An Ascending Auction with Multi-dimensional Signals*

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Abstract

This paper examines a single-unit ascending auction where agents have interdependent values and observe multi-dimensional signals. The challenge is to characterize how the multi-dimensional signals observed by an agent are aggregated onto that agent's one-dimensional bid. The challenge is solved by projecting an agent's private signals onto a one-dimensional *equilibrium statistic*; the equilibrium bidding strategies are constructed as if each agent observed only his own equilibrium statistic. An agent's equilibrium statistic aggregates this agent's private signals while taking into account the additional information deduced from the other agents' bids. The focus is on a symmetric model in which each agent observes two private signals, but the solution method extends to any (possibly asymmetric) *Gaussian* information structure.

The equilibrium characterization reveals important aspects of bidding strategies that arise only when agents observe multidimensional signals. In contrast to one-dimensional environments, an ascending auction may have multiple symmetric equilibria that yield different social surpluses—the result of a strategic complementarity when signals are aggregated onto an agent's bid. The aggregation of signals is also affected by the disclosure of a public signal; the disclosure may thereby jointly decrease the revenue and increase the social surplus. It follows that in multi-dimensional environments the linkage principle fails and public signals have an important effect on the social surplus.

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1 Introduction

Motivation. Auctions have been extensively studied in economics. It is an empirically relevant and a theoretically rich literature: auctions are commonly used to allocate goods across agents, and there is a rich class of models that allows the study of bidding in auctions. A critical assumption in most auction models is that agents observe one-dimensional signals.

In this paper, the two main objectives are: (*i*) to characterize the equilibrium of an auction in which agents observe multi-dimensional signals; and (*ii*) to analyze the differences between auctions with one-dimensional signals and auctions with multi-dimensional signals. As a by-product, we provide predictions of auctions that arise only when agents observe multi-dimensional signals.

Our paper is motivated by the observation that, in many environments, agents' information is naturally a multi-dimensional object. Consider, for example, the auction of an oil field, and suppose that an agent's valuation of the oil field is determined by its size and by the agent-specific cost of extracting oil. Furthermore, assume that each agent privately observes her own cost and all agents observe conditionally independent noisy signals about the oil field's size. This environment is therefore one in which agents observe two-dimensional signals. In most auction environments, agents similarly observe multi-dimensional signals about their valuation of the good (e.g., timber, highway construction procurements, art, real estate).[1]

There is an important conceptual difference between bidding in an auction with one-dimensional signals and in one with multi-dimensional signals. If agents observe one-dimensional signals, then observing agent i's bid is informationally equivalent to observing agent i's signal. Yet, in environments with multi-dimensional signals, observing agent i's bid is *not* informationally equivalent to observing all the signals observed by agent i. It follows that a bid reflects only an aggregate statistic of an agent's signals. In the oil field example, agent j cannot tell whether agent i's low bid is due to a high cost of extracting oil or to agent i's belief that the oil reservoir is small. The extent to which agent i's bidding is driven by his private costs or his beliefs about the oil reservoir's size is critical for agent j to determine

[1] In timber auctions, agents may differ in their harvesting cost and in their estimate of harvest quality (Haile (2001) or Athey and Levin (2001)). In highway construction procurements, the winning bidder faces idiosyncratic cost shocks and common cost shocks (Somaini (2011) or Hong and Shum (2002)). In art and real estate auctions, an agent privately observes his own taste shock and a noisy signal of an unknown common shock, where the latter can represent the good's quality or its future resale value.

her own bidding strategy.[2] After all, agent j's valuation of the oil field is independent of agent i's costs but is not independent of agent i's signal about the size of the oil reservoir. This distinction leads to a crucial difference in the equilibrium bidding.

The conceptual challenge is to account for the feedback between agents' bids: the way agent i's signals are aggregated onto his bid depends on how agent j's signals are aggregated onto her bid. So in our example of an oil field auction, if costs are positively correlated across agents then agent i's costs allow him to disentangle, at least in part, agent j's costs from her private signal about the reservoir's size. This affects i's bid, and we show that it leads him to bid less aggressively with respect to his own private cost—which in turn influences agent j's bidding strategy. This information aggregation problem has impeded the study of auctions with multi-dimensional signals. We provide an equilibrium characterization that allows to understand this problem of information aggregation.

Model. Our model consists of N agents bidding for an indivisible good in an ascending auction. The utility of an agent who wins the object is determined by a common shock and an idiosyncratic shock. Each agent privately observes his own idiosyncratic shock and, additionally, each agent observes a (noisy) signal about the common shock. The valuations are log-normally distributed; the signals are normally distributed. We focus on symmetric environments and symmetric equilibria.

The two-dimensional signals contain elements of a pure–common values environment and a pure–private values environment; our only departure from classic models in the auction literature is the information structure's multi-dimensionality.[3] This allows us to identify those elements of bidding strategies that arise only due to the multi-dimensional signals. However, the solution method extends to any—including possibly asymmetric—Gaussian information structure.

The focus on an ascending auction and on Gaussian signals is useful in fully characterizing a class of equilibria. The ascending auction is a frequently used auction format and the assumption of Gaussian signals has been used in the empirical auction literature (see, for example, Hong and Shum (2002)). Therefore, this is a natural model for the study of auctions with multi-dimensional signals.

[2]To facilitate the exposition, we shall often refer to agents i and j via (respectively) masculine and feminine pronouns.

[3]If agents observed only their idiosyncratic shock then this would be a classic pure–private values environment; if agents observed only the signal of the common shock, this would be a classic pure–common values environment.

Characterization of the Equilibrium. This paper's main result is to characterize a class of equilibria in the ascending auction. In the class of equilibria we characterize the drop-out time of an agent is determined by a linear combination of the signals he observes; we refer to this linear combination of signals as an *equilibrium statistic*. The equilibrium bidding strategies are the same as if each agent observed only his own equilibrium statistic. An agent's equilibrium statistic is a sufficient statistic — for this agent's two private signals— to compute his valuation at the end of the auction (i.e. after all drop-out times are observed); however, an agent's equilibrium statistic does not satisfy this sufficiency property before the auction ends.

The equilibrium characterization is tractable because the drop-out time of an agent is determined by a linear combination of the signals he observes: the equilibrium statistic. The linearity arises because expectations with Gaussian signals are linear. Gaussian signals are commonly used in models in which agents have linear best response.[4] However, the ascending auction is not a linear best response game. In fact, during the auction an agent's (interim) expected valuation is not Gaussian; this is because an agent can only infer a lower bound on the drop-out time of the agents that have not yet dropped out.

The ascending auction allows us to keep the Bayesian updating within the Gaussian family when we evaluate the equilibrium conditions. In the equilibria we characterize, an agent's drop-out time remains optimal even after observing the drop-out time of all other agents.[5] Consequently, we evaluate the best response conditions using the *realized* drop-out time of each agent (and not a lower bound for those times). This property of the equilibria in an ascending auction, in conjunction with the assumed Gaussian nature of signals, renders the problem tractable. For example, a first-price auction with Gaussian signals does not preserve the same tractability because in such an auction it is not possible to evaluate an agent's best-response conditions using the realized bids of other agents.

Novel Predictions. An auction's outcome is ultimately determined by the equilibrium statistic. The analysis of auctions in multi-dimensional environments is different than in one-dimensional environments because the equilibrium statistic is endogenous. As a consequence, there are key features of bidding strategies that pertain only when agents observe multi-

[4]The classic approach in linear best-response games is to conjecture (and later verify) that there is an equilibrium in which the joint distribution of actions is Gaussian.

[5]Formally, the set of equilibria we characterize constitute a posterior equilibrium. This is stronger notion of equilibrium is due to Green and Laffont (1987).

dimensional signals. These additional elements of bidding strategies deliver predictions—of ascending auction outcomes—that would not be possible if agents observed only one-dimensional signals.

In contrast to one-dimensional environments, multi-dimensional environments may feature multiple symmetric equilibria of an ascending auction.[6] The various equilibria yield distinct levels of revenue as well as a different social surplus.[7] The multiplicity of equilibria is due to a strategic complementarity in the weight that agents place—in their bidding strategy—on their own idiosyncratic shock. This complementarity arises because signals must be aggregated onto an agent's bid. In general, if the signals are multi-dimensional then there is no straightforward mapping between the distribution of signals and the social surplus.

In one-dimensional environments, public signals do not change the social surplus but they do increase revenue.[8] However, a public signal about the average idiosyncratic shock across agents overturns both of those predictions: now the public signal increases the social surplus and may also reduce the revenue generated by an auction. Note that a public signal about the average idiosyncratic shock across agents does not change an agent's expected valuation conditional only on her private information.[9] Yet, the public signal allows an agent to disentangle what proportion of other agents' bids is determined by their respective idiosyncratic shocks versus their signal about the common shock. Thus, this public signal does change an agent's beliefs about the realization of the common shock in equilibrium. This alters the equilibrium statistic. More broadly, the comparative statics will be different in one-dimensional than in multi-dimensional environments because any change in the model's primitives will change the equilibrium statistic.

Literature Review. There is an extensive literature on auctions with one-dimensional signals. Much of this research is based on the seminal contribution of Milgrom and Weber (1982), who eloquently describe the assumption as follows:

> "To represent a bidder's information by a single real-valued signal is to make two substantive assumptions. Not only must his signal be a sufficient statistic for all of the information he possesses concerning the value of the object to him, it must

[6]Bikhchandani, Haile, and Riley (2002) show that there is a continuum of symmetric equilibria. Nevertheless, both the allocation and the equilibrium price are the same across equilibria. See Krishna (2009) for a textbook discussion.

[7]By "revenue" we mean ex ante expected revenue, and by "social surplus" we mean ex ante expected surplus.

[8]The ascending auction in one-dimensional environments is efficient, so public signals cannot change the social surplus. That public signals increase revenue is known as the *linkage principle*, which we discuss in Section 5.3.

[9]This is because an agent's valuation is independent of the idiosyncratic shocks of other agents.

also adequately summarize his information concerning the signals received by the other bidders." (p. 1097)

These remarks offer a clear explanation of what is entailed by assuming one-dimensional signals. They also illustrate the difficulty of characterizing an equilibrium when agents observe multi-dimensional signals: in general, an agent's bid is not determined simply by her interim expected valuation.[10]

The literature on auctions with multi-dimensional signals has made progress in two ways. The first way is to make the appropriate assumptions about the distribution of signals such that an agent's bid is determined only by her interim expected valuation. The second way is to provide properties of an auction without having to characterize the equilibrium bids. The distinguishing features of our paper are that we do not impose any assumptions on how signals are correlated across agents and we fully characterize a class of equilibria. This allows us to study how signals are aggregated onto an agent's bid while taking into account the feedback effects between bids of different agents, which ultimately delivers new predictions about ascending auctions. We now discuss the literature on auctions with multi-dimensional signals and interdependent valuations.[11]

Wilson (1998) studies an ascending auction with two-dimensional signals. Wilson (1998) assumes that the random variables are log-normally distributed and also drawn from a diffuse prior.[12] This setup can be seen as a particular limit of our model (see note 22); in this limit, an agent's bid is a function only of her interim expected valuation. Because of its tractability (which our model shares to a great extent), the model studied by Wilson (1998) is often used in empirical work.[13]

Dasgupta and Maskin (2000) studies a generalized VCG mechanism. They show that, if agents' signals are independently distributed across agents, then an agent's interim expected valuation delivers a one-dimensional statistic that can be used to characterize the mechanism's Nash equilibria.[14] The intuition is that an agent has no information about the

[10]An agent's interim expected valuation is his expected valuation conditional only on his private signals.

[11]There is a literature that studies multi-dimensional signals in private values environments (see, for example, Fang and Morris (2006) or Jackson and Swinkels (2005)). These works are primarily based on first-price auctions, and they seek to explain how multi-dimensional signals change the "bid shading" in such auctions. Multi-dimensional signals play a different role in this literature than in our model. In fact, an ascending auction has an equilibrium in dominant strategies when agents have private values.

[12]Because the signals are drawn from a diffuse prior, they are not truly random variables and the updating does not, technically speaking, proceed via Bayes' rule. The updating instead follows a heuristic linear rule, which is akin to Bayesian updating with log-normal random variables.

[13]See Hong and Shum (2003) for further discussion on the use of normal distributions in empirical analysis.

[14]McLean and Postlewaite (2004) studies an interesting variation of the VCG mechanism for environments in which agents

signals observed by other agents and so the impact of the signals on the agent's payoff is a sufficient statistic of all the information this agent observes. Provided the signals are independently distributed, this approach can also be used in many other mechanisms—including a first-price auction and an ascending auction (see Goeree and Offerman (2003) or Levin, Peck, and Ye (2007)). We discuss the assumption of independent signals after we provide the information structure in our model (in Footnote 16).

Jackson (2009) provides an example of an ascending auction for which no equilibrium exists. The model he describes is similar to ours in that it features both a private and a common signal; however, the distribution of signals and payoff shocks in that model have finite support (so they are non-Gaussian). Jackson (2009) establishes that the existence of an equilibrium is not guaranteed in an auction model with multi-dimensional signals. The extent to which it is possible to construct equilibria with multi-dimensional non-Gaussian information structures is still an open question.

Pesendorfer and Swinkels (2000) study a sealed-bid uniform price auction in which there are k goods for sale, each agent has unit demand, and each agent observes two-dimensional signals. They study the limit in which the number of agents grows to infinity. Pesendorfer and Swinkels (2000) are able to provide asymptotic properties of any equilibrium without the need to characterize such an equilibrium or even prove its existence.

Finally, there is a literature that studies mechanism design with multidimensional signals. Jehiel and Moldovanu (2001) show that when agents observe multidimensional independently distributed signals there is no efficient mechanism. The impossibility result in Jehiel and Moldovanu (2001) does not apply to our model because in our model signals are not independently distributed (this is precisely what makes the construction of an equilibrium challenging); however, the Nash equilibrium in our model is still inefficient. Jehiel, Meyer-ter-Vehn, Moldovanu, and Zame (2006) show that when agents observe multidimensional signals then it is not possible to construct a mechanism that has an ex post equilibrium. The equilibria of the ascending auction when agents observe multidimensional signals is not an ex post equilibrium.

The rest of our paper proceeds as follows. Section 2 describes the model and Section 3 studies one-dimensional signals. Section 4 characterizes the equilibrium with two-

observe multi-dimensional signals that are *not* independently distributed.

dimensional signals and Section 5 examines the effect of public signals. Section 6 generalizes the methodology to allow for multi-dimensional asymmetric signals and other mechanisms. We conclude in Section 7 with a summary and some suggestions for further research. Section 8 gives the proofs that are not included in the main text.

2 Model

2.1 Payoffs and Information

We study N agents bidding for an indivisible good in an ascending auction. The utility of an agent $n \in N$ who wins the object at price p is given by

$$u(i_n, c, p) \triangleq \exp(i_n) \cdot \exp(c) - p, \tag{1}$$

where $\exp(\cdot)$ denotes the exponential function, $i_n \in \mathbb{R}$ is an idiosyncratic shock, and c is a common shock. The utility of an agent who does *not* win the good is zero. To make the notation more compact, we define

$$v_n \triangleq i_n + c. \tag{2}$$

The payoff shock v_n summarizes the valuation of agent n; note that $\exp(i_n) \cdot \exp(c) = \exp(v_n)$.

The idiosyncratic shocks and the common shock are jointly normally distributed with mean 0 and with respective variance σ_i^2 and σ_c^2. Assuming that the idiosyncratic and common shock have zero mean reduces the notation but plays no role in the analysis. The idiosyncratic shocks have correlation $\rho_i \in (-1/(N-1), 1)$ across agents and are distributed independently of the common shock.[15]

Agent n observes two signals, of which the first is a perfectly informative signal about her own idiosyncratic shock i_n. The second is a noisy signal about the common shock:

$$s_n \triangleq c + \varepsilon_n; \tag{3}$$

here ε_n is a noise term that is independent across agents, independent of all other random variables in the model, and normally distributed with variance σ_ε^2. The private information

[15]The minimum statistically feasible correlation is $-1/(N-1)$. Hence we impose no restrictions on the set of feasible correlations beyond the requirement that it is an interior correlation.

of agent n is summarized by the pair of random variables (i_n, s_n).[16] If every agent n observed only signal i_n, then this would be a pure–private values model; if each agent n observed only signal s_n, it would be a pure–common values model.

For the oil field example, $\exp(c)$ can be interpreted as the oil field's size and $\exp(i_n)$ as the technology of firm n. The total amount of oil that can be extracted from an oil reserve of size $\exp(c)$ by firm n using technology $\exp(i_n)$ is equal to $\exp(i_n) \cdot \exp(c)$. Li, Perrigne, and Vuong (2000) use log-additive payoffs (as in (1)) to study Outer Continental Shelf wildcat auctions.

If we multiply the utility function by -1 then the model can be interpreted as the "procurement" of a project, where $\exp(i_n) \cdot \exp(c)$ is the cost of delivering that project. Here $\exp(i_n)$ can be interpreted as the total amount of inputs that bidder n needs to complete the project and $\exp(c)$ as a price index of those inputs. Hong and Shum (2002) use log-additive payoffs (as in (1)) to study procurement auctions held by the state of New Jersey.

2.2 Ascending Auction

We study an ascending auction.[17] An auctioneer raises the price continuously. At each moment in time, an agent can drop out of the auction; in that event, the agent neither pays anything nor obtains the good. The object is won by the last agent to drop out of the auction,[18] who pays the price at which the second-to-last agent dropped out of the auction.[19] Because each drop-out time is associated with a unique price, we shall use the terms "price" and "drop-out time" interchangeably.

The outcome of the ascending auction is described by the order in which each agent drops out and the respective prices at which they do so. The number of agents left in the auction when agent n drops out is denoted by a permutation π.[20] For example, the identity of the last agent to drop out of the auction is given by $\pi^{-1}(1)$. The price at which agents drop out

[16] A model of independently distributed signals (as in Dasgupta and Maskin (2000) or Goeree and Offerman (2003)) is recovered by assuming that $\rho_i = 0$ and assuming the noise terms $\{\varepsilon_n\}_{n \in N}$ are *negatively* correlated across agents. In particular, the variance of the noise terms must be large enough and sufficiently negatively correlated so that $cov(s_n, s_m) = \sigma_v^2 + \sigma_\varepsilon^2 \cdot corr(\varepsilon_m, \varepsilon_m) = 0$.

[17] We follow Krishna (2009) in the formal description of the ascending auction.

[18] We assume that the auction continues until all agents have dropped out. The price at which the last agent drops out is clearly payoff irrelevant because he pays only the price at which the second-to-last agent dropped out. This fact allow us to simplify the notation in some parts of the paper because there is always one drop-out time for each agent.

[19] In case of a tie, the good is sold with equal probability to one of the agents who was the last to drop out. In equilibrium, there will be no ties and an agent cannot tie with another agent by unilaterally changing his strategy. Hence, as is standard in an ascending auction, the tie-breaking rule does not matter.

[20] A permutation is a bijective function $\pi : N \to N$.

is denoted $p_1 > \cdots > p_N$. So for any strategy profile, the expected utility of agent n is

$$\mathbb{E}[\mathbb{1}\{\pi^{-1}(1) = n\}(\exp(i_n) \cdot \exp(c) - p_2)],$$

where $\mathbb{1}\{\cdot\}$ is the indicator function. We study the auction's symmetric Nash equilibria.

A symmetric strategy of agent n is a set of functions $\{P_n^k\}_{k \in N}$ with

$$P_n^k : \mathbb{R}^2 \times \mathbb{R}^{N-k} \to \mathbb{R}_+. \tag{4}$$

The function $P_n^k(i, s_n, p_{k+1}, ..., p_N)$ is the drop-out time of agent n when k agents are left in the auction and the observed drop-out times are $p_N < \cdots < p_{k+1}$. The function $P_n^k(i, s_n, p_{k+1}, ..., p_N)$ must satisfy the following inequality:

$$P_n^k(i, s_n, p_{k+1}, ..., p_N) \geq p_{k+1}.$$

That is, agent n cannot drop out of the auction at a price *lower* than the price at which another agent has already dropped out. Note that we restrict attention to symmetric equilibria in symmetric environments. It is therefore sufficient to specify the price at which an agent dropped of the auction yet the agent's identity is irrelevant (see Section 6 for a generalization).

3 Benchmark: One-dimensional Signals

We first study one-dimensional signals; the analysis of this case will be helpful for understanding the analysis of two-dimensional environments. All results in this section are direct corollaries or simple extensions of results that are already well known in the literature.

3.1 Information Structure

We assume agent n observes a one-dimensional signal

$$s_n' = i_n + b \cdot (c + \varepsilon_n), \tag{5}$$

where $b \in \mathbb{R}_+$ is an exogenous parameter. In other words, agent n observes only a linear combination of the two-dimensional signal (i_n, s_n). The one-dimensional signal (5) provides a parameterized class of information structures that include pure private values and pure common values as limits. If $b = 0$, then the model is that of a pure–private values auction; if $b \to \infty$, then we are modeling a pure–common values auction.

The specific form of the signal in (5) illuminates the connections to the model in which an agent observes both signals separately. This class of one-dimensional signals is, in essence, a special case of the model studied by Milgrom and Weber (1982).[21] Although we believe that (5) specifies a natural class of one-dimensional information structures, we are not aware of any other paper that examines this class of signals *except* for the case $b = 1$.[22]

3.2 Characterization of Equilibrium with One-dimensional Signals

To characterize the equilibrium we relabel agents such that the realization of signals satisfy

$$s'_1 > \cdots > s'_N.$$

Since signals are noisy, it follows that the order over valuations may not be preserved. For example, it may be that $v_{n+1} > v_n$ even though, by construction, $s'_{n+1} \leq s'_n$.

If the signals $(s'_1, ..., s'_{n-1})$ are equal to s'_n (i.e., all signals higher than s'_n are equal to s'_n), then we write the expectation of v_n as

$$\mathbb{E}[v_n \mid s'_n, ..., s'_n, s'_{n+1}, ..., s'_N]. \tag{6}$$

For example, if $N = 3$ then $\mathbb{E}[v_2 \mid s'_2, s'_2, s'_3]$ denotes the expected valuation of the agent with the second-highest signal—conditional on (*i*) the realization of her own signal, (*ii*) the realization of agent 3's signal, and (*iii*) the realization of agent 1's signal being equal to s'_2.

[21]More precisely: if $b \leq 1$ then this environment is a particular case of the model studied by Milgrom and Weber (1982), but if $b > 1$ then this environment may fail to satisfy all the assumptions in that paper; even so, their analysis goes through without important changes. For example, an information structure with $b > 1$ and $\sigma_\varepsilon^2 = 0$ would not satisfy the monotonicity assumption in Milgrom and Weber (1982) because, in that case, agent n's utility would be decreasing in the realization of agent m's signal. Yet, the failure of this monotonicity condition is sufficiently "mild" that the analysis in Milgrom and Weber (1982) goes through unchanged.

[22]Hong and Shum (2002) study a model where the payoff environment is as in (1) and where agents observe one-dimensional signals—as in (5) with $b = 1$ (see also Hong and Shum (2003)). In Wilson (1998), agents observe two-dimensional signals as in our model. In that paper, however, the shocks are drawn from a diffuse prior, which corresponds to taking the limits $\sigma_c^2 \to \infty$, $\sigma_i^2 \to \infty$, and $\rho_i \to 1$ at a particular rate. For this reason, his model reduces to a one-dimensional signal as in (5) with $b = 1$ (for a discussion, see Hong and Shum (2002)).

Proposition 1 (Equilibrium of Ascending Auction)**.**

The ascending auction with one-dimensional signals as in (5) *has a Nash equilibrium in which agent n's drop-out time is given by*

$$p_n = \mathbb{E}[\exp(v_n) \mid s'_n, ..., s'_n, s'_{n+1} ..., s'_N]. \tag{7}$$

In equilibrium, agent 1 wins the good and pays $p_2 = \mathbb{E}[\exp(v_2) \mid s'_2, s'_2, s'_3, ..., s'_N]$.

Proposition 1 provides the classic equilibrium characterization found in Milgrom and Weber (1982), which is essentially the unique symmetric equilibrium (see note 6). In equilibrium, the agent with the nth-highest signal drops out of the auction at her expected valuation conditional on the signals observed by agents who have already dropped out (i.e., agents $m > n$) and assuming that the $n-1$ signals that are higher than s'_n are all equal to s'_n.

The equilibrium strategies (see (7)) satisfy the following two conditions: (i) agent 1 does not regret winning the good at price p_2, and (ii) no agent $m > 1$ regrets dropping out of the auction instead of waiting until agent 1 drops out. These conditions are formally expressed as follows:

$$\mathbb{E}[\exp(v_1) \mid s'_1, ..., s'_N] - \mathbb{E}[\exp(v_2) \mid s'_2, s'_2, ..., s'_N] \geq 0; \tag{8}$$

$$\forall m > 1, \quad \mathbb{E}[\exp(v_m) \mid s'_1, ..., s'_N] - \mathbb{E}[\exp(v_1) \mid s'_1, s'_1, s'_2, ..., s'_{m-1}, s'_{m+1}, ..., s'_N] \leq 0. \tag{9}$$

Condition (8) states that the expected valuation of agent 1 conditional on all the signals is greater than the price at which agent 2 drops out of the auction. Therefore, agent 1 does not regret winning the good. Condition (9) states that the expected valuation of agent m conditional on all the signals is less than the price at which agent 1 would drop out of the auction if agent m waited until agent 1 dropped out.[23] Hence agent $m > 1$ does not regret dropping out of the auction even if she observed the realization of all the signals. These considerations lead to an important property: the strategy profile (see (7)) remains a Nash equilibrium even if every agent observes the signals of all other agents.[24]

We now show that the social surplus generated by the auction is decreasing in the weight b.

[23] If agent $m > 1$ waited until all other agents dropped out of the auction, then she would win the good at price $\tilde{p}_2 = \mathbb{E}[\exp(v_1) \mid s'_1, s'_1, s'_2, ..., s'{}_{m-1}, s'{}_{m+1}, ..., s'_N]$. This is the expected valuation of agent 1 conditional on the signals of all agents other than agent m and assuming that agent m observed a signal equal to that observed by agent 1.

[24] Formally, this is an ex post equilibrium; see Section 6.2 for a discussion.

Proposition 2 (Comparative Statics: Social Surplus)**.**
The social surplus $E[\exp(v_1)]$ *is decreasing in* b*.*

Proposition 2 is an intuitive result. As b becomes larger, the correlation between the drop-out time of agent n and the noise term ε_n increase. This relation leads to inefficiencies that reduce the social surplus. If $b \to 0$, then an agent's drop-out time is perfectly correlated with his idiosyncratic shock; hence the auction is efficient. If $b \to \infty$, then an agent's drop-out time is perfectly correlated with the noise term. In this case, the identity of the auction's winner is independent of the realization of idiosyncratic shocks.

4 Characterization of Equilibria

We now characterize a class of equilibria when agents observe two-dimensional signals (i_n, s_n). The first step of the equilibrium characterization is to project the signals onto a one-dimensional object that we call an equilibrium statistic. Then we show that there exists a class of equilibria where each agent behaves as if she observes only her own equilibrium statistic. After characterizing the equilibrium, we offer an intuitive explanation of how that equilibrium statistic is determined. As an illustration of the subtle mapping between the information structure and the auction's outcome, we show that an ascending auction may have multiple symmetric equilibria that generate different social surpluses.

4.1 Equilibrium Statistic

The fundamental object that enables us to characterize an equilibrium is the *equilibrium statistic*, which amounts to the projection of signals that determine agents' respective drop-out times.

Definition 1 (Equilibrium Statistic)**.**
The random variables $\{t_n\}_{n \in N}$ *constitute an equilibrium statistic if there exists a* $\beta \in \mathbb{R}$ *such that, for all* $n \in N$:

$$t_n = i_n + \beta \cdot s_n; \tag{10}$$

$$\mathbb{E}[v_n \mid i_n, s_n, t_1, ..., t_N] = \mathbb{E}[v_n \mid t_1, ..., t_N]. \tag{11}$$

An equilibrium statistic is a linear combination of signals that satisfy the statistical condition (11). The expected value of v_n conditional on all equilibrium statistics $\{t_n\}_{n \in N}$ is equal to the expected value of v_n conditional on (i) all equilibrium statistics $\{t_n\}_{n \in N}$ and (ii) both signals (i_n, s_n). In other words, if agent n knows the equilibrium statistic of other agents, then her equilibrium statistic is a sufficient statistic—for both signals observed by agent n—to compute the expectation of v_n. Note that the weight β is the same for all agents. The reason is that we focus on symmetric equilibria, and hence, all agents use the same weight. Throughout the paper, t_n will denote an equilibrium statistic.

Our next proposition characterizes the set of equilibrium statistics.

Proposition 3 (Equilibrium Statistic)**.**
A linear combination of signals $t_n = i_n + \beta \cdot s_n$ *is an equilibrium statistic if and only if* β *is a root of the cubic polynomial* $x_3 \cdot \beta^3 + x_2 \cdot \beta^2 + x_1 \cdot \beta + x_0$*, where:*

$$x_3 = \frac{1}{(1-\rho_i)(1+(N-1)\rho_i)} \frac{(\sigma_\varepsilon^2 + N \cdot \sigma_c^2)}{\sigma_i^2 \sigma_c^2} \; ; \; x_2 = \frac{-1}{(1-\rho_i)\sigma_i^2} \; ; \; x_1 = \frac{\sigma_\varepsilon^2 + \sigma_c^2}{\sigma_\varepsilon^2 \sigma_c^2} \; ; \; x_0 = \frac{-1}{\sigma_\varepsilon^2}. \tag{12}$$

Moreover, all real roots of the polynomial are between 0 *and* 1*.*

According to this proposition, the set of equilibrium statistics is determined by a cubic equation that always has at least one real root. We first provide the equilibrium characterization of the ascending auction and then describe intuitively how the equilibrium statistic is determined by the information structure.

4.2 Equilibrium Characterization

We show that, for every equilibrium statistic, there exists a Nash equilibrium in which each agent n behaves as if she observed *only* her equilibrium statistic t_n. The characterization of the equilibrium strategies is analogous to the treatment in Section 3, but now we use the equilibrium statistic (instead of (5)). One must bear in mind that the equilibrium statistic is simply an auxiliary element that helps characterize a class of equilibria, but it is not (in general) a sufficient statistic for an agent's private signals.

Much as in the previous analysis of one-dimensional signals, for the two-dimensional case we assume that agents are ordered as follows:

$$t_1 > \cdots > t_N. \tag{13}$$

If there are multiple equilibrium statistics, then there will be a Nash equilibrium for each one. Different equilibrium statistics induce a different order (as in (13)), so the Nash equilibrium is described in terms of the order induced by each equilibrium statistic.

Theorem 1 (Symmetric Equilibrium with Multi-dimensional Signals)**.**
For every equilibrium statistic, there exists a Nash equilibrium in which agent n's drop-out time is given by

$$p_n = \mathbb{E}[\exp(v_n) \mid t_n, ..., t_n, t_{n+1}, ..., t_N]. \tag{14}$$

In equilibrium, agent 1 wins the object and pays $p_2 = \mathbb{E}[\exp(v_2) \mid t_2, t_2, ..., t_N]$.

Theorem 1 establishes that there exists a class of equilibria in which agents project their signals onto a one-dimensional statistic via the equilibrium statistic $t_n = i_n + \beta \cdot s_n$. In equilibrium, every agent n behaves as if he observed *only* t_n, which is a one-dimensional object.

We prove Theorem 1 in two steps: we first provide the equilibrium conditions and then show that these conditions are satisfied. The equilibrium conditions are similar to (8) and (9): an agent's drop-out time remains optimal even after observing the realized drop-out times of all agents in the auction.[25] An important aspect of this setup is that agent n can learn the equilibrium statistic of agent m by observing the latter's drop-out time; but agent n does not learn both of agent m's signals separately. Hence the optimality condition for agent n's drop-out time accounts for (i) both of agent n's observed signals and (ii) the equilibrium statistic $\{t_m\}_{m \neq n}$ of other agents. The equilibrium statistic's properties can be used to show that the optimality conditions are satisfied. Toward that end, we reduce the equilibrium conditions in the two-dimensional environment to the corresponding equilibrium conditions that arise in a one-dimensional environment.

Proof of Theorem 1. We check the following two conditions: (i) agent 1 never regrets winning the object at price p_2 after all agents $m > 1$ drop out of the auction; and (ii) no

[25] Formally, the Nash equilibrium we characterize is also a posterior equilibrium (cf. Green and Laffont (1987)).

agent $m > 1$ regrets dropping out of the auction instead of waiting until all other agents (including agent 1) drop out. Formally, the conditions that must be satisfied are as follows:

$$\mathbb{E}[\exp(v_1) \mid i_1, s_1, t_1, ..., t_N] - \mathbb{E}[\exp(v_2) \mid t_2, t_2, ..., t_N] \geq 0; \tag{15}$$

$$\forall m > 1, \quad \mathbb{E}[\exp(v_m) \mid i_m, s_m, t_1, ..., t_N] - \mathbb{E}[\exp(v_1) \mid t_1, t_1, t_2, ..., t_{m-1}, t_{m+1}, ..., t_N] \leq 0. \tag{16}$$

Condition (15) states that the expected valuation of agent 1—conditional on the two signals she observes and on the information she infers from the drop-out time of other agents—is greater than the price at which agent 2 drops out of the auction. It follows that agent 1 does not regret winning the good. Condition (16) states that the expected valuation of agent m —conditional on the two signals he observes and on the information he infers from the drop-out time of other agents— is less than the price at which agent 1 would drop out of the auction if agent m waits until agent 1 drops out. Therefore, agent $m > 1$ does not regret dropping out of the auction before agent 1.

We now prove that (15) and (16) are satisfied. We can use (11) to write

$$\forall n, \quad \mathbb{E}[\exp(v_n) \mid i_n, s_n, t_1, ..., t_N] = \mathbb{E}[\exp(v_n) \mid t_1, ..., t_N].$$

Note that the expectations in (11) were taken *without* the exponential function. But since all random variables are (assumed to be) Gaussian, it follows that the distribution of v_n conditional on $(i_n, s_n, t_1, ..., t_N)$ is the the same as the distribution of v_n conditional on $(t_1, ..., t_N)$.[26] So if (11) is satisfied, then it is satisfied also for any function of v_n. Hence (15) and (16) are satisfied if and only if

$$\mathbb{E}[\exp(v_1) \mid t_1, ..., t_N] - \mathbb{E}[\exp(v_2) \mid t_2, t_2, ..., t_N] \geq 0; \tag{17}$$

$$\forall m > 1, \quad \mathbb{E}[\exp(v_m) \mid t_1, ..., t_N] - \mathbb{E}[\exp(v_1) \mid t_1, t_1, t_2, ..., t_{m-1}, t_{m+1}, ..., t_N] \leq 0. \tag{18}$$

We remark that checking (17) and (18) is equivalent to checking their counterparts, (8) and (9), in one-dimensional environments. In other words: since we proved (in Section 3) that (8) and (9) are satisfied, it follows—after simply replacing b with β—that (17) and (18)

[26]For any jointly distributed (x, y), we have $x|y \sim \mathcal{N}(\mathbb{E}[x|y], \sigma_x^2 - \text{var}(\mathbb{E}[x|y]))$. Because $\mathbb{E}[v_n \mid i_n, s_n, t_1, ..., t_N] = \mathbb{E}[v_n \mid t_1, ..., t_N]$, we also have that $\text{var}(\mathbb{E}[v_n \mid i_n, s_n, t_1, ..., t_N]) = \text{var}(\mathbb{E}[v_n \mid t_1, ..., t_N])$. As a result, $v_n \mid (i_n, s_n, t_1, ..., t_N) \sim v_n \mid (t_1, ..., t_N)$.

are also satisfied.■

For the class of equilibria characterized by Theorem 1, the analysis in Section 3 can be applied once we replace s'_n with t_n (or, equivalently, b with β). The key element of the characterization that determines an equilibrium's qualitative properties is β—that is, the weight placed by the equilibrium statistic on signals about the common shock. If $\beta \approx 0$, then the auction's outcome will be efficient and the outcome will resemble the one that obtains in a pure–private values environment. As β increases, social surplus is reduced and the model resembles more of an "interdependent values" environment. Note that all equilibrium statistics satisfy $\beta \leq 1$. Yet, if $\beta \approx 1$ and if the variance of the idiosyncratic shock is small enough relative to the variance of the common shock and the noise term, then the model will resemble a pure–common values model. At this point, the natural question that arises is: How does the information structure determine the equilibrium statistic?

4.3 Analysis of the Equilibrium Statistic

We now provide some intuition regarding how β is determined. Analogously to how the Nash equilibrium of any game can be understood by analyzing agents' best-response functions, we can explain how the equilibrium statistic is determined by analyzing how the expectations are determined "out of equilibrium". We fix an exogenous one-dimensional signal,[27]

$$s'_m = s_m + \frac{1}{b} i_m, \tag{19}$$

and define the terms $\gamma_i, \gamma_s, \gamma' \in \mathbb{R}$ implicitly as follows:[28]

$$\mathbb{E}[v_n \mid i_n, s_n, \{s'_m\}_{m \neq n}] = \gamma_i \cdot i_n + \gamma_s \cdot s_n + \frac{\gamma'}{N-1} \cdot \sum_{m \neq n} s'_m. \tag{20}$$

The weight b is an equilibrium statistic if and only if it satisfies

$$b = \frac{\gamma_s}{\gamma_i}.$$

[27] The signal is the same as in Section 3, but we have divided it by $1/b$. The only effect of this maneuver is that it makes some of the subsequent comparisons more transparent.

[28] By symmetry, all signals $\{s'_m\}_{m \neq n}$ are given the same weight (denoted by γ').

We provide some intuition on how the equilibrium statistic is determined by characterizing how γ_i and γ_s change with b, as formalized in Lemma 1.

Lemma 1 (Best Responses)**.**
The weights (γ_i, γ_s) satisfy the following conditions.

(i) γ_s is decreasing in b.

(ii) If $\rho_i > 0$, then γ_i is strictly quasi-convex in b, and $\gamma_i \to 1$ as $b \to 0$ or $b \to \infty$.

This lemma illustrates how agent n modifies the weights she places on her private signals when any agent $m \neq n$ changes the weight he places on his private signals.[29] On the one hand, if agents $m \neq n$ place a larger weight on their signal about the common shock then agent n will place a smaller weight on her signal about the common shock. On the other hand, the weight that agents $m \neq n$ place on their signal about the common shock has a nonmonotonic effect on the weight that agent n places on her idiosyncratic shock. Note that γ_i is decreasing in b (at least within a certain range of b), from which it follows that the weights placed by agents on their respective idiosyncratic shocks exhibit strategic complementarity. In particular, if agent m increases the weight he places on i_m then agent n will likewise increase the weight she places on i_n. This is the key intuition underlying the multiplicity of equilibria we illustrate in the following section. We now provide an intuitive account of parts (i) and (ii) of Lemma 1.

Analysis of γ_s. The informativeness of s'_m about the common shock c increases with b, and the extent to which agent n relies on her own private signal about c is *decreasing* in the amount of additional information that she has about c. Hence γ_s is decreasing in b.

Analysis of γ_i. The analysis of γ_i is more subtle. From agent n's perspective, i_m is a noise term in s'_m. That is, agent n would prefer simply to observe s_m. If $\rho_i = 0$, then γ_i is constant in b and is also equal to 1. This state of affairs is natural given that an agent knows her own idiosyncratic shock, which is independent of the noise in s'_m; hence agent n assigns this signal a weight of 1. The conceptual difference between $\rho_i = 0$ and $\rho_i \neq 0$ is that, in the latter case, i_n influences agent n's beliefs about c. When i_n is correlated with i_m, agent n can use i_n to filter out the noise in s'_m. The nonmonotonicity of γ_i reflects the use of i_n to filter noise.[30]

[29] To facilitate the exposition, we shall often refer to agents n and m via (respectively) feminine and masculine pronouns.

[30] Because agent n would prefer simply to observe s_m, if $b \to \infty$ then n can observe s_m directly; in this case, she need not use

The reason that $\gamma_i < 1$ (when $\rho_i > 0$) is that for an agent the direct effect — on his payoff shock — of observing a high or a low idiosyncratic shock is offset by updating his beliefs about the common shock in the opposite direction. This relation can be clearly illustrated in terms of our oil field example.

Suppose that agents are bidding for an oil field, the technology shocks are correlated ($\rho_i > 0$), and agent n observes a very high technology shock ($i_n \gg 0$). If agent n sees that agent m dropped out early from the auction, then agent n must infer that agent m observed an extremely "bad" signal about the oil field's size ($s_m \ll 0$). After all — because technology shocks are correlated — it makes sense for agent n to expect that agent m also observed a relatively high technology shock. Conversely, if agent n observed a low technology shock then agent n would be less pessimistic about the size of the oil field. Thus the direct effect of observing a high or a low technology shock is offset by updating beliefs about the oil field's size in the opposite direction. Therefore, conditional on the drop-out time of agent m, agent n's observed technology shock is not that informative about her preferences. Hence agents bid less aggressively on their technology shock (equivalently, γ_i decreases), which in turn reduces the social surplus.

Comparative Statics with Respect to ρ_i. As suggested by the preceding discussion, ρ_i plays an important role in determining β. We can show that, if there exists a unique equilibrium, then β is increasing in ρ_i.[31] So if there exists a unique equilibrium, then the auction's efficiency is decreasing in ρ_i.[32] If the idiosyncratic shocks are independently distributed ($\rho_i = 0$), then there is no strategic complementarity—which implies the existence of a unique equilibrium (i.e., within the class of equilibria referenced in Theorem 1). The formal statements and proofs of the aforementioned results are given in the Online Appendix.

4.4 Illustration of the Equilibrium Multiplicity

The cubic polynomial that determines the set of equilibrium statistics (see (12)) may have multiple roots. The implication is that an ascending auction with multi-dimensional signals may have multiple symmetric equilibria—that is, a different equilibrium for every root. As

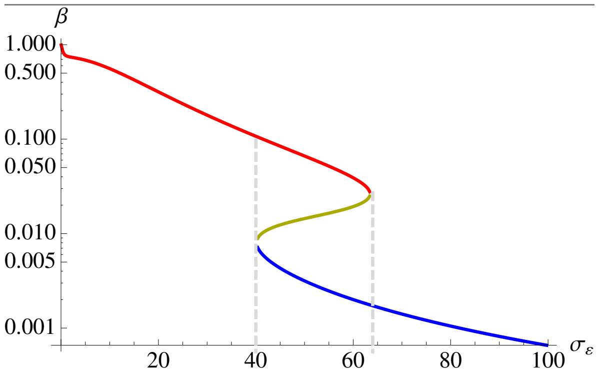
i_n to filter out the noise s'_m and so $\gamma_i = 1$. If $b \to 0$ then the signal s'_m provides no information about c and so agent n does not use s'_m to predict c at all; here, too, $\gamma_i = 1$. Thus it is only for intermediate values of b that agent n uses i_n to predict c.

[31] If there exist three equilibria, then the comparative statics is reversed in one of them.

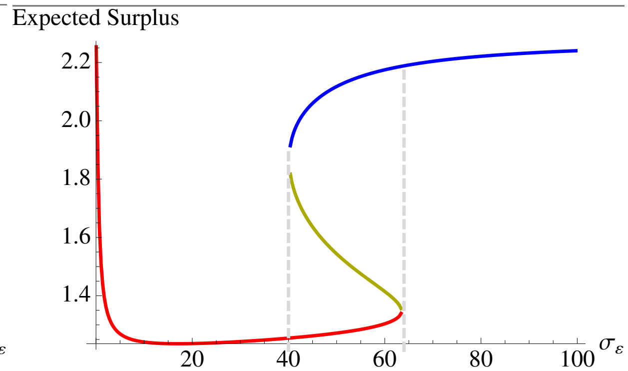
[32] More precisely: as ρ_i increases, the correlation between the identity of the auction's winner and his realization of the idiosyncratic shock decreases. As $\rho_i \to 1$, the good's allocation (i.e., the identity of the winner) is independent of the idiosyncratic shocks. Note that, in the limit $\rho_i = 1$, this is a pure–common values environment and so any allocation is efficient.

discussed in Section 4.3, the multiplicity of equilibria is due to a strategic complementarity in agents' bidding strategy.

We illustrate the multiplicity of equilibria with a parameterized example. In Figure 1(a) we plot the set of equilibrium statistics for different values of the variance in the noise. The graph's different colors correspond to the different roots of the cubic polynomial that determines the set of equilibrium statistics (see (12)). We can see that there are values of the noise term (e.g., $\sigma_\varepsilon = 50$) for which multiple equilibria exist.



(a) Equilibrium Statistic



(b) Expected Social Surplus ($\mathbb{E}[\exp(v_1)]$)

Figure 1: Outcome of ascending auction for $\sigma_c = 5/2$, $\sigma_i = 0.6$, $\rho_i = 3/4$, and $N = 50$.

In Figure 1(b) we plot the expected social surplus generated in the auction corresponding to the equilibrium statistic shown in part (a) of the figure. There is one equilibrium in which β is small (the blue line). This equilibrium will resemble a private values environment: the social surplus generated will be high and the winner's curse will be low. There is also one equilibrium in which β is large (the red line). This equilibrium will resemble a common values environment: the social surplus generated will be low and the winner's curse will be high. Neither the auction's generated revenue nor the buyers' rents is plotted in this graph, but both are qualitatively similar to our plot of the social surplus generated in the auction.

The social surplus generated by the auction is nonmonotonic in the size of the noise term (σ_ε^2) because there two countervailing effects. First, if β is fixed then, as σ_ε^2 increases, the correlation between an agent's drop-out time and the noise term ε_n also increases; the result is a reduction in social surplus. Second, as σ_ε^2 increases, the weight given to s_n decreases and so the weight placed on the noise term ε_n also decreases. The latter effect, in turn, reduces the correlation between an agent's drop-out time and the noise term; the result is an increase in social surplus.

If the noise term is extremely small or large (i.e., as $\sigma_\varepsilon^2 \to 0$ or $\sigma_\varepsilon^2 \to \infty$), then there is a unique equilibrium. This follows because, in both limits, the model approaches a private values model. If the former case, agents know almost perfectly the realization of c simply by virtue of their private information; in the latter case, agents ignore s_n and so the model is again a private values model. Note that in both limits the equilibrium is efficient, which holds not only for this parameterized example; as the Online Appendix shows, in these extreme cases there is always a unique equilibrium.

5 Effects of Public Signals

In this section we study how the precision of a public signal affects the social surplus and revenue generated by an auction. Our analysis reveals that the comparative statics in two-dimensional environments may be different from that in one-dimensional environments. This difference arises because, in the class of equilibria characterized by Theorem 1, agents behave as if they observed only the equilibrium statistic. Yet because the equilibrium statistic is endogenous, this implies that the comparative statics is partially determined by changes in the equilibrium statistic.

5.1 Description of Public Signals

We now study the effect of public information on the equilibrium outcome. In a model with one-dimensional signals, it is natural to consider a public signal about the average valuation across agents. In our environment, the valuation of an agent is determined by two payoff shocks. Hence it is natural likewise to consider both a public signal about the common shock and a public signal about the average idiosyncratic shock.

We assume that agents have access to two public signals (i.e., in addition to (i_n, s_n)). The first signal provides agents with more information about the common shock:

$$\bar{s}^1 = c + \varepsilon^1, \tag{21}$$

where ε^1 is independent of all the random variables defined so far and is normally distributed with variance σ_1^2. This signal $\bar{s}^1$ can be interpreted as disclosing additional information about the good (e.g., more information about the size of an oil field). The second public signal provides agents with information about the average idiosyncratic shock:

$$\bar{s}^2 = \frac{1}{N} \sum_{n \in N} i_n + \varepsilon^2; \tag{22}$$

here ε^2 is independent of all random variables defined so far and is normally distributed with variance σ_2^2. The signal $\bar{s}^2$ can be interpreted as providing more information about bidders' characteristics (e.g., more information about the industry's average cost of extracting oil).[33]

Agent n observes the signals $(i_n, s_n, \bar{s}^1, \bar{s}^2)$. The analysis in Section 4 can be extended in a straightforward manner to accommodate public signals. The only modification required is that the public signals be added as "conditioning variables" in the expectations. That is, our definition of an equilibrium statistic (see (11)) is modified as follows:

$$\mathbb{E}[v_n \mid i_n, s_n, t_1, ..., t_N, \bar{s}^1, \bar{s}^2] = \mathbb{E}[v_n \mid t_1, ..., t_N, \bar{s}^1, \bar{s}^2]. \tag{23}$$

In addition, the strategy of agents (see (14)) is now written as

$$p_n = \mathbb{E}[\exp(v_n) \mid t_n, ..., t_n, t_{n+1}, ..., t_N, \bar{s}^1, \bar{s}^2]. \tag{24}$$

It should be clear that, under these two modifications, all of the analysis in Section 4 is unchanged.

5.2 Effect of Public Signals on Social Surplus

Next we examine how the social surplus is affected by public signals. Social surplus is equal to the expected valuation of the agent who observed the highest equilibrium statistic (i.e., $\mathbb{E}[\exp(v_1)]$).

Proposition 4 (Comparative Statics of Public Signals: Social Surplus)**.**
If the ascending auction has a unique equilibrium, then the social surplus is decreasing in σ_1^2 and also in σ_2^2. In the limit, we have

$$\lim_{\sigma_1^2 \to 0} \mathbb{E}[\exp(v_1)] = \lim_{\sigma_2^2 \to 0} \mathbb{E}[\exp(v_1)] = \mathbb{E}\big[\max_{n \in N}\{\exp(v_n)\}\big].^{34}$$

[33] All our results go through in the same way if, instead of observing a public signal $\bar{s}^2 = \sum_{n \in N} i_n / N + \varepsilon^2$, each agent n observes $N-1$ private signals about the idiosyncratic shocks of agents $m \neq n$—that is, if n observes signals $s_n^m = i_m + \varepsilon_n^m$ for all $m \neq n$.

[34] We study the ex ante expected social surplus, rather than the interim expected social surplus, so that we can avoid taking

Proposition 4 shows that social surplus increases with the precision of public signals. In the limit where one of the public signals is arbitrarily precise, the equilibrium approaches the efficient outcome. Note that, the ascending auction would implement the efficient outcome if agents ignored their private signal about the common shock (i.e. signal s_n). Thus a sufficiently precise public signal reduces the weight that agents place on s_n all the way to zero. If the ascending auction has three equilibria, then the social surplus: (i) increases with the public signal's precision in the equilibria with the highest and the lowest β, yet (ii) decreases with the public signal's precision in the equilibrium with the "medium" β.

The intuition for why social surplus is decreasing in σ_1^2 is simple. As the public information about c becomes more precise, an agent needs to place less weight on her private signal s_n in order to predict c. Hence the correlation between an agent's drop-out time and her realization of the noise term ε_n decreases, which means that social surplus increases.

The reason that $\bar{s}^2$ changes the social surplus is that it changes how agent n's idiosyncratic shock affects her beliefs about the common shock. As explained in Section 4.3, if $\rho_i > 0$ then for an agent the direct effect — on his payoff shock— of observing a high idiosyncratic shock is partially offset by updating beliefs about the common shock in the opposite direction. This offsetting leads agent n to bid less aggressively on her idiosyncratic shock, which reduces the social surplus. If there is a public signal about i_m, then the weight given to i_n when predicting i_m is reduced; the public signal thus reduces the correlation between agent n's idiosyncratic shock and her beliefs about the common shock. Hence agent n trades more aggressively on her idiosyncratic shock, which increases the social surplus.

The equilibrium converges to the efficient outcome as $\bar{s}^2$ becomes arbitrarily precise because, in the limit, the effects described in Section 4.3 are reversed. Namely, the direct effect of observing a high idiosyncratic shock is reinforced by updating beliefs about the common shock in the *same* (rather than the opposite) direction, after which agents trade even more aggressively on their idiosyncratic shocks. This behavior increases social surplus to the efficient levels. This is also what happens when idiosyncratic shocks are negatively correlated.

limits of random variables. The statement holds in the absence of expectations if one considers convergence in probability.

5.3 Effect of Public Signals on Revenue

We now study how revenue is affected by a public signal about the common shock. We shall use $\max^{(2)}\{\cdot\}$ to denote the second-order statistic (i.e., the second maximum).

Proposition 5 (Public Signals about the Common Shock).
If the signal about the common shock becomes arbitrarily precise, then the following equality holds:

$$\lim_{\sigma_1^2 \to 0} \mathbb{E}[p_2] = \mathbb{E}\big[\max_{n \in N}{}^{(2)}\{\exp(v_n)\}\big].$$

Proposition 5 shows that, as the public signal about c becomes arbitrarily precise ($\sigma_1^2 \to 0$), revenue approaches the expected second-highest valuation. The intuition for this claim is that, in the limit, an agent ignores her own private signal s_n. The situation is therefore "as if" the only private signal observed is i_n; in this limit, then, it is as if agents had private values. Now we examine how revenue is affected by a public signal about the average idiosyncratic shock.

Proposition 6 (Public Signals about the Average Idiosyncratic Shock).
If the signal about the average idiosyncratic shock becomes arbitrarily precise, then

$$\lim_{\sigma_2^2 \to 0} \mathbb{E}[p_2] = 0.$$

According to this proposition, as $\bar{s}^2$ becomes arbitrarily precise ($\sigma_2^2 \to 0$), revenue becomes arbitrarily close to zero. Note that the price is nonnegative in every realization of the auction. Therefore, the price converges in distribution to zero.

Here we offer an intuitive account of Proposition 6. To simplify the exposition, suppose that $N = 2$. The fundamental component of our analysis is that the difference between agent 2's valuation and his drop-out time increases with the precision of $\bar{s}^2$. That is,

$$\mathbb{E}[v_2 \mid t_1, t_2, \bar{s}^2] - \mathbb{E}[v_2 \mid t_2, t_2, \bar{s}^2]$$

decreasing with σ_2^2. The reason is that $\bar{s}^2$ provides information about i_1, which makes t_1 more informative about the common shock. Hence agent 2 can place more weight on t_1 when predicting c. Revenue converges to zero in the limit because the weight that the expectation $\mathbb{E}[v_2 \mid t_1, t_2, \bar{s}^2]$ places on t_1 diverges to infinity. The value of $\mathbb{E}[v_2 \mid t_1, t_2, \bar{s}^2]$

does not diverge because the weight that this expectation places on $\bar{s}^2$ diverges to *negative* infinity — this offsets the large weight placed on t_1.

To develop further intuition on why the weight that the expectation $\mathbb{E}[v_2 \mid t_1, t_2, \bar{s}^2]$ places on t_1 diverges to infinity, consider the following. If $\sigma_2^2 \approx 0$, then the equilibrium statistic of agent $n \in \{1, 2\}$ is given by:[35]

$$t_n = i_n + \epsilon \cdot s_n,$$

where $\epsilon \approx 0$ (and, in the limit, is equal to 0). Because $\sigma_2^2 \approx 0$, agent 2 can filter out i_1 from t_1 almost perfectly — which allows him to infer s_1 . So if $\sigma_2^2 \approx 0$, then the expectation can be written (approximately) as follows

$$\mathbb{E}[v_2 \mid t_1, t_2, \bar{s}^2] \approx t_2 + \frac{\sigma_c^2}{\sigma_c^2 + \sigma_\varepsilon^2/2} \cdot \frac{1}{\epsilon} \cdot \left(\frac{t_1 + t_2}{2} - \bar{s}^2 \right). \tag{25}$$

Observe that this expectation places a weight of order $1/\epsilon$ on the last term. The second term in (25) does not diverge because $\sigma_2^2 \approx 0$; therefore, $((t_1 + t_2)/2 - \bar{s}^2) \approx \epsilon \cdot (s_1 + s_2)/2$.

The revenue is calculated by replacing t_1 with t_2 (i.e., by computing $\mathbb{E}[v_2 \mid t_2, t_2, \bar{s}^2]$). So when we compute the revenue, the term $1/\epsilon$ is multiplied by $((t_2 + t_2)/2 - \bar{s}^2) \approx (t_2 - t_1)$, which is negative and does not converge to zero. It follows that $\mathbb{E}[v_2 \mid t_2, t_2, \bar{s}^2]$ diverges to $-\infty$ (which yields 0 when we take the exponential function).[36]

In our model, the public signals $\bar{s}^1$ and $\bar{s}^2$ have opposite effects on revenue. The signal about the common shock *reduces* the weight that agent 2 places on t_1 when computing the expected value of v_2. That reduction, in turn, decreases the difference between agent 2's expected valuation and the time she drops out of the auction (i.e., $(\mathbb{E}[v_2 \mid t_1, t_2, \bar{s}^2] - \mathbb{E}[v_2 \mid t_2, t_2, \bar{s}^2])$) and thereby increases revenue. In contrast, the signal about the average idiosyncratic shock *increases* the weight that agent 2 places on t_1 when computing the expected value of v_2. That increase, in turn, increases the difference between agent 2's expected valuation and the time she drops out of the auction (i.e., $(\mathbb{E}[v_2 \mid t_1, t_2, \bar{s}^2] - \mathbb{E}[v_2 \mid t_2, t_2, \bar{s}^2])$) and thereby reduces revenue.

The preceding discussion shows that evaluating the effect of a public signal on revenue requires that one take the nature of that signal into account. For this purpose, it is also

[35]Proposition 4 implies that, in the limit, $t_2 \approx i_2$.

[36]The explosive behavior is reminiscent of the behavior of ascending auction under common values with asymmetric agents (cf. Bulow, Huang, and Klemperer (1999)).

important to account fully for all the private signals of agents. For a sharper illustration, consider the following signal:

$$\tilde{s}_n \triangleq i_n + \tilde{\varepsilon}_n; \tag{26}$$

here $\tilde{\varepsilon}_n$ is a noise term that is independent of all other random variables in the model, independent across agents, and with a small variance ($\text{var}(\tilde{\varepsilon}_n) \approx 0$). In addition, assume that $\rho_i > 0$. It is easy to check that, if agents observed only $\tilde{s}_n$, then revenue would be strictly increasing in the precision of $\bar{s}^2$ (this is a particular case of Milgrom and Weber (1982)). If agents instead observed $(\tilde{s}_n, s_n)$, then revenue would be strictly decreasing in the precision of $\bar{s}^2$.[37] Thus we have demonstrated how adding a private signal to the information structure of agents can reverse the effect of a public signal.

Failure of the Linkage Principle.[38] Proposition 6 can be interpreted as a failure of the linkage principle.[39] The linkage principle has been shown to fail in other environments.[40] In contrast to the previous literature, we prove that the linkage principle may fail for no other reason than the multi-dimensionality of the information structure. Thus our paper identifies a new channel through which the linkage principle may fail.

Violation of Assumptions in Milgrom and Weber (1982). In our original model, agent n's expected valuation conditional on all signals

$$\mathbb{E}[v_n \mid \{i_m\}_{m \in N}, \{s_m\}_{m \in N}, \bar{s}^2]$$

is nondecreasing in all the conditioning variables, and all signals are positively correlated. Thus all assumptions of Milgrom and Weber (1982) are satisfied *except* their assumption that private signals are one-dimensional. If we consider the reduced information structure in which every agent observes only her equilibrium statistic then the expected valuation of agents is decreasing in the realization of $\bar{s}^2$ (see (25)); this violates a different assumption

[37] This claim follows from Proposition 6 and a continuity argument. In Section 6 we generalize the model to any Gaussian information structure, in which case the equilibrium changes continuously with the variance–covariance matrix of that structure.

[38] The linkage principle states that (*i*) public signals increase an auction's revenue and (*ii*) ascending auctions yield higher revenue than first-price auctions; see Krishna (2009) for a textbook discussion.

[39] In particular, Proposition 6 shows that public signals may reduce revenue. Bergemann, Brooks, and Morris (2017) establish that revenue from a first-price auction is bounded away from 0; hence Proposition 6 also shows that an ascending auction may yield lower revenue than a first-price auction.

[40] Perry and Reny (1999) establish that the linkage principle may fail in multi-unit auctions. The linkage principle has been shown to fail also in environments characterized by an asymmetric payoff structure (Krishna (2009)) or by private values (Thierry and Stefano (2003)). Axelson and Makarov (2016) demonstrate that the linkage principle fails in common values auctions when an agent must take an action after winning the good. As in our model, the bid of an agent in Axelson and Makarov (2016) does not fully reveal the signal observed by that agent; however, in their paper the reason is that the good's final payoff is not strictly monotonic in the realization of signals observed by agents (see also Atakan and Ekmekci (2014)).

of Milgrom and Weber (1982).[41] Hence, the original information structure only violates the assumption that signals are one-dimensional while the "reduce information structure" violates the assumption that expectations are increasing in the realization of public signals.[42]

6 Extensions

Next we discuss how the solution method used in Section 4 can be extended to other environments. We first explain how the analysis can be extended to any Gaussian (and possibly asymmetric) multi-dimensional information structure, after which we show how the same equilibrium statistic can be used to find a class of Nash equilibria in other trading mechanisms. Here we provide an informal discussion; formal results and analysis are given in the Online Appendix.

6.1 General Multi-dimensional Signals

The analysis in Section 4 can be extended in a natural way to any multi-dimensional Gaussian information structure. Suppose that each agent $n \in N$ observes J signals:

$$\mathbf{s}_n = (s_n^1, ..., s_n^J);$$

here bold fonts signify vectors, and the superscripts denote the a signal's number. Agent n's utility from winning the object is

$$u(v_n) - p,$$

where $v_n \in \mathbb{R}$ is a payoff shock and $u(\cdot)$ is a strictly increasing function. In our baseline model we assumed that $u(\cdot) = \exp(\cdot)$. The joint distribution of signals and payoff shocks $(v_1, ..., v_N, \mathbf{s}_1, ..., \mathbf{s}_N)$ is jointly Gaussian, yet it may be asymmetrically distributed.

There is a class of equilibria that can be characterized in the same way as we characterized the class of equilibria with symmetric two-dimensional signals. First, we project the signals.

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[41] Milgrom and Weber (1982) assume that the utility of agents is increasing in the realization of all signals and also that signals are positively correlated (strictly speaking, they assume that signals are "affiliated"—but that amounts to the same thing in a Gaussian environment). Of course, it is possible to change the sign of $\bar{s}^2$; then the public signal would have a positive effect on agent n's valuation. Yet, in that case the public signal $\bar{s}^2$ would be negatively correlated with t_n, which would violate the affiliation property.

[42] Board (2009) studies an indivisible-good auction in a private value environment, but the signals do not satisfy the assumptions in Milgrom and Weber (1982); he shows that public signals can have an effect on the allocation of the good and reduce revenues. In our model, both signals observed by an agent satisfy the affiliation property and the monotonicity condition, the only assumption that is not satisfied is the one-dimensionality of the signals.

For this we must identify a set of weights $(\boldsymbol{\beta}_1, ..., \boldsymbol{\beta}_N) \in \mathbb{R}^{N \times J}$ such that, for all $n \in N$:

$$\mathbb{E}[v_n \mid \mathbf{s}_n, \mathbf{s}_1 \cdot \boldsymbol{\beta}_1, ..., \mathbf{s}_N \cdot \boldsymbol{\beta}_N] = \mathbb{E}[v_n \mid \mathbf{s}_1 \cdot \boldsymbol{\beta}_1, ..., \mathbf{s}_N \cdot \boldsymbol{\beta}_N], \tag{27}$$

where $\boldsymbol{\beta}_n \cdot \mathbf{s}_n$ denotes the dot product. In asymmetric environments, the weights employed by each agent ($\boldsymbol{\beta}_n$) may differ. However, there always exist weights such that (27) is satisfied.

For each projection of signals

$$(t_1, ..., t_N) = (\mathbf{s}_1 \cdot \boldsymbol{\beta}_1, ..., \mathbf{s}_N \cdot \boldsymbol{\beta}_N)$$

that satisfies a regularity condition, there exists an equilibrium in which agent $n \in N$ behaves as if she observes only the one-dimensional signal t_n. That regularity condition is called the *average crossing property*; this property is necessary to ensure that, in asymmetric environments, the ascending auction has an ex post equilibrium when agents observe one-dimensional signals.[43] For brevity, here we simply explain the result, but this is stated and proved in the Online Appendix (see Theorem 2 in the Online Appendix).

We remark that the only applicable constraint is in the equilibrium characterization in a one-dimensional environment;[44] the projection of signals (onto an equilibrium statistic) per se is not a problem, since a projection always exists. For a specific information structure, it is simple to check whether the information structure has an equilibrium statistic that satisfies the average crossing condition. It is worth emphasizing that, in Section 4, we implicitly checked for the existence of an ex post equilibrium when agents observe only their equilibrium statistic. We did this by first solving the model (from Section 3) in which agents observe only one-dimensional signals.

6.2 Other Mechanisms

Here we extend the methodology used in Section 4 to find a class of Nash equilibria when agents observe multi-dimensional Gaussian signals in a larger class of games. The solution method remains the same. We first project the signals onto a one-dimensional equilibrium statistic and then show that an equilibrium exists in which each agent behaves as if he

[43] See Krishna (2009) for a textbook discussion.

[44] The same characterization can be applied if we consider an ascending auction with re-entry (see Section 6.2 for a discussion). Allowing for re-entry relaxes the conditions under which an ex post equilibrium exists when agents observe one-dimensional signals (see Izmalkov (2001)).

observes only his equilibrium statistic. Note that, crucially, the definition of an equilibrium statistic does not change.

Consider a game with N agents. Agent $n \in N$ takes an action $a_n \in A_n$ for which the payoff function is given by

$$u_n(v_n, a_1, ..., a_N),$$

where $v_n \in \mathbb{R}$ is a payoff shock. Agent n observes J signals $(s_n^1, ..., s_n^J)$, and the joint distribution of all signals and payoff shocks is (jointly) normally distributed. An equilibrium statistic is defined the same way as in the previous section (i.e., as in (27)).

Fix an equilibrium statistic, and consider first an auxiliary game in which agent n observes only a one-dimensional signal equal to her equilibrium statistic $(\boldsymbol{\beta}_n \cdot \mathbf{s}_n)$. Suppose there exists in this auxiliary game a strategy profile $\{\hat{\alpha}_n\}_{n \in N}$, with $\hat{\alpha}_n : \mathbb{R} \to A_n$, that is an ex post equilibrium.[45] Then, in the original game (where agent n observes J signals), the strategy profile $\{\alpha_n\}_{n \in N}$ with $\alpha_n : \mathbb{R}^J \to A_n$ defined as[46]

$$\alpha(\mathbf{s}_n) = \hat{\alpha}(\boldsymbol{\beta}_n \cdot \mathbf{s}_n), \tag{28}$$

is a Nash equilibrium. This result is a natural extension of Theorem 1. For brevity, here we simply explain the result, but this is formally stated and proved in the Online Appendix (see Theorem 3 in the Online Appendix).

The methodology can be extended to games that have an ex post equilibrium when agents observe one-dimensional signals. These mechanisms have the property that the optimality condition (as in an ascending auction) can be evaluated using the realized value of all signals when agents observe one-dimensional signals. We briefly review some of the mechanisms that have an ex post equilibrium when agents observe one-dimensional signals.

There are classic trading mechanisms that have an ex post equilibrium when agents observe one-dimensional signals; examples include multi-unit ascending auctions (cf. Ausubel (2004); Perry and Reny (2005)) and generalized VCG mechanisms (cf. Dasgupta and Maskin (2000)). Supply function competition in linear quadratic environments has an ex post equilibria when agents are symmetric (cf. Vives (2011)). In addition to the classic mechanisms,

[45] A Nash equilibrium $(a_1, ..., a_N)$ is also an ex post equilibrium when agent n's action is optimal even if she was aware of the realization of all other agents' signals. See Bergemann and Morris (2005) for a discussion.

[46] As in the ascending auction, this setup constitutes a posterior equilibrium (see Green and Laffont (1987)).

several recent papers offer novel mechanisms that have an ex post equilibria when agents observe one-dimensional signals.[47]

The equilibrium statistic can also be used to understand models of competitive equilibrium under asymmetric information. Ganguli and Yang (2009) and Manzano and Vives (2011) study a rational expectations equilibrium in which agents observe two-dimensional signals. Amador and Weill (2010) study a micro-founded macro model with informational externalities. The fixed point that determines an equilibrium in these models is closely related to the equilibrium statistic defined in our paper. Heumann (2016) studies the properties of a competitive economy when agents observe multi-dimensional signals using the equilibrium statistic defined here.

It is worth mentioning some mechanisms that do *not* have an ex post equilibria when agents observe one-dimensional signals. Two classic examples are first-price auctions and Cournot competition. In a first-price auction, each agent tries to anticipate the bid of other agents in order determine how much he wants to "shade" his bid. This is in contrast to an ascending auction, where an agent's drop-out time remains optimal even if he knew the drop-out time of other agents (and so he need not anticipate those drop-out times). In Cournot competition, an agent does attempt to anticipate the quantity submitted by other agents because these quantities ultimately determine the equilibrium price. Yet, in supply function competition an agent can condition his purchased quantity on the equilibrium price and so she does not need to anticipate the demand of other agents. Understanding a first-price auction or Cournot competition when agents observe multi-dimensional signals would require techniques other than those developed in this paper.[48]

7 Conclusions

The auction literature relies heavily on the assumption that agents observe one-dimensional signals. In this paper we offer a tractable model of an ascending auction in which agents

[47] Ausubel, Crampton, and Milgrom (2006) propose a "combinatorial clock auction" for the case of auctioning many related items. Sannikov and Skrzypacz (2014) study a variation of supply function equilibria in which each agents can condition their purchases on the quantity bought by other agents. Kojima and Yamashita (2014) study a variation of the double auction that improves on it along several dimensions. All of these cited mechanisms have an ex post equilibrium when agents observe one-dimensional signals.

[48] Lambert, Ostrovsky, and Panov (2014) study a static version of a Kyle (1985) trading model under multi-dimensional Gaussian signals; this is strategically similar to Cournot competition. For the reasons already mentioned, our methodology is not suitable for the study of trading models such as the one described in Lambert, Ostrovsky, and Panov (2014) and vice versa.

observe multi-dimensional signals. Our main conceptual contribution is that, in multi-dimensional environments, an agent's bid is determined by the (endogenous) equilibrium statistic. We show that there may be multiple symmetric equilibria in such environments and overturn some classic results on the effects of public signals. These novel predictions clearly illustrate two broader points concerning multi-dimensional environments: (*i*) there is no simple mapping between the model's primitives and an auction's outcome; and (*ii*) the comparative statics is not the same as in one-dimensional environments. This paper provides a set of tools that can be used to increase our understanding of multi-dimensional environments and how they differ from their one-dimensional counterparts.

Our model relies on two critical assumptions: agents bid in an ascending auction, and signals are normally distributed. Extending our analysis to other auction formats will require the development of new techniques because different auction formats provide different incentives for bidders. In first-price auctions, for instance, an agent has an incentive to shade her bid. We therefore surmise that characterizing the equilibrium of a first-price auction with multi-dimensional signals would yield substantive new insights. To extend our analysis to non-Gaussian signals it is necessary to overcome two challenges. First, the definition of an equilibrium statistic must allow for arbitrary functions of a signal (and not simply linear combinations); proving the existence of an equilibrium statistic in such a case is challenging. Second, the resulting equilibrium statistic must satisfy the monotonicity condition that guarantees the existence of an equilibrium when agents observe only their own equilibrium statistic; proving this is challenging without an explicit expression for the equilibrium statistic (in the linear case this was proven in Section 3). Although we believe that our novel predictions do not hinge on the assumption of Gaussian signals, it is certainly possible that further new predictions could be derived by forgoing that assumption.

8 Appendix: Proofs

Preamble. We first provide explicit expressions for the expectations with normal random variables. To do this, we use the definition of one-dimensional signal given in (5).

Since $(v_1, ..., v_N, s'_1, ..., s'_N)$ are jointly Gaussian, it follows that the distribution of $(v_1, ..., v_N)$

conditional on $(s'_1, ..., s'_N)$ is also jointly Gaussian. We can therefore write:[49]

$$\mathbb{E}[\exp(v_n) \mid s'_1..., s'_N] = \exp\big(\mathbb{E}[v_n \mid s'_1, ..., s'_N] + \frac{1}{2} \cdot \mathrm{var}(v_n \mid s'_1, ..., s'_N)\big).$$

Similarly, after replacing $(s'_1, ..., s'_N)$ with $(s'_n, ..., s'_n, s'_{n+1}, ..., s'_N)$ we obtain

$$\mathbb{E}[\exp(v_n) \mid s'_n, ..., s'_n, s'_{n+1}..., s'_N] = exp\Bigg(\mathbb{E}[v_n \mid s'_n, ..., s'_n, s'_{n+1}..., s'_N] + \frac{var(v_n \mid s'_1, ..., s'_N)}{2}\Bigg). \tag{29}$$

Note that $\mathbb{E}[\exp(v_n) \mid s'_n, ..., s'_n, s'_{n+1}..., s'_N]$ is computed as if the realization of $(s'_1, ..., s'_n)$ were equal to $(s'_n, ..., s'_n)$. Because the conditional variance of normal random variables is independent of the realization of the random variables, we can write $\mathrm{var}(v_n \mid s'_1, ..., s'_N) = \mathrm{var}(v_n \mid s'_n, ..., s'_n, s'_{n+1}..., s'_N)$.

We now explicitly compute the coefficients of the Bayesian updating with the normal random variables. We have that

$$\mathbb{E}[v_n \mid s'_1, ..., s'_N] = \kappa \cdot \left(s'_n + \lambda \sum_{m=1}^{N} s'_m\right), \tag{30}$$

where

$$\kappa \triangleq \frac{(1-\rho_i)\sigma_i^2}{(1-\rho_i)\sigma_i^2 + b^2 \cdot \sigma_\varepsilon^2}; \tag{31}$$

$$\lambda \triangleq \frac{1}{N}\Bigg(\frac{((1-\rho_i) + \rho_i \cdot N) \cdot \sigma_i^2 + b \cdot N \cdot \sigma_c^2}{((1-\rho_i) + \rho_i \cdot N) \cdot \sigma_i^2 + b^2(N \cdot \sigma_c^2 + \sigma_\varepsilon^2)} \frac{(1-\rho_i)\sigma_i^2 + b^2 \cdot \sigma_\varepsilon^2}{(1-\rho_i)\sigma_i^2} - 1\Bigg). \tag{32}$$

This is just computing the coefficients of the Bayesian updating. To ensure that the coefficients λ and κ are correctly computed, it is sufficient to check that

$$\forall m \in N, \quad \mathrm{cov}(v_n - \mathbb{E}[v_n \mid s'_1, ..., s'_N], s'_m) = 0; \tag{33}$$

for this we use (30) and the definitions of κ and λ.[50] Finally, note that $\kappa > 0$ and that, for

[49] If y is normally distributed, then: $\mathbb{E}[\exp(y)] = \exp\big(\mathbb{E}[y] + \frac{1}{2} \cdot \mathrm{var}(y)\big)$. This is just the mean of a log-normal random variable.

[50] That is, λ and κ solve the following system of equations:

$$\sigma_i^2 + b\sigma_c^2 = \kappa\left(\sigma_i^2 + b^2(\sigma_c^2 + \sigma_\varepsilon^2) + \lambda\left(\sigma_i^2 + b^2(\sigma_c^2 + \sigma_\varepsilon^2) + (N-1)(\rho_i \cdot \sigma_i^2 + b^2 \cdot \sigma_c^2)\right)\right),;$$

$$\rho_i \cdot \sigma_i^2 + b\sigma_c^2 = \kappa\left(\rho_i\sigma_i^2 + b^2 \cdot \sigma_c^2 + \lambda\left(\sigma_i^2 + b^2(\sigma_c^2 + \sigma_\varepsilon^2) + (N-1)(\rho_i \cdot \sigma_i^2 + b^2 \cdot \sigma_c^2)\right)\right),$$

which corresponds to (33) for $m = n$ and $m \neq n$, respectively.

all $n \in N$,

$$(1+n\cdot\lambda)=\frac{n}{N}\frac{((1-\rho_i)+\rho_i\cdot N)\cdot\sigma_i^2+b\cdot N\cdot\sigma_c^2}{((1-\rho_i)+\rho_i\cdot N)\cdot\sigma_i^2+b^2(N\cdot\sigma_c^2+\sigma_\varepsilon^2)}\frac{(1-\rho_i)\sigma_i^2+b^2\cdot\sigma_\varepsilon^2}{(1-\rho_i)\sigma_i^2}+\frac{N-n}{N}>0.$$

Proof of Proposition 1. The proof is standard in the literature (see e.g. Krishna (2009)). Nevertheless, we give the proof here for completeness and to check that all the conditions are satisfied. In particular, we check the following three conditions.

1. According to the equilibrium strategies (see (7)), agent $n+1$ drops out of the auction before agent n does. This is a necessary condition for an equilibrium because, according to (7), agent n's equilibrium strategy is conditioned on the signals $(s'_{n+1},...,s'_N)$. Hence it is necessary to check that agents with higher signals drop out later in the auction.

   By (30), the following expression holds:

$$\mathbb{E}[v_{n-1}\mid s'_{n-1},...,s'_{n-1},s'_n,...,s'_N]-\mathbb{E}[v_n\mid s'_n,...,s'_n,s'_{n+1},...,s'_N]=\kappa(1+\lambda\cdot(n-1))(s'_{n-1}-s'_n)>0.$$

   The equality is based on (30); the inequality reflects that $\kappa>0$, $(1+(n-1)\lambda)>0$ (as previously shown), and $(s'_{n-1}-s'_n)>0$ by construction. It now follows from (29) that, for all $n\in\{2,...,N\}$,

$$\mathbb{E}[\exp(v_{n-1})\mid s'_{n-1},...,s'_{n-1},s'_n...,s'_N]-\mathbb{E}[\exp(v_n)\mid s'_n,...,s'_n,s'_{n+1}...,s'_N]>0. \tag{34}$$

   Therefore, agent n drops out of the auction before agent $n-1$.

2. We now check that agent 1 does not regret winning the auction (this is (8)).

   Using (30), we note that

$$\mathbb{E}[v_1\mid s'_1,...,s'_N]-\mathbb{E}[v_2\mid s'_2,s'_2,...,s'_N]=\kappa(1+\lambda)(s'_1-s'_2)>0.$$

   It is clear that the inequality is satisfied also if we take the exponential of v_1 and v_2, so (8) is satisfied. Hence agent 1 does not regret winning the auction.

3. Finally, we check that agent $m>1$ does not regret waiting until agent 1 drops out of the auction (this is (9)).

By (30),

$$\mathbb{E}[v_m \mid s'_1, ..., s'_N] - \mathbb{E}[v_1 \mid s'_1, s'_1, s'_2, ..., s_{m-1}, s_{m+1}, ..., s'_N] = \kappa(1+\lambda)(s'_m - s'_1) < 0$$

holds. The inequality will also be satisfied if we take the exponential of v_m and v_1, from which it follows that (9) is satisfied. Therefore, agent $m > 1$ does not regret waiting until agent 1 drops out of the auction.

We thus conclude that the equilibrium strategies constitute an ex post equilibrium.■

Proof of Proposition 2. In order to characterize how the ex ante expected social surplus is determined by $b \in \mathbb{R}$, we first provide an orthogonal decomposition of signals and payoff shocks (this decomposition will also be used later in subsequent proofs). We define

$$\bar{v} \triangleq \frac{1}{N} \sum_{n \in N} v_n; \quad \Delta v_n \triangleq v_n - \bar{v}; \quad \bar{s}' \triangleq \frac{1}{N} \sum_{n \in N} s'_n; \quad \Delta s'_n \triangleq s'_n - \bar{s}'. \tag{35}$$

Variables marked with an overbar represent the *average* of the variable over all agents, and variables preceded by a Δ correspond to the *difference* between that variable and the average variable; we refer to the former (resp., the latter) as the "common" (resp., the "orthogonal") component of a random variable. For example, $\bar{v}$ is the common component of v_n and Δv_n is the orthogonal component of v_n. It is crucial for our purposes that a random variable's common component is always independent of its orthogonal component. Thus, for instance, $\text{cov}(\Delta v_n, \bar{s}') = 0$; this expression entails the independence of those two random variables in our Gaussian environment.[51]

By construction, agent 1 wins the good and so the expected social surplus—given the realization of the signals—can be written as

$$S(s'_1, ..., s'_N) \triangleq \mathbb{E}[\exp(v_1) \mid s'_1, ..., s'_N].$$

We can now write the expected social surplus as follows:

$$\mathbb{E}[S(s'_1, ..., s'_N)] = \mathbb{E}[\mathbb{E}[\exp(v_1) \mid s'_1, ..., s'_N]] = \mathbb{E}[\mathbb{E}[\exp(\bar{v}) \cdot \exp(\Delta v_1) \mid \bar{s}', \Delta s'_1, ..., \Delta s'_N]].$$

[51] To check this, note that $\sum_{n \in N} \Delta v_n = 0$ by construction. Then, by symmetry, for all $n, m \in N$ we have $\text{cov}(\Delta v_n, \bar{s}') = \text{cov}(\Delta v_m, \bar{s}')$, and from the collinearity of the covariance it follows that $\sum_{n \in N} \text{cov}(\Delta v_n, \bar{s}') = 0$. Clearly, then, we must have $\text{cov}(\Delta v_n, \bar{s}') = 0$. The argument can be repeated for the common and orthogonal components of any random variable.

Because the common components of any random variable are independent of its orthogonal components, we have

$$\mathbb{E}[S(s'_1,...,s'_N)] \;=\; \mathbb{E}[\mathbb{E}[\exp(\bar{v})|\bar{s}'] \cdot \mathbb{E}[\exp(\Delta v_1) \mid \Delta s'_1,...,\Delta s'_N]].$$

From the law of iterated expectations it follows that $\mathbb{E}[\mathbb{E}[\exp(\bar{v})|\bar{s}']] = \mathbb{E}[\exp(\bar{v})] = \exp\left(\frac{1}{2}\sigma_{\bar{v}}^2\right)$. Therefore,

$$\mathbb{E}[S(s'_1,...,s'_N)] = \exp\left(\frac{1}{2}\sigma_{\bar{v}}^2\right) \times \mathbb{E}[\mathbb{E}[\exp(\Delta v_1) \mid \Delta s'_1,...,\Delta s'_N]]. \tag{36}$$

Since the equilibrium is efficient, we have

$$\mathbb{E}[\mathbb{E}[\exp(\Delta v_1) \mid \Delta s'_1,...,\Delta s'_N]] = \mathbb{E}\big[\max_{n\in N} \{\ \mathbb{E}[\exp(\Delta v_n) \mid \Delta s'_1,...,\Delta s'_N]\ \}\big]$$

Note that $\Delta v_n = \Delta i_n$ and so

$$\Delta s'_n = \Delta v_n + \beta \cdot \Delta\varepsilon_n.$$

It is now clear that, as β increases, $(\Delta s'_1,...,\Delta s'_N)$ becomes less informative (in the Blackwell sense) about $(\Delta v_1,...,\Delta v_N)$. Therefore, $\mathbb{E}[\mathbb{E}[\exp(\Delta v_1) \mid \Delta s'_1,...,\Delta s'_N]]$ is decreasing in β. ■

Proof of Proposition 3. The proof proceeds in two steps.

Step 1. We first establish that $t_n = i_n + \beta \cdot s_n$ is an equilibrium statistic if and only if

$$\mathrm{cov}(v_n - \mathbb{E}[v_n \mid t_1,...,t_N], i_n) = 0 \tag{37}$$

and $\beta \neq 0$.

"Only If". Clearly, $\beta = 0$ is not an equilibrium statistic (simply note that $\mathbb{E}[v_n \mid i_1,...,i_N] \neq \mathbb{E}[v_n \mid s_n, i_1,...,i_N]$). By the construction of the expectation, we have

$$\mathrm{cov}(v_n - \mathbb{E}[v_n \mid i_n, s_n, t_1,...,t_N], i_n) = 0. \tag{38}$$

So if $\mathbb{E}[v_n \mid i_n, s_n, t_1,...,t_N] = \mathbb{E}[v_n \mid t_1,...,t_N]$, then (37) must be satisfied.

"If". Note that, by construction of the expectation,

$$\forall m \in N, \quad \mathrm{cov}(v_n - \mathbb{E}[v_n \mid t_1,...,t_N], t_m) = 0. \tag{39}$$

Now, by the collinearity of the covariance, if (37) is satisfied and if (39) is satisfied with $\beta \neq 0$, then also

$$\operatorname{cov}(v_n - \mathbb{E}[v_n \mid t_1, ..., t_N], s_n) = 0. \tag{40}$$

This follows because s_n is a linear combination of t_n and i_n. Therefore, if (37) is satisfied then (39) and (40) are satisfied by construction. As a result, the covariance of $(i_n, s_n, t_1, ..., t_N)$ and $(v_n - \mathbb{E}[v_n \mid t_1, ..., t_N])$ is equal to 0. Hence we must have

$$\mathbb{E}[v_n \mid t_1, ..., t_N] = \mathbb{E}[v_n \mid i_n, s_n, t_1, ..., t_N]. \tag{41}$$

Thus $t_n = i_n + \beta \cdot s_n$ is an equilibrium statistic, which completes the proof's first step.

Step 2. We now prove that β satisfies (37) if and only if β solves the cubic polynomial (12). It is clear that $\operatorname{cov}(v_n, i_n) = \sigma_i^2$. By (30), we have

$$\operatorname{cov}(\mathbb{E}[v_n \mid t_1, ..., t_N], i_n) = \kappa((\lambda + 1)\sigma_i^2 + \lambda(N-1)\rho_i\sigma_i^2).$$

Hence we can rewrite (37) as

$$1 - \kappa((\lambda + 1) + \lambda(N-1)\rho_i) = 0.$$

Multiplying both sides by:

$$-\frac{(\beta^2\sigma_\varepsilon^2 - (\rho_i - 1)\sigma_i^2)\left(\beta^2(N\sigma_c^2 + \sigma_\varepsilon^2) + \sigma_i^2((N-1)\rho_i + 1)\right)}{\beta(\rho_i - 1)\sigma_c^2\sigma_i^2\sigma_\varepsilon^2((N-1)\rho_i + 1)}$$

yields the cubic polynomial (12), which proves the result.■

Proof of Theorem 1. This theorem was proved in the main text.■

Proof of Lemma 1. In order to write the expectations in terms of conditionally independent signals, define

$$\hat{s} \triangleq \frac{1}{N-1} \sum_{m \neq n} \left(s'_m - \frac{\rho_i \cdot i_n}{b} \right) = c + \frac{1}{N-1} \sum_{m \neq n} \left(\varepsilon_m + \frac{1}{b}(i_m - \rho_i \cdot i_n) \right). \tag{42}$$

Note that $\hat{s}$ is independent of i_n and that, by symmetry, $\hat{s}$ is a sufficient statistic for $\{s'_m\}_{m \neq n}$

to predict v_n. Expectation (20) can now be written as

$$\mathbb{E}[v_n \mid i_n, s_n, \{s'_m\}_{m \neq n}] = i_n + \mathbb{E}[c \mid s_n, \hat{s}], \tag{43}$$

where $(s_n, \hat{s})$ are conditionally independent signals of c. Using standard formulas of expectations with Gaussian random variables, we can write

$$\gamma_s = \frac{1/\sigma_\varepsilon^2}{1/\sigma_\varepsilon^2 + 1/\sigma_c^2 + 1/\text{var}(\hat{s}|c)}; \qquad \gamma' = \frac{1/\text{var}(\hat{s}|c)}{1/\sigma_\varepsilon^2 + 1/\sigma_c^2 + 1/\text{var}(\hat{s}|c)}.$$

Using (42), it is easy to check that

$$\begin{aligned} \text{var}(\hat{s}|c) &= \text{var}\left(\frac{1}{N-1}\sum_{m \neq n}\left(\varepsilon_m + \frac{1}{b}(i_m - \rho_i \cdot i_n)\right)\right) \\ &= \frac{1}{N-1}\left(\sigma_\varepsilon^2 + \frac{1}{b^2}(1-\rho_i^2)\sigma_i^2 + \frac{1}{b^2}(N-2)(\rho_i - \rho_i^2)\sigma_i^2\right). \end{aligned}$$

After replacing and simplifying terms, we obtain:

$$\gamma_s = \frac{b^2\sigma_\varepsilon^2\sigma_c^2 + (1-\rho_i)\sigma_i^2\sigma_c^2((N-1)\rho_i + 1)}{b^2\sigma_\varepsilon^2(N\sigma_c^2 + \sigma_\varepsilon^2) + (1-\rho_i)\sigma_i^2((N-1)\rho_i + 1)(\sigma_c^2 + \sigma_\varepsilon^2)};$$

$$\gamma' = \left(\frac{b^2(N-1)\sigma_c^2\sigma_\varepsilon^2}{b^2\sigma_\varepsilon^2(N\sigma_c^2 + \sigma_\varepsilon^2) + (1-\rho_i)\sigma_i^2((N-1)\rho_i + 1)(\sigma_c^2 + \sigma_\varepsilon^2)}\right).$$

Finally, we have that:

$$\gamma_i = 1 - \frac{\rho_i \cdot \gamma'}{b}.$$

This equality is a direct consequence of γ_i being equal to 1 *plus* the weight given to i_n by the prediction of c. Yet, from (42) it follows that this weight is equal to $-\rho_i \cdot \gamma'/b$. Therefore,

$$\gamma_i = 1 - \frac{\rho_i \cdot \gamma'}{b} = 1 - b \cdot \frac{(N-1)\rho_i\sigma_c^2\sigma_\varepsilon^2}{b^2\sigma_\varepsilon^2(N\sigma_c^2 + \sigma_\varepsilon^2) + (1-\rho_i)\sigma_i^2((N-1)\rho_i + 1)(\sigma_c^2 + \sigma_\varepsilon^2)}.$$

It is easy to check that γ_s is decreasing in b and that, in the limits $b \to 0$ and $b \to \infty$, we have (respectively) $\sigma_c^2/(\sigma_c^2 + \sigma_\varepsilon^2)$ and $\sigma_c^2/(N\sigma_c^2 + \sigma_\varepsilon^2)$. We can similarly check that $\gamma_i < 1$ and that, if either $b \to 0$ or $b \to \infty$, then $\gamma_i \to 1$. It is also possible to check that γ_i is quasi-convex in b. ■

Proof of Proposition 4. Before giving the proof, we make some relevant observations.

Remark 1. We use the notation defined in (35), and we extend the definition to all other random variables. So as before, variables with an overbar (resp., preceded by Δ) correspond to the average of that variable over all agents (resp., the difference between that variable and the average variable). We also note that:

$$\sigma_{\bar{v}}^2 = \sigma_v^2 \left(\frac{1 + (N-1)\rho_v}{N} \right) \quad \text{and} \quad \sigma_{\Delta v}^2 = \sigma_v^2 \frac{(N-1)(1-\rho_v)}{N},$$

where ρ_v is the correlation of payoff shocks across agents. For any random variable, the variances of the common and orthogonal components are determined by the correlation of the random variable across agents in the same way. If we multiply all terms in the cubic polynomial (12) by $N/(N-1)$, then (12) can be written in terms of the common and orthogonal components of a random variable as follows:

$$x_3 = \frac{(\sigma_{\Delta i}^2 + \sigma_{\bar{\imath}}^2)(\sigma_{\bar{\varepsilon}}^2 + \sigma_c^2)}{\sigma_{\Delta i}^2 \sigma_{\bar{\imath}}^2 \sigma_c^2}; \quad x_2 = \frac{-1}{\sigma_{\Delta i}^2}; \quad x_1 = \frac{\sigma_{\Delta\varepsilon}^2 + \sigma_{\bar{\varepsilon}}^2 + \sigma_c^2}{\sigma_{\Delta\varepsilon}^2 \sigma_c^2}; \quad x_0 = \frac{-1}{\sigma_{\Delta\varepsilon}^2}. \tag{44}$$

Remark 2. We now show that, in the model with public signals, a linear combination of signals $t_n = i_n + \beta \cdot s_n$ is an equilibrium statistic if and only if β is a root of the cubic polynomial $x_3 \cdot \beta^3 + x_2 \cdot \beta^2 + x_1 \cdot \beta + x_0$ with coefficients:

$$x_3 = \frac{(\sigma_{\Delta i}^2 + \sigma_{\bar{\imath}'}^2)(\sigma_{\bar{\varepsilon}}^2 + \sigma_{c'}^2)}{\sigma_{\Delta i}^2 \sigma_{\bar{\imath}'}^2 \sigma_{c'}^2}; \quad x_2 = \frac{-1}{\sigma_{\Delta i}^2}; \quad x_1 = \frac{\sigma_{\Delta\varepsilon}^2 + \sigma_{\bar{\varepsilon}}^2 + \sigma_{c'}^2}{\sigma_{\Delta\varepsilon}^2 \sigma_{c'}^2}; \quad x_0 = \frac{-1}{\sigma_{\Delta\varepsilon}^2} \tag{45}$$

for

$$\sigma_{\bar{\imath}'}^2 = \sigma_{\bar{\imath}}^2 - \frac{\sigma_{\bar{\imath}}^4}{\sigma_{\bar{\imath}}^2 + \sigma_2^2} \quad \text{and} \quad \sigma_{c'}^2 = \sigma_c^2 - \frac{\sigma_c^4}{\sigma_c^2 + \sigma_1^2}. \tag{46}$$

In other words, analyzing the equilibrium with public signals is equivalent to redefining the variances of the common shocks and the common component of the idiosyncratic shocks.

We prove this claim by first defining

$$i_n' \triangleq i_n - \mathbb{E}[i_n | \bar{s}^2] \quad \text{and} \quad c' \triangleq c - \mathbb{E}[c | \bar{s}^1].$$

Note that $\mathbb{E}[i_n|\bar{s}^2]$ is the same across agents and so $\Delta i_n = \Delta i_n'$. That is, public signals do not change the idiosyncratic component of a random variable. The variance of $\bar{\imath}'$ and $\bar{c}'$ is given by (46). Analogously, define

$$s'_n \triangleq c' + \varepsilon_n \quad \text{and} \quad t'_n \triangleq i'_n + \beta \cdot (c' + \varepsilon_n). \tag{47}$$

Observe that, for the purpose of this proof, s'_n is defined differently than in (5). All variables marked with a prime are orthogonal to $(\bar{s}^1, \bar{s}^2)$. Hence the linear combination of signals $t'_n = i'_n + \beta \cdot s'_n$ is an equilibrium statistic if and only if

$$\mathbb{E}[i'_n + c' \mid t'_1, ..., t'_N] = \mathbb{E}[i'_n + c' \mid i'_n, s'_n, t'_1, ..., t'_N].$$

We can therefore use the characterization of an equilibrium statistic in Proposition 3 while using the variables with primes. This approach corresponds to changing the variance of $\bar{\imath}'$ and c' according to (46).

Remark 3. Much as in the proof of Proposition 2, the expected social surplus can be written as

$$\mathbb{E}[S(s_1, ..., s_N)] = \exp\left(\frac{1}{2}\text{var}(\bar{v})\right) \times \mathbb{E}[\mathbb{E}[\exp(\Delta v_1) \mid \Delta t_1, ..., \Delta t_N]], \tag{48}$$

where

$$\Delta t_n = \Delta i_n + \beta \cdot \Delta \varepsilon_n.$$

As in Proposition 2, it is easy to check that the expected social surplus is decreasing in β and that, if $\beta \to 0$, then the equilibrium approaches the efficient outcome.

Main Step. Define the polynomial

$$q(\beta, \sigma_2^2, \sigma_1^2) \triangleq x_3 \cdot \beta^3 + x_2 \cdot \beta^2 + x_1 \cdot \beta + x_0 \tag{49}$$

with x_3, x_2, x_1, x_0 as defined in (45) (note that these coefficients depend on (σ_2^2, σ_1^2)). Let $\beta^*(\sigma_2^2, \sigma_1^2)$ be a root of (49), and let this root be unique. It is now easy to check that

$$\frac{\partial \beta^*(\sigma_2^2, \sigma_1^2)}{\partial \sigma_2^2} = -\frac{\frac{\partial q(\beta^*(\sigma_2^2,\sigma_1^2),\sigma_2^2,\sigma_1^2)}{\partial \sigma_2^2}}{\frac{\partial q(\beta^*(\sigma_2^2,\sigma_1^2),\sigma_2^2,\sigma_1^2)}{\partial \beta}}, \tag{50}$$

and similarly for the derivative with respect to σ_1^2. We can also clearly see that $q(\beta, \sigma_2^2, \sigma_1^2)$ is decreasing in $\sigma_{\bar{\imath}'}^2$ and $\sigma_{c'}^2$ (and thus also decreasing in σ_2^2 and σ_1^2, respectively). It follows that the numerator in (50) is negative. If $q(\beta,\sigma_2^2,\sigma_1^2)$ has a unique root, then $q(\beta,\sigma_2^2,\sigma_1^2)$ is increasing at this root and so the denominator in (50) must be positive. Therefore, if $q(\beta,\sigma_2^2,\sigma_1^2)$ has a unique root then this root is increasing in σ_2^2 and σ_1^2. The implication is that social surplus is decreasing in σ_1^2 and ${\sigma_2}^2$.

Now note that, in the limit $\sigma_{\bar{i}'}^2 \to 0$ or $\sigma_{c'}^2 \to 0$, every root of the polynomial $q(\beta)$ must converge to zero. Hence the social surplus is equal to the social surplus of the efficient outcome.■

Proof of Proposition 5. The proof is similar to the proof given for Proposition 4. We use all the definitions and arguments therein, extending them to show the results on revenue. We start by making some additional observations.

Remark 1. The coefficients λ and κ (defined in (31) and (32)) can be rewritten—in terms of the common and orthogonal components of the random variables—as follows:

$$\kappa = \frac{\sigma_{\Delta i}^2}{\sigma_{\Delta i}^2 + \beta^2 \cdot \sigma_{\Delta\varepsilon}^2}; \tag{51}$$

$$\lambda = \frac{1}{N}\left(\frac{\sigma_{\bar{i}}^2 + \beta \cdot \sigma_c^2}{\sigma_{\bar{i}}^2 + \beta^2(\sigma_c^2 + \sigma_{\bar{\varepsilon}}^2)} \frac{\sigma_{\Delta i}^2 + \beta^2 \cdot \sigma_{\Delta\varepsilon}^2}{\sigma_{\Delta i}^2} - 1\right). \tag{52}$$

Remark 2. We can use the definition of s_n' in (47) to write the expectations as

$$\mathbb{E}[v_n \mid s_1', ..., s_N', \bar{s}^1, \bar{s}^2] = \kappa' \cdot \left(s_n' + \lambda' \sum_{m=1}^{N} s_m'\right) + \mathbb{E}[v_n \mid \bar{s}^1, \bar{s}^2]. \tag{53}$$

Here κ' and λ' are defined as

$$\kappa' \triangleq \frac{\sigma_{\Delta i}^2}{\sigma_{\Delta i}^2 + \beta^2 \cdot \sigma_{\Delta\varepsilon}^2} \quad \text{and} \tag{54}$$

$$\lambda' \triangleq \frac{1}{N}\left(\frac{\sigma_{\bar{i}'}^2 + \beta \cdot \sigma_{c'}^2}{\sigma_{\bar{i}'}^2 + \beta^2(\sigma_{c'}^2 + \sigma_{\bar{\varepsilon}}^2)} \frac{\sigma_{\Delta i}^2 + \beta^2 \cdot \sigma_{\Delta\varepsilon}^2}{\sigma_{\Delta i}^2} - 1\right), \tag{55}$$

where $\sigma_{\bar{i}'}^2$ and $\sigma_{c'}^2$ are as defined in (46).

Remark 3. The price paid in the auction is equal to $\mathbb{E}[\exp(v_2) \mid t_2, t_2, ..., t_N, \bar{s}^1, \bar{s}^2]$. From

(29) and (53) it follows that

$$\begin{aligned} p_2 &= \mathbb{E}[\exp(v_2) \mid t_2, t_2, ..., t_N, \bar{s}^2, \bar{s}^1] \\ &= \exp(\mathbb{E}[v_2 \mid t_1, ..., t_N] - \kappa'\lambda'(t_1 - t_2) + \frac{1}{2}\mathrm{var}(v_2 \mid t_1, ..., t_N, \bar{s}^2, \bar{s}^1)) \\ &= \exp(-\kappa' \cdot \lambda' \cdot (t_1 - t_2)) \times \mathbb{E}[\exp(v_2) \mid t_1, ..., t_N, \bar{s}^2, \bar{s}^1] \end{aligned} \tag{56}$$

Main Step. We now give the main part of the proof. We know from Proposition 4 that $\beta \to 0$ in the limit $\sigma_1^2 \to 0$. In this case $\lambda' \to 0$ and $\kappa' \to 1$ (note that $\sigma_{\bar{\imath}'}^2$ is strictly above 0 in the limit $\sigma_1^2 \to 0$). Now (56) implies that, in the limit,

$$\lim_{\sigma_1^2 \to 0} \mathbb{E}[p_2] = \mathbb{E}[\mathbb{E}[\exp(v_2) \mid i_1, ..., i_N, \bar{s}^2, \bar{s}^1]];$$

here v_2 is the valuation of the agent who observed the *second* maximum over $(t_1, ..., t_N)$. Yet, since $\beta \to 0$, it follows that $t_n \to i_n'$, and thus agent 2 also has the second-highest valuation. ■

Proof of Proposition 6. The proof is similar to the proof of Proposition 5. We shall use all of the definitions and arguments from that proof but extend them to establish:

$$\lim_{\sigma_2^2 \to 0} \mathbb{E}[p_2] = 0.$$

In the limit $\sigma_2^2 \to 0$, we have $\beta \to 0$ and $\sigma_{\bar{\imath}} \to 0$. In this limit, we thereby have $\lambda' \to \infty$ and $\kappa' \to 1$ (see (54) and (55)). Note that $t_1 - t_2$ is always positive because agents are relabeled such that this stipulation is satisfied. Moreover, in the limit $\sigma_2^2 \to 0$ we also have $t_1 - t_2 \to i_1 - i_2$; it follows that $(t_1 - t_2)$ has positive variance. Also, $\mathbb{E}[\exp(v_2) \mid t_1, ..., t_N, \bar{s}^2, \bar{s}^1]$ is finite because this is the expected valuation of the agent with the second-highest equilibrium statistic. Therefore,

$$\lim_{\sigma_2^2 \to 0} \mathbb{E}\big[\exp(-\kappa' \cdot \lambda' \cdot (t_1 - t_2)) \times \mathbb{E}[\exp(v_2) \mid t_1, t_2, ..., t_N, \bar{s}^2, \bar{s}^1]\big] = 0$$

because $\lambda' \to \infty$ and $(t_1 - t_2)$ has positive variance. ■

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Online Appendix to: An Ascending Auction with Multi-dimensional Signals

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This online appendix contains results that are complementary to those in the main paper. It is organized as follows. Section A provides additional results on the baseline model (with two-dimensional signals). Section B shows how to characterize a class of Nash equilibria in an ascending auction for any Gaussian information structure. Section C details how to characterize a class of Nash equilibria in a broader class of mechanisms when agents observe multi-dimensional Gaussian signals. Finally, proofs of the results in this appendix are given in Section D.

A Additional Results from the Baseline Model

Throughout this section, we retain the model described in Section 2. We shall identify conditions that guarantee the existence of a unique equilibrium, and we also provide additional comparative statics.

A.1 Uniqueness of Equilibrium

Here we describe what is required for there to be a unique equilibrium (within the class of equilibria studied in Theorem 1). We do this by stipulating conditions under which the cubic equation (12) has a unique root.

Proposition 7 (Multiplicity of Equilibria).
The auction has multiple equilibria if $18x_3x_2x_1x_0 - 4x_2^3x_0 + x_2^2x_1^2 - 4x_3x_1^3 - 27x_3^2x_0^2 > 0$; *the auction has a unique equilibrium — within the class of equilibria studied in Theorem 1— when that expression is strictly less than* 0 *(here* x_3, x_2, x_1, *and* x_0 *are as defined in* (12)*).*[52]

Proposition 7 characterizes the environments in which the cubic polynomial (12) has multiple roots. This proposition will be used to derive corollaries that are easier to interpret. We show that the following conditions are necessary for there to be multiple equilibria: (*i*) the idiosyncratic shocks are positively correlated; and (*ii*) the noise term in the signal s_n is large but not *too* large.

***(i)* Correlated idiosyncratic shocks.** Multiple equilibria arise only if idiosyncratic shocks are positively correlated ($\text{corr}(i_n, i_m) > 0$).

Corollary 1 (Correlation in Idiosyncratic Shocks).
If the idiosyncratic shocks are independently distributed ($\rho_i = 0$), *then the ascending auction has a unique equilibrium within the class of equilibria studied in Theorem 1.*

Corollary 1 states that correlated idiosyncratic shocks is a necessary condition for there to exist multiple equilibria in the ascending auction. If the idiosyncratic shocks are independently distributed, then agent n's beliefs about c are independent of i_n. Hence there is no complementarity in the weights that agents place on i_n.

[52] The case in which the focal expression is *equal* to 0 must be considered independently; in this case there is a unique equilibrium if and only if $x_2 = 3x_3x_1$.

***(ii)* Intermediate size of the noise terms.** We show that there is a unique equilibrium for large enough σ_ε^2 or small enough σ_ε^2.

Corollary 2 (Uniqueness of Equilibrium)**.**
If either $\sigma_\varepsilon^2 \to 0$ or $\sigma_\varepsilon^2 \to \infty$, then there exists a unique equilibrium within the class of equilibria studied in Theorem 1.

According to this corollary, if the noise term is large enough or small enough then there exists a unique equilibrium. If $\sigma_\varepsilon^2 \to 0$, then agents have complete information about the common shock. This limit corresponds to a private values environment and so there must be a unique equilibrium. If $\sigma_\varepsilon^2 \to \infty$ then agent n ignores s_n; this limit, too, corresponds to a private values environment. Corollary 2 states that multiple equilibria arise only for an intermediate level of noise. However, that statement provides no intuition regarding the magnitudes that should be considered.

Multiple equilibria arise when the signal s_n is noisy enough that agent n does not learn "too much" about c from s_n. At the same time, s_n must be sufficiently precise that the collection of all signals $(s_1, ..., s_N)$ is informative about c. This requirement can be loosely formulated as follows:

$$\text{corr}(s_n, c) \approx 0 \quad \text{and} \quad \text{corr}(\sum_{n \in N} s_n, c) \approx 1. \tag{57}$$

Suppose that (57) is satisfied. In that event, if agent n observes only s_n then she cannot make a precise prediction of c. However, an agent who observed $(s_1, ..., s_N)$ could make a precise prediction of c.

As the number of agents increases, the likelihood that (57) is satisfied for very noisy signals increases. This is because $\text{corr}(\sum_{n \in N} s_n, c)$ converges to 1 as $N \to \infty$. So if we take the limits $N \to \infty$ and $\sigma_\varepsilon^2 \to \infty$ at the appropriate rates, then $\text{corr}(\sum_{n \in N} s_n, c)$ converges to 1 and $\text{corr}(s_n, c)$ converges to 0. This occurs when N grows faster than σ_ε^2.

Corollary 3 (Multiplicity of Equilibria)**.**
If $\sigma_\varepsilon^2 \to \infty$ and $N \to \infty$ (with N diverging faster than σ_ε^2), then there are multiple equilibria if and only if

$$\sigma_c^2 \geq \frac{4(1-\rho_i)}{\rho_i} \sigma_i^2. \tag{58}$$

Corollary 3 shows that it is possible to have multiple equilibria, even in the limit $\sigma_\varepsilon^2 \to \infty$, provided that N grows faster than σ_ε^2. It follows that the model has an interesting discontinuity as we approach large markets ($N \to \infty$). In the limits $N \to \infty$ and $\sigma_\varepsilon^2 \to \infty$, the model does not approach a model of private values. The reason is that, if the number of agents grows faster than σ_ε^2, then the collection of the signals observed by all the agents is still informative about c.

A.2 Comparative Statics with Respect to ρ_i.

Finally, we provide the comparative statics with respect to ρ_i.

Proposition 8 (Comparative Statics with Respect to ρ)**.**
If there is a unique equilibrium statistic, then β increases with ρ_i.

According to Proposition 8, β decreases with the correlation of the idiosyncratic shocks. This relation implies that an auction's efficiency is decreasing in the correlation of idiosyncratic shocks across agents (see note 32).

B General Multi-dimensional Signals

Here we extend the methodology developed in Section 4 to allow for general Gaussian signals. We follow the same approach: after computing an equilibrium statistic, we compute the equilibrium as if agents observe only that statistic.

B.1 Payoff Environment and Information Structure

As in Section 2, we study N agents bidding for an indivisible good in an ascending auction. The utility of agent $n \in N$ if she wins the object at price p is given by

$$u(v_n, p) \triangleq \exp(v_n) - p, \tag{59}$$

where $v_n \in \mathbb{R}$ is a payoff shock. The utility of an agent who does *not* win the good is set equal to 0. Agent n observes J signals,

$$\mathbf{s}_n = (s_n^1, ..., s_n^J),$$

where, as before, bold fonts denote vectors. The joint distribution of signals and payoff shocks is jointly Gaussian and may be asymmetrically distributed. That is, $(v_1, ..., v_N, \mathbf{s}_1, ..., \mathbf{s}_N)$ is jointly normally distributed. The description of the auction is the same as in Section 2, although here we do not restrict attention to equilibria in which agents use symmetric strategies.

B.2 One-dimensional Signals

We begin by studying one-dimensional signals. If agents observe one-dimensional signals and if the average crossing condition is satisfied, then the ascending auction has an efficient ex post equilibrium (see Krishna (2003)). The average crossing condition is defined as follows.

Definition 2 (Average Crossing Condition)**.**
The one-dimensional information structure $(s_1, ..., s_N, v_1, ..., v_N)$ *satisfies the average crossing condition if, for all* $\mathscr{A} \subset \{1, ..., N\}$ *and for all* $n, m \in \mathscr{A}$ *with* $n \neq m$,

$$0 < \frac{\partial \mathbb{E}[v_n \mid s_1, ..., s_N]}{\partial s_m} \leq \frac{1}{|\mathscr{A}|} \sum_{h \in N} \frac{\partial \mathbb{E}[v_h \mid s_1, ..., s_N]}{\partial s_m}.$$

The average crossing condition guarantees that the effect of agent m's signal on agent n's valuation is not too large. The comparison is with respect to the *average* effect that agent m's signal has on any group of agents that contains agent n.

To characterize the equilibrium, we assume that the agents are ordered as follows:

$$\mathbb{E}[v_n \mid s_1, ..., s_N] > \cdots > \mathbb{E}[v_N \mid s_1, ..., s_N]. \tag{60}$$

That is, we assume that agents are ordered according to their expected valuation conditional on the signals of all agents. We define $\tilde{s}_1 \in \mathbb{R}$ as follows:

$$\begin{aligned} \tilde{s}_1 \triangleq \quad & \arg\min_{s' \in \mathbb{R}} \quad && \mathbb{E}[v_1 | s', s_2, ..., s_N] \\ & \text{subject to} \quad && \forall n \in N, \quad \mathbb{E}[v_1 \mid s', s_2, ..., s_N] \geq \mathbb{E}[v_n \mid s', s_2, ..., s_N]; \end{aligned} \tag{61}$$

here $\tilde{s}_1$ is the signal that yields the lowest expected payoff shock to agent 1 while keeping that shock higher than the expected payoff shocks of other agents.

Proposition 9 (Equilibrium for One-dimensional Signals)**.**
The ascending auction with one-dimensional signals has a Nash equilibrium in which agent 1 wins the object and pays the following price:

$$p_2 = \mathbb{E}[v_1 \mid \tilde{s}_1, s_2, ..., s_N]. \tag{62}$$

The ascending auction has an equilibrium in which the agent with the highest expected valuation wins the good. The price paid for that good is the expected valuation of the auction's winner—but evaluated at the *minimum* signal this agent could have observed and still win the good.

B.3 Equilibrium Statistic

As in section 4, our characterization of the equilibrium relies on projecting the signals an agent observes onto a one-dimensional statistic. In equilibrium, agents behave as if they observed only their equilibrium statistic. The only change here is that we must allow for higher-dimensional objects for the weights. Since different signals are generally given different weights, it is convenient to work with vectors. We denote the dot product between $\boldsymbol{\beta} \in \mathbb{R}^J$ and $\mathbf{s}_n \in \mathbb{R}^J$ by $\boldsymbol{\beta} \cdot \mathbf{s}_n$.

We first define an equilibrium statistic for general Gaussian information structures.

Definition 3 (Equilibrium Statistic)**.**
The random variables $\{t_n\}_{n \in N}$ constitute an equilibrium statistic if there exist $(\boldsymbol{\beta}_1, ..., \boldsymbol{\beta}_N) \in \mathbb{R}^{N \times J}$ such that, for all $n \in N$,

$$t_n = \boldsymbol{\beta}_n \cdot \mathbf{s}_n; \tag{63}$$

$$\mathbb{E}[v_n \mid t_1, ..., t_N] = \mathbb{E}[v_n \mid \mathbf{s}_n, t_1, ..., t_N]. \tag{64}$$

This definition of an equilibrium statistic is a natural extension of the main text's Definition 1 but in which we allow for general J-dimensional signals. Note that the weights $\boldsymbol{\beta}_n$ placed on the signals of agent n may differ from the weights $\boldsymbol{\beta}_m$ placed on agent m's signals. Recall that the equilibrium statistic is the fundamental object that allows us to characterize the equilibrium in multi-dimensional environments.

B.4 Equilibrium with Multi-dimensional Signals

We now characterize a class of equilibria in which agents observe multi-dimensional signals. We fix an equilibrium statistic $(t_1, ..., t_N)$, and the agents are ordered as follows:

$$\mathbb{E}[v_1 \mid t_1, ..., t_N] > \cdots > \mathbb{E}[v_N \mid t_1, ..., t_N]. \tag{65}$$

Thus, as before, we assume that agents are ordered according to their expected valuation conditional on the equilibrium statistic of all agents. We define $\tilde{t}_1 \in \mathbb{R}$ as

$$\begin{aligned} \tilde{t}_1 \triangleq \quad & \arg\min_{t' \in \mathbb{R}} && \mathbb{E}[v_1 \mid t', t_2, ..., t_N] \\ & \text{subject to} && \forall n \in N, \quad \mathbb{E}[v_1 \mid t', t_2, ..., t_N] \geq \mathbb{E}[v_n \mid t', t_2, ..., t_N]; \end{aligned} \tag{66}$$

here $\tilde{t}_1$ is the analogue of $\tilde{s}_n$ but uses the equilibrium statistic.

Theorem 2 (Equilibrium for Multi-dimensional Signals)**.**
If the equilibrium statistic $\{t_n\}_{n \in N}$ satisfies the average crossing condition, then the ascending auction has a Nash equilibrium in which agent 1 wins the object and pays a price equal to

$$p_2 = \mathbb{E}[v_1 \mid \tilde{t}_1, t_2, ..., t_N].$$

Theorem 2 characterizes a class of equilibria in which agents behave as if they observe only one-dimensional signals. This is a natural extension of Theorem 1. Our characterization requires that the equilibrium statistic satisfy the average crossing condition. In applications, it is straightforward to check whether the average crossing condition is satisfied.

The same characterization can be applied if we consider an ascending auction with re-entry (see Section C for a formal argument). Allowing for re-entry relaxes the conditions under which an ex post equilibrium exists when agents observe one-dimensional signals (see Izmalkov (2001)).

C Other Mechanisms

We now extend the methodology to find—in a wider group of games—a class of Nash equilibria where agents observe multi-dimensional Gaussian signals. The solution method remains the same. We first project the signals onto a one-dimensional equilibrium statistic, and then we show that an equilibrium exists in which agents behave as if they observe only their respective equilibrium statistics. We emphasize that the definition of an equilibrium statistic does not change.

C.1 General Games

We consider a game with N agents. Agent $n \in N$ takes action $a_n \in A_n$, where A_n is assumed to be a metric space. The payoff for agent $n \in \{1, ..., N\}$ depends on the realization of her payoff shock $v_n \in \mathbb{R}$ and on the actions taken by all agents. Agent n's payoff is denoted by

$$u_n(v_n, a_1, ..., a_N).$$

As before, vectors are denoted in bold font. We define a profile of actions as follows:

$$\mathbf{a} \triangleq (a_1, ..., a_N),$$

and we let $(a'_n, \mathbf{a}_{-n})$ denote the action profile

$$(a_1, ..., a_{n-1}, a'_n, a_{n+1}, ..., a_N).$$

We retain the information structure used in Section B. The definition of an equilibrium statistic is the same as in Definition 3.

We distinguish between the payoff environment and the information structure. This distinction is necessary because we want to compute the set of equilibria for a fixed payoff environment but under different information structures. The payoff environment P consists of the actions available to each agent and their utility functions. The information structure $\mathscr{I}$ is the joint distribution of signals and payoff shocks. The game is defined by the payoff environment and the information structure—that is, by $(P, \mathscr{I})$. Given an equilibrium statistic $(t_1, ..., t_N) \in \mathbb{R}^N$, the information structure in which agent n observes only t_n is the *reduced-form* information structure $\widehat{\mathscr{I}}$.

In game $(P, \mathscr{I})$, a strategy for agent n is defined by a function $\alpha_n : \mathbb{R}^J \to A_n$; in game $(P, \widehat{\mathscr{I}})$, a strategy for agent n is a function $\hat{\alpha}_n : \mathbb{R} \to A_n$. Define the strategy profile

$$(\boldsymbol{\alpha}(\mathbf{s})) \triangleq (\alpha_1(\mathbf{s}_1), ..., \alpha_N(\mathbf{s}_N)).$$

We denote by $(a_n', \boldsymbol{\alpha}_{-n}(\mathbf{s}_{-n}))$ the strategy profile in which all agents play according to $(\boldsymbol{\alpha}(\mathbf{s}))$ *except* for agent n, who takes action a_n' for all realizations of the signals that she observes. That is,

$$(a_n', \boldsymbol{\alpha}_{-n}(\mathbf{s}_{-n})) \triangleq \big(\alpha_1(\mathbf{s}_1), ..., \alpha_{n-1}(\mathbf{s}_{n-1}), a_n', \alpha_{n+1}(\mathbf{s}_{n+1}), ..., \alpha_N(\mathbf{s}_N)\big).$$

C.2 Solution Concepts

Throughout the paper we worked with Nash equilibrium as our solution concept; this is one of the most widely used solution concept in game theory (thus, there is no need for a discussion). However, in this section, we work with solution concepts that are stronger than that of a Nash equilibrium; doing so will allow us to provide sharper results. The use of stronger solution concepts also allows us to sharpen the intuitions of the results. We next define formally a posterior equilibrium.

Definition 4 (Posterior Equilibrium)**.**
A strategy profile $(\alpha_1, ..., \alpha_N)$ forms a posterior equilibrium if—for all agents $n \in N$, all signal realizations $(\mathbf{s}_1, ..., \mathbf{s}_N) \in \mathbb{R}^J$, and all actions $a_n' \in A_n$—the following inequality holds:

$$\mathbb{E}[u_n(v_n, \boldsymbol{\alpha}(\mathbf{s})) \mid \mathbf{s}_n, \boldsymbol{\alpha}(\mathbf{s})] \geq \mathbb{E}[u_n(v_n, (a_n', \boldsymbol{\alpha}_{-n}(\mathbf{s}_{-n}))) \mid \mathbf{s}_n, \boldsymbol{\alpha}(\mathbf{s})]. \tag{67}$$

In a posterior equilibrium (whose definition is due to Green and Laffont (1987)), the strategy of agent n remains optimal even if she knew the actions taken by all other agents. As compared with a Nash equilibrium, here the information set with respect to which the action must be optimal is augmented. Thus the action taken by agent n remains optimal even if he knew the action taken by other agents. We observe that, if a strategy profile is an posterior equilibrium, then it is also a Nash equilibrium.

It is convenient to compare the posterior equilibrium with the ex post equilibrium. A strategy profile $(\alpha_1, ..., \alpha_N)$ forms an ex post equilibrium if—for all agents $n \in N$, all signal realizations $(\mathbf{s}_1, ..., \mathbf{s}_N) \in \mathbb{R}^J$, and all actions $a_n' \in A_n$—the following inequality holds:

$$\mathbb{E}[u_n(v_n, \boldsymbol{\alpha}(\mathbf{s})) \mid \mathbf{s}_1, ..., \mathbf{s}_N] \geq \mathbb{E}[u_n(v_n, (a_n', \boldsymbol{\alpha}_{-n}(\mathbf{s}_{-n}))) \mid \mathbf{s}_1, ..., \mathbf{s}_N]. \tag{68}$$

In an ex post equilibrium, agent n's action is optimal even if she knew the realization of the signals of all other agents. This definition of ex post equilibrium is standard in the literature.[53]

The difference between posterior equilibria and ex post equilibria is the amount of information with respect to which a strategy is optimal. In other words, the difference lies in the conditioning variables appearing in (68) and (67). The action taken by agent n is less informative than the signals that she observes. As a consequence, if an equilibrium is an ex post equilibrium then it is also a posterior equilibrium.

C.3 General Characterization of Equilibria

We now show how to compute a class of posterior equilibria in the game $(P, \mathscr{I})$. For this purpose, we establish an equivalence between ex post equilibria in game $(P, \widehat{\mathscr{I}})$ and posterior equilibria in game $(P, \mathscr{I})$.

Theorem 3 (Equivalence)**.**
Let $(\boldsymbol{\beta}_1 \cdot \mathbf{s}_1, ..., \boldsymbol{\beta}_N \cdot \mathbf{s}_N) \in \mathbb{R}^N$ *be an equilibrium statistic, and let the strategy profile* $\{\hat{\alpha}_n\}_{n\in N}$ *be an ex post equilibrium in game* $(P, \widehat{\mathscr{I}})$*. Then the strategy profile* $\{\alpha_n\}_{n\in N}$*, defined as*

$$\alpha_n(\mathbf{s}_n) = \hat{\alpha}_n(\boldsymbol{\beta}_n \cdot \mathbf{s}_n), \tag{69}$$

is a posterior equilibrium in game $(P, \mathscr{I})$*.*

Theorem 3 shows that equilibria can be computed using a two-step procedure. The first step is using (64) to find the one-dimensional equilibrium statistic. The second step is to compute a posterior equilibrium "as if" agents observed only the equilibrium statistic.

In order to characterize a posterior equilibrium when agents observe multi-dimensional signals, the mechanism must have an ex post equilibrium when agents observe only the equilibrium statistic. Yet, because the equilibrium statistic is endogenous, applying our paper's methodology requires the mechanism to have an ex post equilibrium for a broad class of one-dimensional signals. For example, if a *direct revelation mechanism* has an ex post equilibrium under a specific one-dimensional signal then, in general, this same mechanism (i.e., this mapping from messages to outcomes) may no longer have an ex post equilibrium under a different joint distribution of signals and payoff shocks. Therefore, the methodology can

[53] The notion of an ex post equilibrium has been studied in various contexts in many papers; for a discussion, see Bergemann and Morris (2005).

be applied to a large class of indirect mechanisms that have an ex post equilibrium—that is, irrespective of the information structure's precise description. If a mechanism has an ex post equilibrium for some but not for all one-dimensional information structures, then we must check whether the mechanism has an ex post equilibrium when agents observe only the equilibrium statistic.

D Proofs of the Results in the Online Appendix

Proof of Proposition 7. A standard property of cubic polynomials is that they have a unique root if and only if their discriminant is positive. For (12) this reduces to the condition in Proposition 7, which proves the result.■

Proof of Corollary 1. We show this by proving that, if $\rho_i = 0$, then the discriminant of the polynomial (12) is negative (see Proposition 7). If $\rho_i = 0$, then

$$x_3 = \frac{\sigma_\varepsilon^2 + N \cdot \sigma_c^2}{\sigma_i^2 \sigma_c^2}; \quad x_2 = \frac{-1}{\sigma_i^2}; \quad x_1 = \frac{\sigma_\varepsilon^2 + \sigma_c^2}{\sigma_\varepsilon^2 \sigma_c^2}; \quad x_0 = \frac{-1}{\sigma_\varepsilon^2}. \tag{70}$$

Substituting this into $18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2$ now yields

$$\frac{1}{\sigma_\varepsilon^6 \sigma_i^6 \sigma_c^8} \Bigg(-4\sigma_\varepsilon^8 \sigma_i^4 + \sigma_\varepsilon^2 \sigma_i^2 \sigma_c^6 \left(\left(-27N^2 + 18N + 1\right) \sigma_c^2 - 4(3N+1)\sigma_i^2 \right)$$

$$-4\sigma_\varepsilon^6 \sigma_i^2 \sigma_c^2 \left((N+3)\sigma_i^2 + 2\sigma_c^2 \right) - 4\sigma_\varepsilon^4 \sigma_c^4 \left((9N-5)\sigma_i^2 \sigma_c^2 + 3(N+1)\sigma_i^4 + \sigma_c^4 \right) - 4N\sigma_i^4 \sigma_c^8 \Bigg)$$

Because $N \geq 1$ it is easy to check all terms are negative, from which it follows that the ascending auction has a unique equilibrium.■

Proof of Corollary 2. We prove the result using the characterization in Proposition 7. In the limit $\sigma_\varepsilon^2 \to 0$, we have $x_0 \to \infty$ and $x_1 \to \infty$. The term that dominates is clearly $-4x_1^3 x_3$ and so, in the limit, the discriminant is negative. Hence the cubic polynomial has a unique root. In the limit $\sigma_\varepsilon^2 \to \infty$, we have $x_3 \to \infty$. Here the term that dominates is clearly $-27x_3^2 x_0^2$ and so, in the limit, the discriminant is negative. Therefore, the cubic polynomial has a unique root.

Proof of Corollary 3. This result is also proved using the characterization in Proposition 7.

We take the limits $\sigma_\varepsilon^2 \to \infty$ and $N \to \infty$ (with N diverging faster than σ_ε^2). In the limit $N \to \infty$,

$$x_3 = \frac{1}{(1-\rho_i)\rho_i}\frac{1}{\sigma_i^2}.$$

Considering this value for x_3, we obtain

$$\begin{aligned}\lim_{\sigma_\varepsilon^2\to\infty} \quad & 18x_3x_2x_1x_0 - 4x_2^3x_0 + x_2^2x_1^2 - 4x_3x_1^3 - 27x_3^2x_0^2 \\ = \quad & \frac{-4(1-\rho_i)^2\cdot\sigma_i^4 - 4(1-\rho_i)\rho_i\cdot\sigma_i^4 + \rho_i\cdot\sigma_i^2\sigma_c^2}{(1-\rho_i)\rho_i\cdot\sigma_i^4\sigma_c^6}. \qquad (71)\end{aligned}$$

We can see that (71) is positive if and only if

$$\sigma_c^2 \geq \frac{4(1-\rho_i)}{\rho_i}\sigma_i^2.$$

The result follows after re-arranging terms. ■

Proof of Proposition 8. Define the polynomial

$$q(\beta,\rho_i) \triangleq x_3\cdot\beta^3 + x_2\cdot\beta^2 + x_1\cdot\beta + x_0, \qquad (72)$$

where x_3, x_2, x_1, x_0 are as defined in (12) (note that these coefficients depend on ρ_i). Let $\beta^*(\rho_i)$ be the unique root of (72). It is easy to check that

$$\frac{\partial\beta^*(\rho_i)}{\partial\rho_i} = -\frac{\frac{\partial q(\beta^*(\rho_i),\rho_i)}{\rho_i}}{\frac{\partial q(\beta^*(\rho_i),\rho_i)}{\beta}}. \qquad (73)$$

If $q(\rho_i)$ has a unique root, then $q(\rho_i)$ is increasing at this root and so the denominator of (73) is positive. Later we will check that $q(\rho_i)$ is decreasing in ρ_i, in which the numerator of (73) is negative. So if $q(\beta,\rho_i)$ has a unique root, then this root is increasing in ρ_i.

Proof that $q(\beta,\rho_i)$ is decreasing in ρ_i. We first provide bounds on the value of the root $\beta^*(\rho_i)$. On the one hand, it is easy to see that

$$\beta > \frac{(1+(N-1)\rho_i)\sigma_c^2}{\sigma_\varepsilon^2 + N\sigma_c^2} \implies x_3\cdot\beta^3 + x_2\beta^2 > 0.$$

On the other hand,

$$\beta > \frac{\sigma_c^2}{\sigma_\varepsilon^2+\sigma_c^2} \implies x_1\cdot\beta + x_0 > 0.$$

Hence the root must satisfy the following inequality:

$$\beta^*(\rho_i) \leq \max\left\{\frac{(1+(N-1)\rho_i)\sigma_c^2}{\sigma_\varepsilon^2 + N\sigma_c^2}, \frac{\sigma_c^2}{\sigma_\varepsilon^2+\sigma_c^2}\right\}.$$

To see that $q(\beta, \rho_i)$ is decreasing in ρ_i, note that

$$\frac{\partial q(\beta,\rho_i)}{\partial \rho_i} = \beta^2\left(\frac{(-(N-1)(1-\rho_i)+(1+(N-1)\rho_i))}{(1-\rho_i)^2(1+(N-1)\rho_i)^2}\frac{(\sigma_\varepsilon^2 + N\cdot\sigma_c^2)}{\sigma_i^2\sigma_c^2}\beta - \frac{1}{(1-\rho_i)^2\sigma_i^2}\right). \quad (74)$$

If (74) is negative when evaluated at

$$\beta = \frac{(1+(N-1)\rho_i)\sigma_c^2}{\sigma_\varepsilon^2+N\sigma_c^2} \quad \text{and} \quad \beta = \frac{\sigma_c^2}{\sigma_\varepsilon^2+\sigma_c^2}, \quad (75)$$

then it will continue to be negative at the root β^*. This conclusion follows because the term inside the large parentheses in (74) is an affine function of β with a negative intercept. Therefore, the term inside those parentheses either is an increasing function of β or is negative for all positive values of β. So if the term inside parentheses is negative when evaluated at (75) then it will continue to be negative at the root β^* (because that root is smaller than both terms in (75)).

If we plug $\beta = \frac{(1+(N-1)\rho_i)\sigma_c^2}{(\sigma_\varepsilon^2+N\sigma_c^2)}$ into (74) and simplify terms, we obtain

$$-\beta^2\frac{N-1}{(1-\rho_i)\sigma_i^2((N-1)\rho_i+1)},$$

which is negative. If we plug $\beta = (1+(N-1)\rho_i)\sigma_c^2/\sigma_\varepsilon^2 + N\sigma_c^2$ into (74) and simplify terms, then the result is

$$-\beta^2\frac{(N-1)\big((N-1)(\rho_i^2\sigma_\varepsilon^2+(\rho_i-1)^2\sigma_c^2)+\sigma_\varepsilon^2\big)}{(\rho_i-1)^2\sigma_i^2((N-1)\rho_i+1)^2(\sigma_c^{2\,2}+\sigma_\varepsilon^2)},$$

which is also negative. Thus (74) is negative when evaluated at the root $\beta^*(\rho_i)$. Hence $q(\rho_i)$ is decreasing in ρ_i, which proves the result. ■

Proof of Proposition 9. Krishna (2003) (Theorem 2) shows that the ascending auction has an efficient ex post equilibrium. Therefore, our next step is to prove that the price paid in the efficient ex post equilibrium is (62)—yet this is a standard result in the literature (see e.g. Ausubel (1999)). Finally, Perry and Reny (1999) give a revenue equivalence theorem for ex post equilibria; in particular, if two mechanisms implement the same allocation as an ex post equilibrium, then the payments must be the same. It follows that (62) must also be the payment in the ascending auction's outcome, which proves the result. ■

Proof of Theorem 2. This is a direct corollary of Theorem 3 and the fact that the equilibrium characterized in Proposition 9 is an ex post equilibrium. ■

Proof of Theorem 3. It is clear that, for any equilibrium statistic, the joint distribution of the random variables $(v_1, ..., v_N, \mathbf{s}_1, ..., \mathbf{s}_N, t_1, ..., t_N)$ is jointly normally distributed. We first list the proof's main steps and then explain each one in detail. If $\hat{\alpha}_n : \mathbb{R} \to A_n$ is an ex post equilibrium of game $(P, \widehat{\mathcal{I}})$, then

$$\forall(n \in N, \boldsymbol{t} \in \mathbb{R}^N, a'_n \in A_n), \quad \mathbb{E}[u_n(\hat{\alpha}(t_n), \hat{\alpha}(t_{-n}), v_n) \mid \boldsymbol{t}] \geq \mathbb{E}[u_n(a'_n, \hat{\alpha}(t_{-n}), v_n) \mid \boldsymbol{t}] \tag{76}$$

$$\Rightarrow \forall(n \in N, \boldsymbol{t} \in \mathbb{R}^N, \mathbf{s}_n \in \mathbb{R}^J, a'_n \in A_n), \quad \mathbb{E}[u_n(\hat{\alpha}(t_n), \hat{\alpha}(t_{-n}), v_n) \mid \boldsymbol{t}, \mathbf{s}_n] \geq \mathbb{E}[u_n(a'_n, \hat{\alpha}(t_{-n}), v_n) \mid \boldsymbol{t}, \mathbf{s}_n] \tag{77}$$

$$\Rightarrow \forall(n \in N, \boldsymbol{t} \in \mathbb{R}^N, \mathbf{s}_n \in \mathbb{R}^J, a'_n \in A_n), \quad \mathbb{E}[u_n(\hat{\alpha}(t_n), \hat{\alpha}(t_{-n}), v_n) \mid \mathbf{s}_n, \hat{\alpha}_1(t_1), ..., \hat{\alpha}_N(t_N)] \geq \mathbb{E}[u_n(a'_n, \hat{\alpha}(t_{-n}), v_n) \mid \mathbf{s}_n, \hat{\alpha}_1(t_1), ..., \hat{\alpha}_N(t_N)] \tag{78}$$

$$\Rightarrow \alpha^* : \mathbb{R}^J \to M \text{ with } \alpha^*(\mathbf{s}_n) = \hat{\alpha}(t_n) = \hat{\alpha}(\boldsymbol{\beta}_n \cdot \mathbf{s}_n) \text{ a posterior equilibrium of game } G. \tag{79}$$

Step (76)**.** This follows from the definition of ex post equilibria in the game $(P, \widehat{\mathcal{I}})$.

Step (77)**.** First, observe that the expectations are over the random variable v_n. We must therefore prove that,

$$\forall(\boldsymbol{t} \in \mathbb{R}^N, \mathbf{s}_n \in \mathbb{R}^J), \quad v_n|_{\boldsymbol{t}} = v_n|_{\boldsymbol{t},\mathbf{s}_n}.$$

That is to say, the distribution of v_n conditional on $\boldsymbol{t}$ is the same same as the conditional distribution of v_n conditional on $\mathbf{t}$ and $\mathbf{s}_n$. Because the random variables are normally distributed, it suffices to prove that

$$\forall(\boldsymbol{t} \in \mathbb{R}^N, \mathbf{s}_n \in \mathbb{R}^J), \quad \mathbb{E}[v_n \mid \boldsymbol{t}] = \mathbb{E}[v_n \mid \boldsymbol{t}, \mathbf{s}_n]; \tag{80}$$

$$\forall(\boldsymbol{t} \in \mathbb{R}^N,\, \mathbf{s}_n \in \mathbb{R}^J), \quad \mathrm{var}(v_n \mid \boldsymbol{t}) = \mathrm{var}(v_n \mid \boldsymbol{t}, \mathbf{s}_n). \tag{81}$$

Here (80) is true by the definition of an equilibrium statistic and (81) holds because the variables are jointly Gaussian, from which it follows that

$$\mathrm{var}(v_n \mid \boldsymbol{t}) = \mathrm{var}(v_n) - \mathrm{var}(\mathbb{E}[v_n \mid \boldsymbol{t}]) = \mathrm{var}(v_n) - \mathrm{var}(\mathbb{E}[v_n \mid \boldsymbol{t}, \mathbf{s}_n]) = \mathrm{var}(v_n \mid \boldsymbol{t}, \mathbf{s}_n).$$

Step (78)**.** Note that $\hat{\alpha}_n(t_n)$ is measurable with respect to t_n. Therefore,

$$\mathbb{E}[u_n(a'_n, \hat{\alpha}(t_{-n}), v_n) \mid \boldsymbol{t}, \mathbf{s}_n] = \mathbb{E}[u_n(a'_n, \hat{\alpha}(t_{-n}), v_n) \mid \mathbf{s}_n, \boldsymbol{t}, \hat{\alpha}_1(t_1), ..., \hat{\alpha}_N(t_N)]; \tag{82}$$

$$\mathbb{E}[u_n(\hat{\alpha}(t_n), \hat{\alpha}(t_{-n}), v_n) \mid \boldsymbol{t}, \mathbf{s}_n] = \mathbb{E}[u_n(\hat{\alpha}(t_n), \hat{\alpha}(t_{-n}), v_n) \mid \mathbf{s}_n, \boldsymbol{t}, \hat{\alpha}_1(t_1), ..., \hat{\alpha}_N(t_N)]. \tag{83}$$

That is, we can add $\hat{\alpha}_n(t_n)$ as conditioning variable. Hence (77) can be written as follows:

$$\mathbb{E}[u_n(\hat{\alpha}(t_n), \hat{\alpha}(t_{-n}), v_n) \mid \mathbf{s}_n, \boldsymbol{t}, \hat{\alpha}_1(t_1), ..., \hat{\alpha}_N(t_N)] \geq \mathbb{E}[u_n(a'_n, \hat{\alpha}(t_{-n}), v_n) \mid \mathbf{s}_n, \boldsymbol{t}, \hat{\alpha}_1(t_1), ..., \hat{\alpha}_N(t_N)].$$

Taking the expectation of this expression conditional on $(\mathbf{s}_n, \hat{\alpha}_1(t_1), ..., \hat{\alpha}_N(t_N))$ and then using the law of iterated expectations, we can write

$$\mathbb{E}[u_n(\hat{\alpha}(t_n), \hat{\alpha}(t_{-n}), v_n) \mid \mathbf{s}_n, \hat{\alpha}_1(t_1), ..., \hat{\alpha}_N(t_N)] \geq \mathbb{E}[u_n(a'_n, \hat{\alpha}(t_{-n}), v_n) \mid \mathbf{s}_n, \hat{\alpha}_1(t_1), ..., \hat{\alpha}_N(t_N)].$$

Hence, we prove the step.

Step (79)**.** The equality $\boldsymbol{\beta}_n \cdot \mathbf{s}_n = t_n$ yields the definition of posterior equilibria.Thus we have proved (76)–(79), which completes the proof of Theorem 3. ■

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