider a higher efficiency parameter, say  $\omega = 2$ , if we want to consider a more developed country than Chile.

For the initial natural exports to capital ratio,  $x_0$ , we need an estimate of  $X(0)$  and  $K(0)$ . The total copper exports for Chile were  $X(0) = \text{US\$ } 14.9$  billions in 2005. For the initial capital stock, we find  $K(0)$  such that the output in equation (1) is the Chilean GDP in 2005. The output of the country considers the optimal demand of oil from Proposition 1 and needs an estimate for the crude oil price. Using that the GDP was US\$ 118.9 billions in 2005 and that the average crude oil WTI price that year was US\$ 56.5 per barrel, we obtain an estimate for the Chilean capital stock of US\$ 2015.5 billions. These estimates yield an initial natural exports to capital ratio of  $x_0 = 0.7\%$ . Interestingly, these figures imply that the optimal imports of crude oil,  $S(0)Q^*(0)$ , is US\$ 4.1 billions which is very close to the Chilean fuel and energy imports of US\$ 3.6 billions in 2005. To estimate the volatility of the natural exports returns,  $\sigma_x$ , and its correlation with the crude oil shocks,  $\rho_u$ , we assume that the Chilean copper production changes at a constant rate. This implies that the second moments are due only to variations in copper prices. We consider the closest maturity futures price to be a proxy for the spot price for both, copper and crude oil. Using monthly data from 1995 to 2007, we find that the annualized volatility of copper returns is 21.2% and the correlation between copper and crude oil shocks is 28.9%. The selection of the export's depreciation rate,  $\phi$ , is the most arbitrary one. We choose 5% as the annual depreciation rate, but try different values later. A positive rate captures that the natural resource is exhaustible and that the developing country diversifies its exports over time. Finally, we assume that the country's risk-aversion parameter,  $\gamma$ , is 5.0, and that its impatience parameters,  $\beta$ , is 5%. These values are standard in the literature, but given that the country's risk-aversion has a great repercussion in consumption and in the productive hedging demands, we do a sensitivity analysis with respect to it.

## 4.2 Consumption Strategy

One of the main objectives of the paper is to study the effect of crude oil in the country's consumption decision. For this reason we concentrate on the parameters related to the economy and to the oil dynamics, and their effect on the consumption-wealth ratio in equation (33). It is important to remember that the relative size of the natural exports,  $x_0$ , has a direct effect on consumption through an increase in wealth, but it has no first-order effect on the consumption-wealth ratio.

Figure 1 present the consumption strategy with respect to the technology parameters that determine the impact of the oil in the economy. The figure has three plots, each one for a different country's risk-aversion parameter. Each plot shows the consumption-wealth ratio,  $c^*$ , as a function of the oil share of input,  $\eta$ , for four different situations: one is the Merton case (i.e.  $\eta = x_0 = 0$ ) and the others represent countries with different efficiency of oil usage,  $\omega$ . The plots confirm that risk aversion has a decisive effect on consumption. For  $\gamma < 1$ , the oil share of input has a positive effect on today's consumption (upper plot), while for  $\gamma > 1$  this effect is negative

(lower plot). The intuition is that for our parameters,  $\eta$  has a negative effect on the productivity of capital.<sup>25</sup> Therefore,  $\eta$  has a negative income effect and a positive substitution effect. As always, the substitution effect dominates for  $\gamma < 1$  and the income effect prevails if  $\gamma > 1$ . The opposite happens with respect to the efficiency parameter  $\omega$ , because for higher  $\omega$  less barrels of oil are demanded for production.<sup>26</sup> The empirical evidence supports the fact that more developed economies are more efficient in the usage of oil than less developed countries. This means that if  $\gamma < 1$ , a developed country consumes a lower fraction of its wealth than a developing economy (upper plot for a fixed  $\eta$ ). The reverse occurs if  $\gamma > 1$  (lower plot for a fixed  $\eta$ ). Also, there's an overall effect of risk aversion in the level of  $c^*$  that can be observed by comparing the Merton cases across plots. Today's consumption is increasing on risk-aversion for the Merton case. This occurs because we have calibrated a total factor productivity,  $\alpha$ , that is higher than the impatience parameter,  $\beta$ . A relatively high  $\alpha$  implies a positive income effect and a negative substitution effect of this parameter, so the consumption-wealth ratio is higher for  $\gamma > 1$ . Of course, all income and substitution effects cancel out if  $\gamma = 1$ , implying that no variable changes the consumption-wealth ratio that is fixed at  $\beta = 5\%$  (middle plot).

The effect of the crude oil price and its dynamics is shown in Figure 2. We consider again different risk-aversion. Each plot shows the consumption-wealth ratio,  $c^*$ , as a function of the crude oil price,  $S$ , for 3 different sets of parameters. As we mentioned before, each set has a different assumption for the mean-reversion parameter  $\psi_u$ . The plots confirm that crude oil prices have an effect on consumption that depends on risk-aversion and on the persistence of the shocks. Crude oil is an input to the production technology, therefore it has a negative income effect and a positive substitution effect. As before, this means that for  $\gamma < 1$  today's consumption-wealth ratio increases (upper plot), and for  $\gamma > 1$ , consumption decreases (lower plot). Higher degrees of mean-reversion (i.e. lower persistence) tend to decrease these effects because shocks are short lived and the price reverts faster to its long term mean. Finally, for a country with log utility, the net effect of these variables disappear.

### 4.3 Hedging Strategy

To study the hedging strategy we use the ratio between the dollar amount invested in the futures contracts and the capital stock from equation (34). Because oil has only one risk factor, we consider that at every point in time the country takes positions in one futures contract. We assume that this contract expires 3 months from now and its position is rebalanced continuously.

Figure 3 shows the different sources of the hedging strategy as a function of the correlation between the crude oil and the natural exports shocks,  $\rho_u$ . The figure has three plots, each one

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<sup>25</sup>We can show from equation (21) that the productivity of capital is decreasing in  $\eta$ , i.e.  $\frac{\partial \mu^*}{\partial \eta} < 0$ , if the input ratio  $\frac{w}{K}$  is less than 1. For our parameters this condition is violated only for extremely low crude oil prices ( $S < 0.00279$ ).

<sup>26</sup>Again, from equation (21) we get that  $\frac{\partial \mu^*}{\partial \omega} > 0$ . The income effect w.r.t  $\omega$  is positive while its substitution effect is negative.

for a different relative size of the exports,  $x_0$ . The myopic demand is represented by  $\pi_1$  which is always zero, because the parameter Set 1 assumes that the oil price risk premium,  $\lambda_u$ , is zero. The productive hedging demand,  $\pi_3^*$ , is independent of the exports and the correlation  $\rho_u$ . This demand is positive in all plots ( $\pi_3^* = 1.4\%$ ), because the risk-aversion degree for the country is greater than 1. This value implies that the country takes long positions in the futures contracts. The interesting term here is the pure hedging or statistical hedging component,  $\pi_2^*$ . As we noticed before,  $\pi_2^*$  is decreasing in the correlation  $\rho_u$  and proportional to the relative size of the exports,  $x_0$ . In particular, this demand is more negative for positive correlations and high exports, implying a larger short position in the futures contracts (see the lower plot for higher correlations). Figure 3 is also useful to understand up to what extent the natural exports can be used to hedge the crude oil risk. For example, consider the case of Chile which corresponds the plot in the middle ( $x_0 = 0.7\%$ ). We have calibrated a value of 28.9% for  $\rho_u$ , which means that the net hedging position,  $\pi^*$ , is almost zero. The short positions due to this positive correlation offset the long positions in the contract from the productive hedging demand. Therefore, for this case, the exports are indeed a natural hedge against crude oil shocks.

Figure 4 shows the hedging strategy as a function of the oil share of input,  $\eta$ , for different parameter sets (Sets 1, 2 & 3 from Table 1). This figure is good to analyze the productive hedging demands, because this is the only demand that varies with  $\eta$ .<sup>27</sup> Each one of the three plots in the figure is for a different risk-aversion degree. The upper plot shows that the hedging strategy is decreasing in  $\eta$  if  $\gamma < 1$ . In this case, the productive hedging demand is negative. If  $\gamma < 1$  a negative shock to crude oil prices decreases today's consumption (see figure 2). This is specially true if oil is more important for the economy (higher  $\eta$ 's). This higher sensitivity forces the country to take a larger short position in the futures contract. The reverse happens for  $\gamma > 1$  (lower plot). In both cases, the effect of  $\eta$  decreases for a higher mean-reversion degree, i.e. the slope of the curves becomes flatter. This occurs because the effect of the crude oil shock is less persistent, so fewer contracts are needed to hedge against this scenario. Finally, for the log case (middle plot), the productive hedging demand is zero and the hedging strategy is independent from  $\eta$ . In this case, the demands are different for each parameter set, because the oil returns volatilities and mean-reversion estimates changes. The statistical hedging demand is decreasing in  $\sigma_u$  and increasing in  $\psi_u$  through the volatility of the futures returns. For the middle plot the volatility effect dominates implying that the set with a higher volatility has a lower hedging strategy.

In order to quantify if the size of the futures demands are significant or not, we express the strategy in terms of the imports of oil,  $SQ^*$ . We define  $\theta^*$  as the dollar amount in the futures contracts over the dollar amount of the crude oil imports, i.e.  $\theta^* \equiv I_F(SQ^*)^{-1}p^*$ . For the one-factor model, the interpretation of this variable is simple. This hedge ratio represents the portion of the oil imports being hedged. Figure 5 shows the hedge ratio against the correlation  $\rho_u$  for different levels of relative natural exports,  $x_0$ . We have already presented the effect of these variables in the hedging strategy, so here we limit the discussion to the measurement of the hedge ratio. Let's

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<sup>27</sup>Recall from equation (34) that the productive hedging demands are proportional to  $\eta$ .

consider the case when the natural exports are high ( $x_0 = 1.4\%$ ). When the correlation between oil and exports shocks is high ( $\rho_u \sim 0.5$ ), the hedging decision is to take a short position in the futures contract for approximately 30% of the imports of crude oil. If the correlation is zero, the optimal strategy is to hedge around 10% of the total oil imports. If the correlation is negative and large ( $\rho_u \sim -0.5$ ), the hedge ratio can be as large as 55% of the oil imports.

Finally, figure 6 shows the hedging strategy with respect to the depreciation rate of the exports,  $\phi$ . We do a sensitivity analysis with respect to this rate, because it was one of the few parameters that was arbitrarily chosen for the calibration. Of course, this parameter is only relevant for the case where  $x_0 > 0$ . The hedge ratio increases with  $\phi$ , because the negative statistical hedging component decreases in absolute terms with this parameter.<sup>28</sup> This occurs because for a higher  $\phi$ , the present value of the exports ( $A_2^{-1}X(t)$ ) is lower, thus it decreases its weight in the economy.

## 5 Conclusions

We study the dynamic consumption and hedging strategies of an oil-importing developing country that confronts exogenous crude oil shocks. These countries differ from more developed economies in that their technologies are more intense and less efficient in the use of energy. Also, their economies typically rely on the export of a small number of primary commodities that can potentially be correlated with the crude oil shocks.

The developing country optimally chooses consumption, the physical crude oil imports and the hedging strategy for the existing crude oil futures contracts. Less efficient countries with high degrees of risk aversion consume less than more developed countries, because inefficiencies generate a negative income effect. For countries that care less about consumption-smoothing the opposite may occur, because there exists a substitution effect that makes today's consumption relatively cheaper than tomorrow's consumption. The crude oil price has an overall negative effect for oil-importing economies, because of a lower productivity of capital. The income and substitution effects balance the effect of the crude oil in the economy. Countries with relative risk-aversion degrees greater than one, decrease their consumption if a positive price shock occurs. Shocks to other crude oil factors, such as the convenience yield, are also studied. The impact of these shocks on consumption is through their effect in the expected crude oil price. The more persistent these shocks are, the greater are their impact in the economy.

The long/short positions in the futures contracts are used for hedging purposes, but they also enhance the country's investment opportunity set. The demand for these contracts can be decomposed into the standard myopic demand, a pure hedging or statistical hedging component and the productive hedging demands. The relative size of the natural exports and their correlation with the crude oil shocks are essential for the pure hedging component. This hedging term helps us

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<sup>28</sup>The statistical hedging demand is negative, because we consider a positive  $\rho_u$  for the benchmark case.

understand up to what extent the natural exports can be used to hedge the crude oil risk. We find productive hedging demands with respect to each crude oil factor. These demands hedge against future changes in the productivity of capital rather than against changes in the investment opportunity set as in Breeden (1979). The country's risk-aversion degree, the effect of each oil risk factor in consumption and the shocks persistence drive the size and direction of these demands. Finally, we choose Chile as a benchmark economy and estimate a one-factor model for crude oil prices. We find that the country's copper exports act as a significant natural hedge against oil shocks.

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# Appendix

## A Proofs of Propositions

This appendix contains the proofs of Propositions 1 to 3.

### A.1 Proof of Proposition 1

Proposition 1 shows the optimal consumption, demand for oil and hedging strategies for the developing country. The optimal strategies are obtained from the standard first order conditions of the HJB equation in (18) with respect to each of the controls  $C$ ,  $Q$  and  $p$ .

### A.2 Proof of Proposition 2

Proposition 2 presents a closed-form approximation of the value function of the problem that faces the developing country. To obtain this result, we replace equation (22) and the optimal controls  $\{C^*, Q^*, p^*\}$  from Proposition 1 in equation (18) to get the following non-linear second-order PDE

$$\begin{aligned}
0 &= G(z, Y) \\
&= \frac{\gamma}{1-\gamma} A_1 e^{\frac{\gamma-1}{\gamma} h} (1-zh_z)^{\frac{\gamma-1}{\gamma}} + \left( \alpha(1-\eta)(\alpha\omega e^{-u})^{\frac{\eta}{1-\eta}} e^\epsilon + x_0 z \right) (1-zh_z) - \phi zh_z \\
&\quad + (\kappa_0 - \kappa_Y Y + \sigma_Y \lambda_Y)^\top h_Y + \left( 2(\gamma(1-2zh_z) - z^2((1-\gamma)h_z^2 + h_{zz})) \right)^{-1} \left( (1-zh_z)\lambda_Y \right. \\
&\quad \left. + \sigma_X \rho_Y ((1-\gamma)(1-zh_z)zh_z - (zh_z + z^2h_{zz})) + \sigma_Y^\top ((1-\gamma)(1-zh_z)h_Y - zh_{zY}) \right)^\top \\
&\quad \left( (1-zh_z)\lambda_Y + \sigma_X \rho_Y ((1-\gamma)(1-zh_z)zh_z - (zh_z + z^2h_{zz})) + \sigma_Y^\top ((1-\gamma)(1-zh_z)h_Y - zh_{zY}) \right) \\
&\quad + \frac{1}{2} \sigma_X^2 z^2 ((1-\gamma)h_z^2 + h_{zz}) + \sigma_X \rho_Y^\top \sigma_Y^\top ((1-\gamma)zh_z h_Y + zh_{zY}) \\
&\quad + \frac{(1-\gamma)h_Y^\top \sigma_Y \sigma_Y^\top h_Y + \text{Tr}[\sigma_Y \sigma_Y^\top h_{YY}]}{2} - \frac{\beta}{1-\gamma}
\end{aligned} \tag{A1}$$

As far as we know, there is no closed-form solution for this equation. However, a solution exists for the case where  $\Theta = 0$  with  $\Theta^\top = (\eta, \epsilon, x_0)$ . In this case, the problem converges to the standard Merton problem and  $h(z, Y; 0) = 0$ . Therefore, we pursue an asymptotic expansion around this solution to obtain an approximated closed-form expression for  $h(z, Y)$ . First, we assume that  $h(z, Y)$  is linear on  $\Theta$  (see equation (25)). We replace this guess in equation (A1) and do a Taylor expansion of  $G(z, Y; \Theta)$  around  $\Theta = 0$ . Considering only the first-order terms we get

$$0 = G(z, Y; 0) + \Theta^\top \nabla G_\Theta(z, Y; 0) \tag{A2}$$

where  $\nabla G_\Theta(\cdot)$  is the gradient vector of  $G(\cdot)$  with respect to  $\Theta$ . We need to find the functions  $h^{(\eta)}(z, Y)$ ,  $h^{(\epsilon)}(z, Y)$  and  $h^{(x_0)}(z, Y)$  such that equation (A2) is satisfied. The existence of a solution when  $\Theta = 0$  implies that  $G(z, Y; 0) = 0$ . Also, since (A2) is valid for any (small)  $\eta$ ,  $\epsilon$  and  $x_0$ , the problem reduces to finding the  $h^{(j)}(z, Y)$  such that

$$\nabla G_\Theta(z, Y; 0) = 0 \tag{A3}$$

We guess an affine structure for the  $h^{(j)}(z, Y)$  functions,

$$h^{(\eta)}(z, Y) = M_0^{(\eta)} - M_z^{(\eta)} z - M_Y^{(\eta)\top} Y \tag{A4}$$

$$h^{(\epsilon)}(z, Y) = M_0^{(\epsilon)} - M_z^{(\epsilon)} z - M_Y^{(\epsilon)\top} Y \tag{A5}$$

$$h^{(x_0)}(z, Y) = M_0^{(x_0)} - M_z^{(x_0)} z - M_Y^{(x_0)\top} Y \tag{A6}$$

After replacing these equations in the expansion (A2) we obtain that,

$$\nabla G_{\Theta}(z, Y; 0) = - \left( \alpha L + M_0 A_1 + M_Y{}^\top \left( \kappa_0 + \frac{\sigma_Y \lambda_Y}{\gamma} \right) \right) + (\iota_3 + A_2 M_z) z - \left( \alpha I_{11} - M_Y{}^\top (A_1 I_n + \kappa_Y) \right) Y \quad (\text{A7})$$

where  $A_2 = \alpha + \phi + \sigma_X \rho_Y{}^\top \lambda_Y$ ,  $M_0{}^\top = (M_0^{(\eta)}, M_0^{(\epsilon)}, M_0^{(x_0)})$ ,  $M_z{}^\top = (M_z^{(\eta)}, M_z^{(\epsilon)}, M_z^{(x_0)})$ ,  $M_Y{}^\top = (M_Y^{(\eta)}, M_Y^{(\epsilon)}, M_Y^{(x_0)})$ ,  $L{}^\top = (1 - \log(\alpha\omega), -1, 0)$ ,  $\iota_1{}^\top = (1, 0, 0)$ ,  $\iota_3{}^\top = (0, 0, 1)$ ,  $I_{11} = \iota_1 \iota_1{}^\top$  and  $I_n$  is the identity matrix with rank  $n$ . Since equation (A3) has to be valid for any  $z$  and  $Y$ , the following equations must hold:

$$\alpha L + M_0 A_1 + M_Y \left( \kappa_0 + \frac{\sigma_Y \lambda_Y}{\gamma} \right) = 0 \quad (\text{A8})$$

$$\iota_3 + A_2 M_z = 0 \quad (\text{A9})$$

$$\alpha I_{11} - M_Y (A_1 I_n + \kappa_Y) = 0 \quad (\text{A10})$$

These equations imply that

$$M_0 = -A_1^{-1} \left( \alpha L + M_Y \left( \kappa_0 + \frac{\sigma_Y \lambda_Y}{\gamma} \right) \right) \quad (\text{A11})$$

$$M_z = -A_2^{-1} \iota_3 \quad (\text{A12})$$

$$M_Y = \alpha I_{11} (A_1 I_n + \kappa_Y)^{-1} \quad (\text{A13})$$

or

$$\begin{aligned} M_0^{(\eta)} &= -A_1^{-1} \left( \alpha(1 - \log(\alpha\omega)) + M_Y^{(\eta)\top} \left( \kappa_0 + \frac{\sigma_Y \lambda_Y}{\gamma} \right) \right), & M_0^{(\epsilon)} &= A_1^{-1} \alpha, & M_0^{(x_0)} &= 0 \\ M_z^{(\eta)} &= 0, & M_z^{(\epsilon)} &= 0, & M_z^{(x_0)} &= -A_2^{-1} \\ M_Y^{(\eta)} &= \alpha (A_1 I_n + \kappa_Y)^{-1\top} \iota_1, & M_Y^{(\epsilon)} &= 0, & M_Y^{(x_0)} &= 0 \end{aligned} \quad (\text{A14})$$

To obtain the results in Proposition 2 we replace the  $M$ 's from equation (A14) in our original guess (A4)-(A6).

### A.3 Proof of Proposition 3

Proposition 3 presents approximated closed-form expressions for the optimal strategies of the developing country. First, we define the consumption-wealth ratio as

$$c^* \equiv W^{-1} C^* \quad (\text{A15})$$

$$= \left( K + \frac{J_X}{J_K} X \right)^{-1} J_K^{-1/\gamma} \quad (\text{A16})$$

$$= A_1 e^{\frac{\gamma-1}{\gamma} h} (1 - zh_z)^{\frac{\gamma-1}{\gamma}} \quad (\text{A17})$$

and the ratio of the dollar amount invested in the futures contracts to the capital stock as

$$\pi^* \equiv I_F K^{-1} p^* \quad (\text{A18})$$

$$= K^{-1} (\sigma_Y \sigma_F^{-1})^\top \left( \sigma_Y^{-1} \lambda_Y \frac{-J_K}{J_{KK}} + \sigma_X \sigma_Y^{-1} \rho_Y \frac{-X J_{KX}}{J_{KK}} + \frac{-J_{KY}}{J_{KK}} \right) \quad (\text{A19})$$

$$\begin{aligned} &= (\sigma_Y \sigma_F^{-1})^\top \left( \sigma_Y^{-1} \lambda_Y \frac{1 - zh_z}{1 - 2zh_z - (1 - \gamma)(1 - zh_z)^2 - z^2 h_{zz}} \right. \\ &\quad \left. - \sigma_X \sigma_Y^{-1} \rho_Y z \frac{\gamma h_z + z(1 - \gamma)h_z^2 + zh_{zz}}{1 - 2zh_z - (1 - \gamma)(1 - zh_z)^2 - z^2 h_{zz}} \right) \end{aligned}$$

$$+ \frac{(1 - \gamma)h_{\gamma}(1 - zh_z) - zh_{z\gamma}}{1 - 2zh_z - (1 - \gamma)(1 - zh_z)^2 - z^2h_{zz}} \Big) \tag{A20}$$

If we use Proposition 2 and replace equations (22) and (25) in the optimal controls in (A17) and (A20), we obtain complex expressions that are difficult to interpret. These approximated expressions are non-linear on  $\eta$ ,  $\epsilon$  and  $x_0$ , however, we can obtain more tractable solutions at no cost. Following Kogan and Uppal (2003), we present the approximations in a asymptotically equivalent representation by applying a new Taylor expansion to the approximated optimal controls. Note that we do not need any extra assumption, because we are already considering that  $\eta$  and  $x_0$  are *small*. The new expansions are asymptotically equivalent to the original ones in the sense that both converge to Merton's solutions in the limit.

Table 1: Maximum-Likelihood Estimates for the One-Factor Crude Oil Pricing Model

Maximum-likelihood estimates for the one-factor model for crude oil weekly prices from 1990 to 2005. Sets 1, 2 and 3 restrict the risk-premium parameter to zero, i.e.,  $\lambda_u = 0$ . Sets 2 and 3 fix the crude-oil mean-reversion degree to 0 and 1, respectively.

	Set 0	Set 1	Set 2	Set 3
Parameter	Crude Oil Estimate (Std. Error)	Crude Oil Estimate (Std. Error)	Crude Oil Estimate (Std. Error)	Crude Oil Estimate (Std. Error)
$r$	0.030	0.030	0.030	0.030
$\psi_0$	-0.207 (0.002)	-0.207 (0.001)	0.031 (0.000)	-2.963 (0.006)
$\psi_u$	0.076 (0.001)	0.076 (0.000)	0.000	1.000
$\lambda_u$	1.674 (0.973)	0.000	0.000	0.000
$\sigma_u$	0.247 (0.005)	0.248 (0.005)	0.245 (0.006)	0.427 (0.009)
error auto-corr.	0.809 (0.008)	0.809 (0.010)	0.840 (0.009)	0.883 (0.008)
Log-likelihood	26017.3	26015.9	25613.2	24980.8

Table 2: Developing Country Parameters

Calibrated parameters used for the benchmark country, Chile.

Parameter	Estimates
$\alpha$	9.3%
$\eta$	3.0%
$\omega$	1.0
$x_0$	0.7%
$\sigma_x$	21.2%
$\rho_u$	28.9%
$\phi$	5.0%
$\gamma$	5.0
$\beta$	5.0%

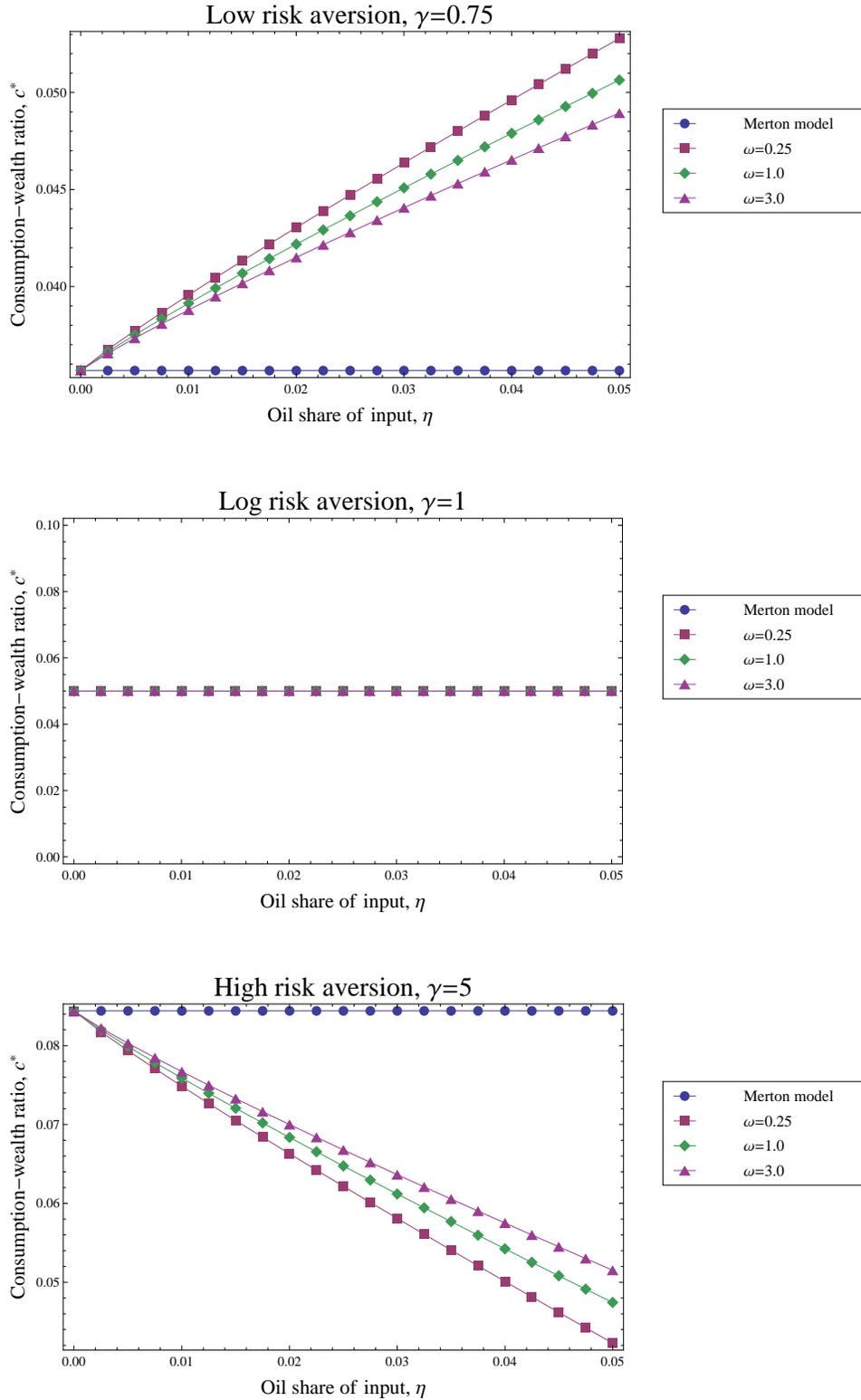


Figure 1: **Consumption-wealth ratio and technology parameters related to the crude oil.** Consumption-wealth ratio,  $c^*$ , as a function of the oil share of input,  $\eta$ , the efficiency of oil usage,  $\omega$ , and the country's risk aversion,  $\gamma$ . The top figure is for  $\gamma = 0.75$ , the plot in the middle is for  $\gamma = 1$  and the one below is for  $\gamma = 5$ . The *Merton model* lines correspond to the case where  $\eta = x_0 = 0\%$ . For the crude oil dynamics we use the parameters from Set 1 in Table 1 and for the country's technologies we use the parameters from Table 2.

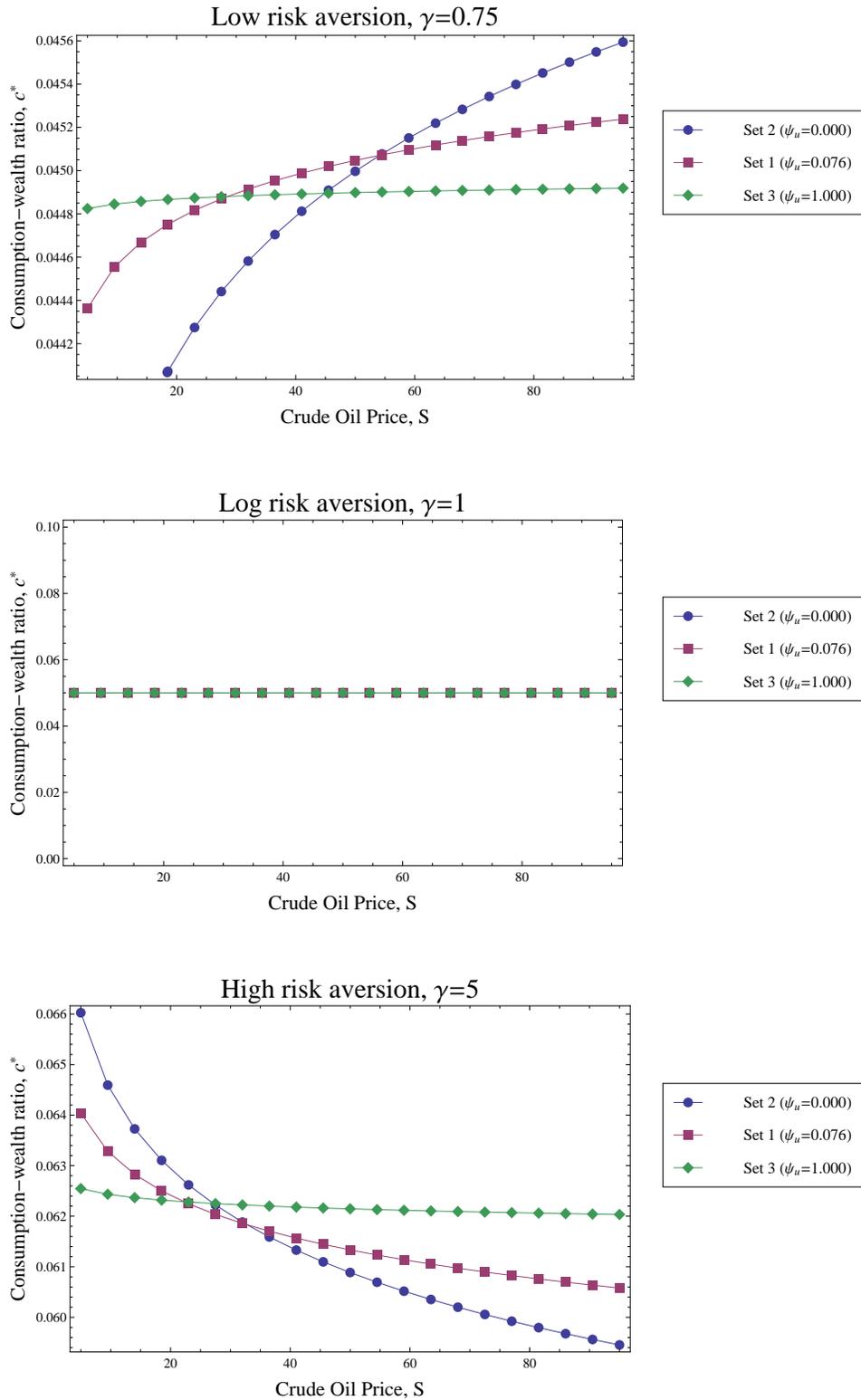


Figure 2: **Consumption-wealth ratio and crude oil prices.** Consumption-wealth ratio,  $c^*$ , as a function of the crude oil price,  $S$ , different sets of crude oil parameters and the country's risk aversion,  $\gamma$ . The top figure is for  $\gamma = 0.75$ , the plot in the middle is for  $\gamma = 1$  and the one below is for  $\gamma = 5$ . Set 1 has the default crude oil parameters. To obtain Sets 2 and 3, we fix the mean reversion degree in  $\psi_u = 0$  and  $\psi_u = 1$ , respectively, and estimate the other parameters. For the country's technologies we use the parameters from Table 2.

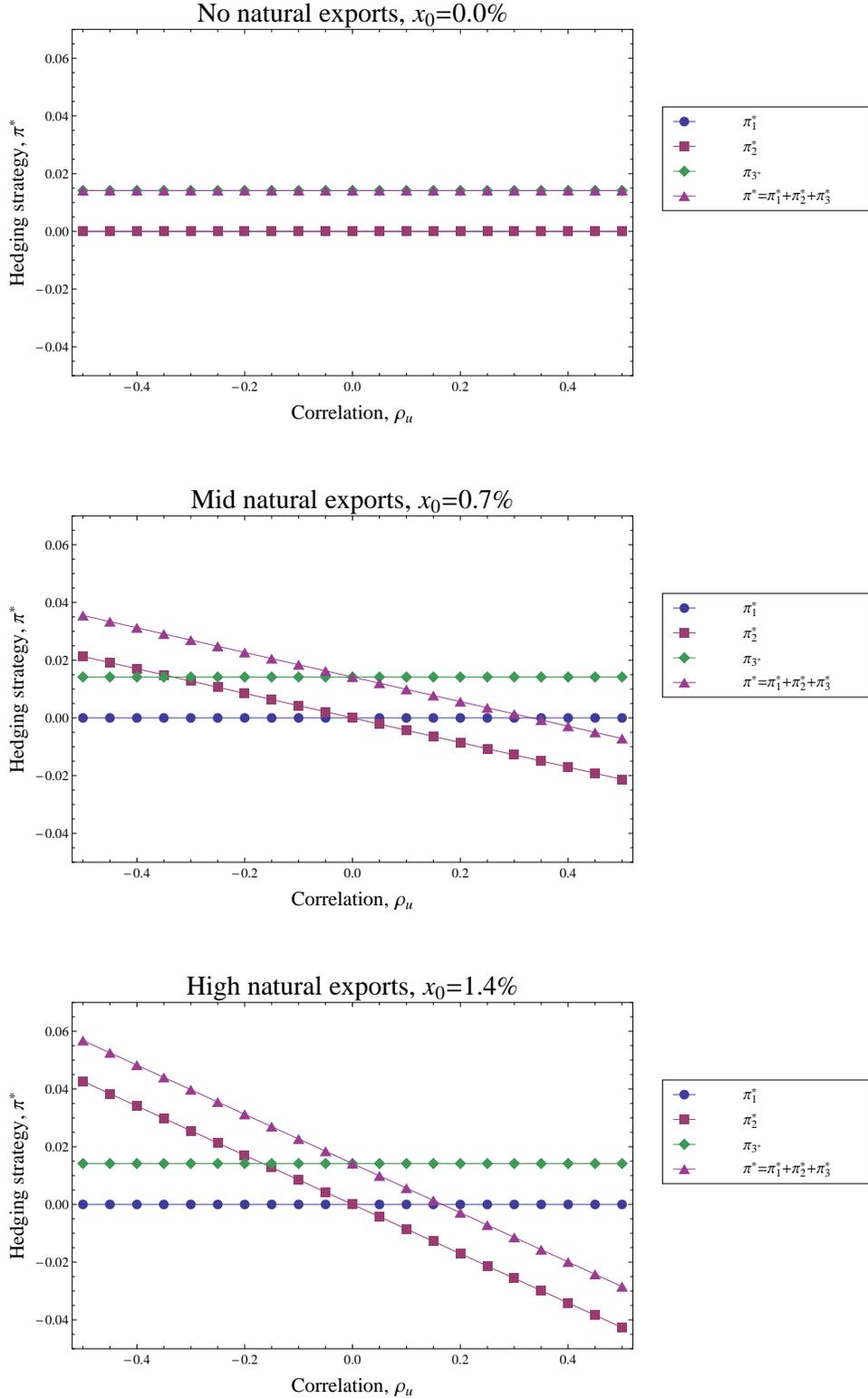


Figure 3: **Hedging strategy, correlation and exports.** Sources of the hedging strategy,  $\pi^*$ , as a function of the correlation between crude oil and natural exports shocks,  $\rho_u$ , and the relative size of the natural exports,  $x_0$ .  $\pi_1$  is the myopic demand,  $\pi_2$  is the statistical hedging component and  $\pi_3$  is the productive hedging demand. The top figure is for  $x_0 = 0.0\%$ , the plot in the middle is for  $x_0 = 0.7\%$  and the one below is for  $x_0 = 1.4\%$ . For the crude oil dynamics we use the parameters from Set 1 in Table 1 and for the country's technologies we use the parameters from Table 2.

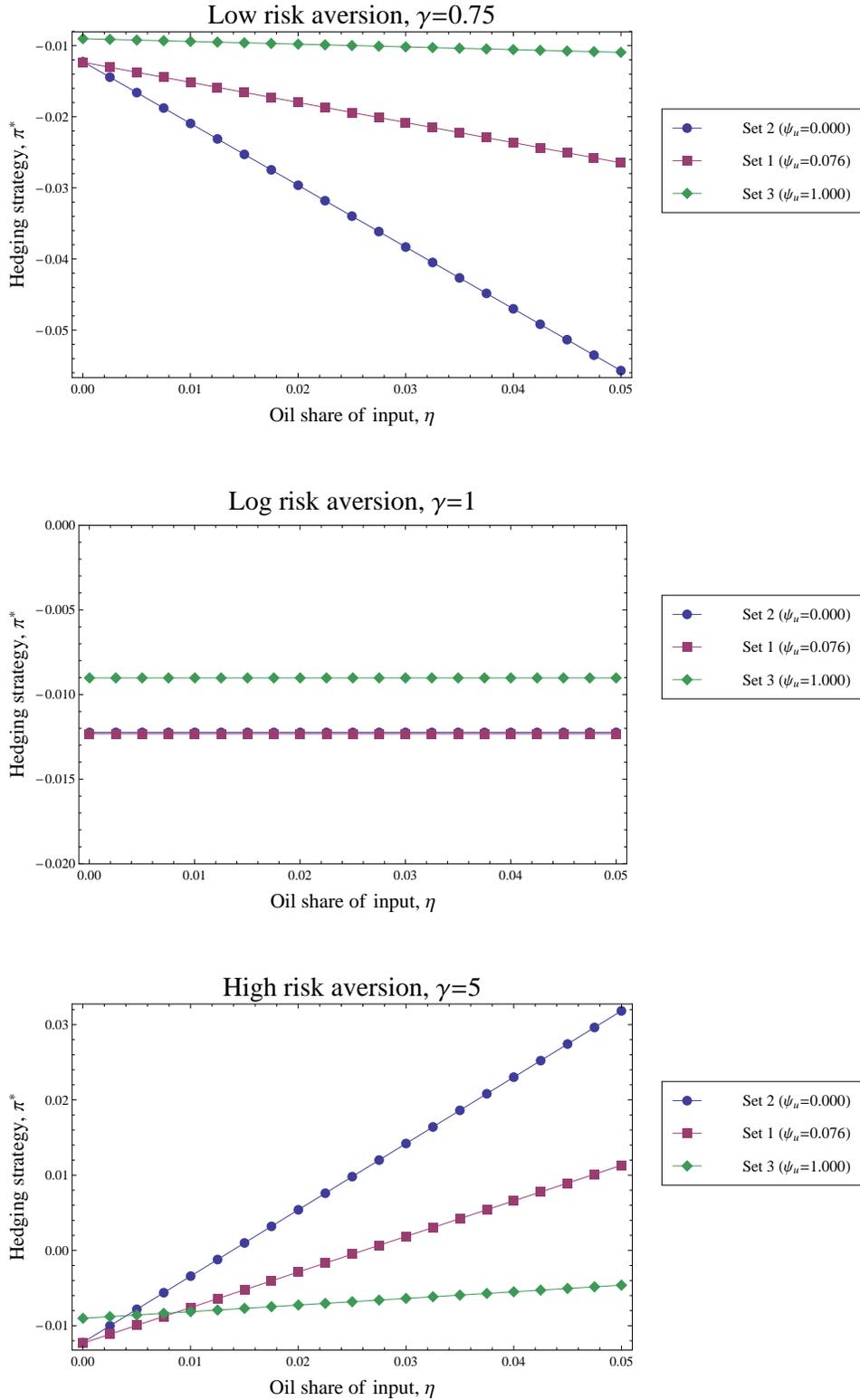


Figure 4: **Hedging strategy and crude oil.** Hedging strategy,  $\pi^*$ , as a function of the oil share of input,  $\eta$ , different sets of crude oil parameters and the country's risk aversion,  $\gamma$ . The top figure is for  $\gamma = 0.75$ , the plot in the middle is for  $\gamma = 1$  and the one below is for  $\gamma = 5$ . Set 1 has the default crude oil parameters. To obtain Sets 2 and 3, we fix the mean reversion degree in  $\psi_u = 0$  and  $\psi_u = 1$ , respectively, and estimate the other parameters. For the country's technologies we use the parameters from Table 2.

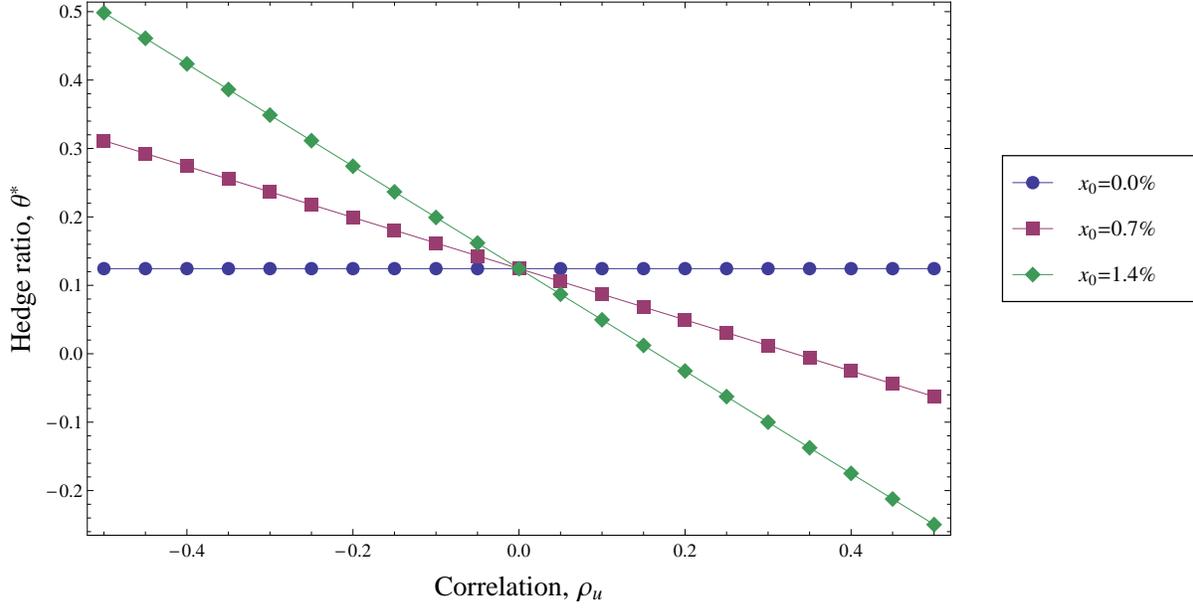


Figure 5: **Hedge ratio, correlation and exports.** Total fraction of crude oil imports being hedged,  $\theta^*$ , as a function of the correlation between crude oil and natural exports shocks,  $\rho_u$ , and the relative size of the natural exports,  $x_0$ . For the crude oil dynamics we use the parameters from Set 1 in Table 1 and for the country's technologies we use the parameters from Table 2.

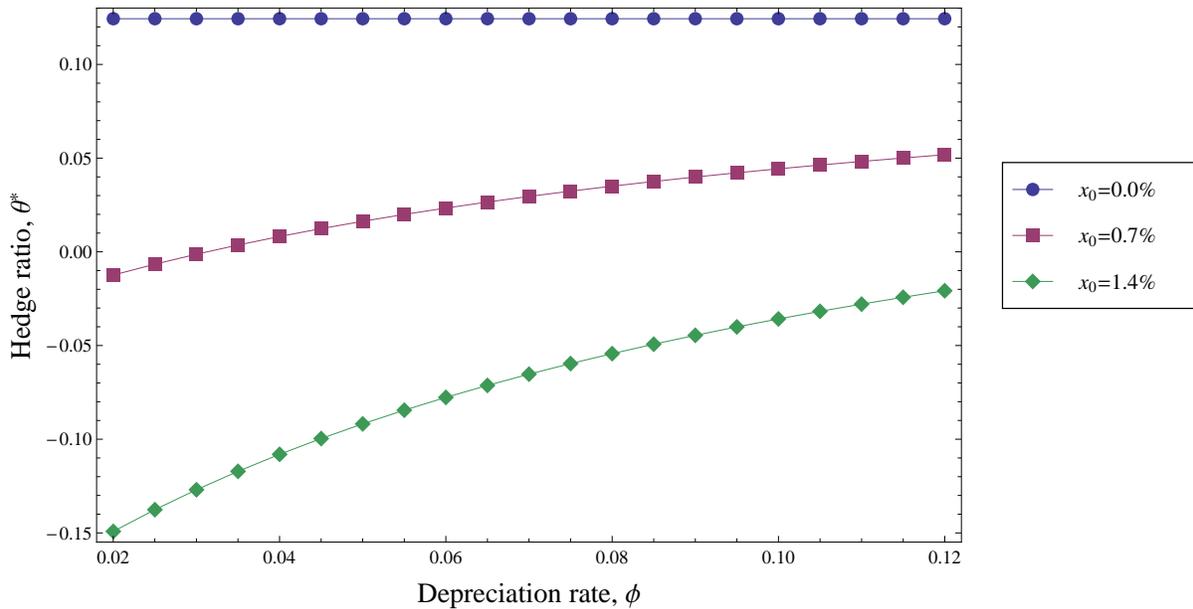


Figure 6: **Hedge ratio and depreciation.** Total fraction of crude oil imports being hedged,  $\theta^*$ , as a function of the exports depreciation rate,  $\phi$ , and the relative size of the natural exports,  $x_0$ . For the crude oil dynamics we use the parameters from Set 1 in Table 1 and for the country's technologies we use the parameters from Table 2.