Risk Classification in Insurance Markets with Risk and Preference Heterogeneity*

Vitor Farinha Luz[†], Piero Gottardi[‡] and Humberto Moreira[§]

April 21, 2021

Abstract

We consider a competitive model of insurance provision where consumers are privately informed about their risk level and preferences. The presence of two-dimensional heterogeneity introduces novel distribution effects absent from the one-dimensional model typically studied in the literature. In this environment we investigate the welfare effects of the use of demographic characteristics in pricing (risk classification) and the effect of changes to the risk distribution. To this end we analyze the effects of the public release of a signal that is informative about individual risk and show that this improves welfare of all consumer types if, and only if the signal structure satisfies a monotonicity property. We also analyze the effects of changes in the risk distribution in the population and show that an increase in risk, according to the monotone likelihood ratio property (MLRP), leads to higher prices for all coverage levels and lower welfare for all consumer types, while increases in risk in the sense of first order stochastic dominance (FOSD) can be beneficial for some consumers.

*We thank seminar participants at the 2017 and 2019 SAET meetings, INFORMS 2019, FGV/EPGE, UBC, Penn State, University of Pittsburg/Carnegie Mellon, 2019 Southern Economic Association meeting, the 2019 LACE-LAMES and 42^o Meeting to the Brazilian Econometric Society. We are also in debt with Andrea Attar, Eduardo Azevedo, Carlos da Costa, Daniel Gottlieb, Li Hao, Pierre-André Chiappori, Wei Li, Michael Peters, Bernard Salanié, Sergei Severinov and André Trindade for useful discussions. Moreira thanks CNPq and Faperj for financial support. This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001. This research was also supported by the Social Sciences and Humanities Research Council of Canada.

†University of British Columbia. email: vitor.farinhaluz@ubc.ca

‡University of Essex. email: piero.gottardi@essex.ac.uk

§FGV EPGE Brazilian School of Economics and Finance. email: humberto.moreira@fgv.br

1 Introduction

Risk classification is a natural consequence of profit maximization by insurance providers in competitive markets. It consists of using individual information to predict or assess the risk level of potential insurees, which may involve both medical information (e.g, pre-existing conditions or medical history) as well as demographic characteristics (e.g., income, age or gender). As highlighted in Handel et al. (2015) (henceforth HHW), allowing for more precise risk classification has the potential of reducing the information asymmetry between consumers and firms and hence alleviate the problem of adverse selection, but at the same time it implies that consumers with different observables may be offered very different premia for the same coverage. This dispersion in prices and, as a consequence, in consumption may have adverse effects on welfare. Hence, a trade-off arises between reducing adverse selection and increasing price dispersion. A few seminal papers have brought to the data the analysis of this tradeoff (see Finkelstein et al. (2009) and HHW for the study of gender-based and age-based discrimination, respectively - considering environments where only one or two coverage levels are available for trade in the market).[1] A theoretical understanding of this trade-off, particularly when we have a large degree of heterogeneity in the population and where all possible coverage levels are available for trade, is still very limited.[2] In this case a significant amount of screening occurs in the market equilibrium and as long as that is not is perfect, the richness of the effects of the use of observable characteristics in pricing comes from the changes in the risk levels of the different types pooling on each distinct coverage level.

In this paper, we study a competitive insurance market in which consumers are privately informed about both their risk level and risk preferences, the two main dimensions of heterogeneity emphasized in the empirical literature (see, for example, Cohen and Einav (2007)). More specifically, consumers have normally distributed losses and constant absolute risk aversion (following Einav et al. (2013) and Azevedo and Gottlieb (2017) – henceforth AG, among others). Consumers' private information concerns both the expected value of the loss they face (capturing their risk level), which has a continuous distribution, and their risk aver-

[1] In particular, HHW focus on a market with two exogenous coverage levels and a mandate and compare numerically, for a distribution of risk calibrated on the level of welfare attained when firms have no information about consumers' risks (referred to as community rating) with the one attained when instead –the information asymmetry between firms and consumers is completely removed (perfect health-based pricing).

[2] See Garcia and Tsur (2021) for a more complete theoretical analysis of an insurance market where a single coverage option is available and a regulator can design an informative signal about risks that is disclosed to firms. In this case, no screening can occur in the market, all agents active in the market trade the same insurance contract, as in Akerlof (1970)'s model.

sion, which has a binary distribution. The difference between the two levels of risk aversion is a key parameter of our model, which captures the amount of preference heterogeneity.

We follow Dubey and Geanakoplos (2002), Bisin and Gottardi (2006) and, more recently, AG in assuming that consumers and firms act as price takers, and firms have consistent beliefs regarding the types of consumers trading any contract which may be offered. In equilibrium all contracts that are traded generate zero profits. We differ from the previous literature mentioned above, by allowing for a continuum of different coverages.

Besides being in line with the empirical literature, the presence of two-dimensional private information is instrumental to the study of risk classification, as it allows for non-trivial effects of the risk distribution on equilibrium outcomes even when contracts providing all coverage levels can be offered. When this happens, in the case of one-dimensional heterogeneity in risk levels considered in a large part of the literature on insurance markets with adverse selection following Rothschild and Stiglitz (1976), competitive equilibria lead to full separation, which implies that the price of each contract is determined by the unique risk type choosing it and, as a consequence, only depends on the risk distribution through its support (see Dubey and Geanakoplos (2002) and AG).[3] A direct implication is that risk classification may become irrelevant: if the set of possible risk levels for both men and women in the population are the same, the introduction of gender-based pricing has zero effect.

On the other hand, in the presence of multiple dimensions of private information, consumers of different types may share the same willingness to pay for coverage at the margin and, as a consequence, equilibria may feature pooling. As each insurance contract is priced based on the average risk level of consumers choosing it, any change in the type distribution has a direct effect on prices since it affects the relative frequency of the different types purchasing the same contract. It also has an indirect effect, as price changes affect contract choices and hence which consumer types are pooled together.

Our main results are the following. First, we show that equilibria always exist where a convex set of contracts is traded and exhibit the following pattern of trades: extreme coverage levels are chosen only by one consumer type (that is, exhibit separation), while each intermediate coverage level is selected by more than consumer type (partial pooling occurs) (Section 3). The equilibrium is not unique, a common feature when some pooling

[3] The same outcome is also derived as a free-entry equilibrium in Rothschild and Stiglitz (1976), the unique pure-strategy equilibrium outcome in a game-theoretic version of that model (Farinha Luz (2017)), and as a directed search equilibrium in the presence of search frictions (Gale (1992), Guerrieri et al. (2010)). While a competitive equilibrium always exists, free-entry equilibria and pure strategy equilibria fail to exist for some risk distributions.

occurs. We present then a method to obtain meaningful comparative static results in spite of the presence of equilibrium multiplicity and clean analytical expressions for the impact of changes in the type distribution on equilibrium prices and allocations by focusing on the case of small preference heterogeneity (Section 4). Third, we investigate the effect of risk classification by studying the effects on the equilibrium outcome of the availability of a public signal that is informative about consumers' risk and identify a novel property of signal structures, referred to as monotonicity, that is necessary and sufficient for the disclosure of the signal to be beneficial for all types of consumers (Section 5). Finally, we study the effects on equilibrium allocations and welfare of more general changes in the distribution of risk in the market (Section 6).

More specifically, in Section 3 we show that the set of coverages traded in equilibrium can be partitioned into three subsets. Contracts with high or with low levels of coverage feature separation, with each contract in these two regions is purchased by a single type, respectively, with high risk and high risk aversion, or with low risk and low risk aversion. Contracts with intermediate levels of coverage are purchased instead by two types (an outcome referred to as discrete pooling), which have different levels of risk but have the same marginal willingness to pay for coverage. Finally, at the boundary between each of the two separating regions and the discrete pooling region there is (at most) one coverage level featuring continuous pooling, i.e., a contract that is chosen by a positive mass of types. We also show that a continuum of equilibria of this kind exist.

In section 4, we focus on the case where preference heterogeneity is small, in contrast to risk heterogeneity. We characterize how equilibrium prices and allocations vary as a function of the level of preference heterogeneity and show that a well defined approximation of the equilibrium exists, in the sense of which exhibits the important property of being independent of the equilibrium selection. In the remainder of the paper, we focus on the case of small, yet positive and fixed, preference heterogeneity. These approximation results are used to obtain novel comparative statics results that are robust to equilibrium selection.

Section 5 studies the price and welfare effects of the disclosure of a public signal that provides some information to firms regarding consumers' risk level. For example, if the signal corresponds to certain demographic characteristic of consumers, the disclosure of this information to firms will not affect any consumer's assessment of his own risk but will allow firms to learn something about the agent's type. The availability of this signal leads to market segmentation, since firms will treat agents differently according to their public signal realization. We analyze the welfare impact of the signal release from an interim perspective, by comparing the utility of any consumer's risk-and-preference type in equilibrium without and with public signal, taking for the latter the expectation across signal realizations.

Our main result identifies a property of signal structures, monotonicity, that is necessary and sufficient for its release to be interim welfare improving, for any type distribution. Monotonicity is expressed in terms of the Kullback-Leibler (KL) divergence measure, capturing relative entropy. A signal structure is monotonic if the probability distributions over signal realizations conditional on two different risk levels become more "distinct" - in the sense of the KL measure - as the difference between the two risk levels increases. For binary signals, monotonicity is equivalent to the MLRP property of the signal distribution and requires that the probability of each signal realization, conditional on risk, be monotonic on risk]. For richer signal structures, monotonicity is shown to be weaker than MLRP. Monotonicity can be verified empirically as long as sufficient information is available regarding the joint distribution of risks and signals (or characteristics) and hence can be useful in practice.

To gain some intuition for our finding, consider a binary informative signal structure and a level of coverage x_H close to full coverage. The disclosure of this signal leads to two distinct prices for x_H: one signal realization leads to a lower price, which we call the good signal, and the other leads to a higher price, which we call the bad signal. Now consider a low-risk consumer who originally chose a level of coverage below x_H and exhibits so a level of risk below that purchasing x_H. Monotonicity of the signal structure implies that this consumer type receives the good signal with higher probability than all those consumers. Since the price of coverage x_H is lower in the market for consumers who received a good signal, incentive compatibility requires that the prices for lower coverage contracts must also be lower, which benefits the low-risk consumer under consideration. We refer to this indirect channel through which the distribution affects prices as the top-down property of prices. Monotonicity guarantees that such an argument can be applied to almost all consumers.

Section 6 studies then the comparative statics properties of equilibria with respect to changes in the type distribution in the population. We focus on changes to the type distribution that increase risk in the population in the sense while maintaining the support the same (that is, feature larger mass on riskier types). Note that such changes would have no effect in the standard one-dimensional model. This question is also relevant for the analysis of the effects of risk classification. Consider an insurance market that is segmented on the basis of an observable binary characteristic, with one group having more risk - in the above sense - than the other. The consequence of precluding risk classification based on this characteristic is that these two segmented markets would be merged into one, with the new merged distribution determining equilibrium prices. Would consumers from the riskier group benefit from this intervention?

The answer depends on the ordering used to compare group riskiness. First, we look at comparisons using monotone likelihood ratio (MLRP) and find that a MLRP increase in the risk distribution of the population leads to an increase of prices for almost all coverage levels and, as a consequence, almost all consumers are worse off. We then look at the weaker notion of first order stochastic dominance (FOSD) and show, through an example, that a FOSD increase in the risk distribution may be beneficial for a positive mass of consumers. This is a surprising result and hinges on the nature of pooling that arises in our model.

Finally, we also study changes in the distribution of risk preferences in the population and show that an increase in the share of high-risk-aversion agents in the population leads to lower equilibrium prices for almost all contracts and higher utility levels for almost all types. Note that both changes in the distribution of types in the population in the sense of increasing risk or risk aversion can be interpreted as positive demand shocks (for insurance) and our results show that the effects of such a shock depends critically on its source: a positive demand shock induced by a (MLRP) increase in risk leads to higher prices for insurance, while a positive demand shock induced by an increase in the share of risk averse consumers leads to lower prices.

Related literature

This paper related both to the empirical insurance literature as well to the theoretical work on competitive price taking model and multidimensional incentive theory.

HHW study risk classification in health insurance, allowing for consumers to be heterogeneous with respect to their level of risk and preferences. Their model considers contracts with two exogenously given coverage levels, being motivated by health care exchanges under the U.S. Affordable Care Act. Our model allows for richer screening of types with a continuum of possible coverage levels, and our results show that the finer separation of types across contracts lead to novel risk classification and comparative static results.

Using data from a single large employer, they estimate the joint distribution of demographic characteristics, preferences and risk levels in the population and perform numerical simulations to analyze multiple counterfactuals. Crucially, welfare is evaluated in ex-ante terms, looking at the expected discounted utility of a 25-year old agent over the life cycle and hence incorporates future potential health and characteristics transitions. Our characterization differs in considering an interim welfare criterion, and characterizing signals which are interim welfare improving for any type distribution.

In particular they consider: (i) "perfect" discrimination of consumers and (ii) the use of age-contingent pricing. Exercise (i) can be seen as fully revealing the private information of consumers and lead to gains from larger coverage which are dominated by the cost of premium uncertainty. Exercise (ii) partially removes information asymmetry regarding risks since agents with same age are not explicitly differentiated. In this case, their model predicts complete unraveling (i.e., all consumers purchasing low coverage). With age-based pricing, premia are increasing with age which might be beneficial since young consumers who might be borrowing constrained have reduced payments and more liquidity. Our model, in contrast, obtains explicit conditions on the joint distribution of age, interpreted as a signal, and risk types to insure that their disclosure is interim welfare improving.

Several other papers have studied the issue of discriminatory pricing risk in insurance markets. Crocker and Snow (1986) study the payoff set generated from incentive compatible and budget-balanced allocations with and without the presence of a costless public informative signal. While it is easy to see that the introduction of such signal weakly enlarges the feasible payoff set, they show that an imperfect signal always generates some Pareto optimal payoffs that are not feasible in the absence of the signal and that categorical discrimination, together with an adequately designed system of transfers, can be welfare improving.

Rothschild (2011) considers a model where allowing for firms to use a costly informative signals and a well-designed government intervention, in the form of a supplemental insurance policy, is beneficial: it leads to strict Pareto improvements or has no effect at all. Both papers differ from our analysis in several technical aspects, but mainly by allowing for more general government interventions.

Finkelstein et al. (2009) study the effects of gender-based pricing in the UK annuity market, where consumers may be privately informed about their survival probabilities (risk types). They use survival data to calibrate a model with binary risk types, treating gender as a partially informative signal about risk. This calibration is used to perform counter-factual analysis, focusing on constrained-efficient allocations (one of which is the MWS allocation). Their analysis shows that the presence of screening allows firms to inefficiently separate consumers with different characteristics, which substantially weakens the ability of price restrictions to redistribute across genders.

The presence of multidimensionality of consumer heterogeneity has been established more broadly in several papers involving structural demand estimation in these markets. See Cohen and Einav (2007) and Einav et al. (2013) for examples. While these papers provide rich estimates of the sources of consumer heterogeneity and drivers of demand, they do not use equilibrium analysis to evaluate counterfactuals.

On the theoretical literature, our paper is related to the very broad competitive screening literature following the works of Rothschild and Stiglitz (1976); Miyazaki (1977); Riley (1979). Our contribution follows a branch of the literature using models of price-taking agents to study adverse selection in such markets, such as Dubey and Geanakoplos (2002); Bisin and Gottardi (2006); Guerrieri et al. (2010); Azevedo and Gottlieb (2017). Our equilibrium concept coincides with the ones proposed in the more recent contributions of Dubey and Geanakoplos (2002) and AG, who use perturbation arguments to justify certain restrictions on prices of non-traded contracts. Equilibrium outcomes may be constrained efficient and are coincide with Riley (1979)'s outcome in the one-dimensional version of our model. Guerrieri et al. (2010) study a model of competitive search and show that search frictions may lead to the same restrictions on the prices of non-traded contracts and, as a consequence, also obtain Riley (1979)'s equilibrium outcome in equilibrium. Finally, Guerrieri and Shimer (2014) and Chang (2018) study competitive search models with multi-dimensional private information. In both models, one can define a one-dimensional sufficient statistic for agents' types, i.e., an exogenously determined real function of agents' types which summarizes agents' willingness to trade. This property allows for preferences to be rewritten as a function of a one-dimensional parameter. Preferences in our model do not satisfy this property: the relevance of each dimension for marginal willingness to trade is endogenous as it depends on equilibrium objects.

A small branch of this literature has studied two-dimensional heterogeneity models through the lens of different strategic equilibrium concepts and illustrate that equilibria may have qualitative differences relative to the fully separating equilibrium outcome from RS. Wambach (2000) considers a two-by-two model with heterogeneity in wealth and risk, using the same no-entry equilibrium concept as RS. Welfare analysis is hindered by the fact that equilibria might not exist. The paper shows that partial risk pooling can arise in equilibrium, as in our model, when heterogeneity in risk levels is sufficiently high. Smart (2000) also considers a two-by-two model with both risk and risk aversion heterogeneity, while using the reactive equilibrium concept of Riley (1979). The paper also shows that agents with different risk levels may not be perfectly separated in equilibrium, which is in line with our characterization, and show that a unique reactive equilibrium exists. Both papers are silent on the question of risk classification and comparative statics with respect to the type distribution.

Laffont et al. (1987) and Deneckere and Severinov (2015) study a monopoly model where a buyer has two-dimensional private information and menus of quantity-price pairs can be offered. The preferences considered here coincide with the quadratic case in their models, but they assume private values. The revenue maximizing allocation features pooling, where consumers with same marginal willingness to pay are pooled together and, in contrast to the competitive model, may feature exclusion.

2 Model

A continuum of agents (insurees or consumers) face income uncertainty due to the possibility of negative shocks. More specifically, suppose that agents have income W and can suffer a loss $\widetilde{l}$ distributed according to the normal distribution $N(\mu, 1)$. Agents can purchase insurance contracts which are characterized by a pair $(x, p) \in [0,1] \times \mathbb{R}_+$, with $x \in (0,1)$ denoting the insurance coverage (i.e., the fraction of the loss reimbursed) and $p \geq 0$ the premium paid.[4] Agent's preferences are described by a constant absolute risk aversion (CARA) formulation with parameter $\rho > 0$. Hence, the expected utility of choosing contract (x, p) is given by[5]

$$\begin{aligned} v(x, p; \mu, \rho) &\equiv \mathbb{E}\left\{-\exp\left[-\rho\left(W - p - (1-x)\widetilde{l}\right)\right]\right\} \\ &= -\exp\left\{-\rho\left[W - (1-x)\mu - \frac{\rho}{2}(1-x)^2 - p\right]\right\}, \end{aligned} \tag{1}$$

where the second equality is a simple consequence of the exponential structure of both the CARA utility and the normal distribution and is frequently used in the insurance literature (see Einav et al. (2013); Azevedo and Gottlieb (2017), for example).

We assume that agents are privately informed about both their risk level μ and their level for risk aversion ρ, which together constitute their types. Types are randomly distributed, with the set of possible risk levels being denoted by $[\mu_L, \mu_H]$ and, for tractability, risk aversion takes one of two values in $\{\rho_l, \rho_h\}$. As we suggested in the introduction, we propose

[4] Our model assumes that policy reimbursements are linear in losses and do not allow for randomizations. In the context of principal-agent moral hazard problem where average loss is under control of the consumer, Holmstrom and Milgrom (1987) show that the optimality of linear contracts. It is beyond the scope of this paper to show that the linearity is without loss in this adverse selection competitive setting. Allowing for randomizations over coverage levels could potentially allow for fully separating incentive compatible allocations, but the restriction to deterministic coverage is natural in applications.

[5] The model can be extended to allow for private information about loss variance, as well as its mean. If $\widetilde{l} \sim N(\mu, \sqrt{\sigma})$, it can be shown that agents utility generated by any contract (p, x) only depends on types σ and ρ through the product $\sigma\rho$. In which case we can, without loss, consider $\sigma = 1$ with the parameter ρ representing the product of risk aversion and risk variance.

a methodology to study models with two-dimensional heterogeneity by approximating the simplified one-dimensional risk model. Hence the continuous-discrete type model is a parsimonious model to accomplish this task. We will often describe the pair of risk aversion levels by the midpoint level ρ_0 and a risk preference heterogeneity level $\delta > 0$, which means that $\rho_l \equiv \rho_0 - \delta/2$ and $\rho_h \equiv \rho_0 + \delta/2$. Hence, the type space is $\Theta \equiv [\mu_L, \mu_H] \times \{l, h\}$ and types are denoted by $\theta = (\mu, i)$.

Using expression (1), consumer preferences over contracts can be represented by their certainty equivalent associated with contract (x, p), which is given by

$$u(x, \theta) - p, \tag{2}$$

where $u(x, \theta) = x\mu - \frac{\rho_i}{2}(1-x)^2$. From now on, we work directly with this linear-quadratic utility specification.

The insurees' marginal rate of substitution between coverage and price, which determines their willingness to pay for coverage, is

$$u_x(x, \theta) = \mu + \rho_i (1-x), \tag{3}$$

where u_x stands for the partial derivative notation.

From the marginal utility expression (3), an agent's high willingness to pay for coverage can be due to a high risk level (high μ) or to a high risk aversion level (high ρ). Importantly, the degree to which risk aversion affects one's willingness to pay depends on the (endogenous) level of coverage. For example, risk aversion has a very small effect on one's marginal willingness to pay for coverage if the starting point is close to full coverage.[6] As a consequence, the pools of agent's types are endogenously determined in equilibrium.

Firms are risk neutral and their expected profit of offering contract (x, p) to an agent with parameters $\theta = (\mu, i)$ is given by the price minus the cost of coverage provision for risk type:

$$p - c(x, \theta),$$

where $c(x, \theta) = \mu x$. This means that risk aversion ρ is a private-value component of the private information while risk μ is a common-value component of the private information

[6] As mentioned in the introduction, this feature distinguishes our model from Guerrieri and Shimer (2014) and Chang (2018), where a one-dimensional sufficient statistic for types' effect on willingness to trade exists. More precisely, it is impossible to find functions $g : \Theta \mapsto \mathbb{R}$ and $\widetilde{u} : [0,1] \times \mathbb{R}$ such that (i) $u(x; \theta) = \widetilde{u}(x; g(\theta))$, for all $\theta \in \Theta$ and $x \in [0, 1]$, (ii) $\widetilde{u}(\cdot)$ satisfies the single-crossing property.

of an individual as it affects the expected costs of firms providing insurance to him. The number of firms is irrelevant, as they are price takers and have constant returns to scale in insurance provision.

We consider a competitive equilibrium notion in the Walrasian sense, meaning that all individuals and firms take prices for all contracts as given. A price function $p : [0,1] \to \mathbb{R}_+$ specifies the cost of each contract in equilibrium.[7] An allocation is a measurable function $t : \Theta \to [0,1]$ representing a selection of agents' optimal coverage choices, i.e., $t(\theta)$ solves

$$U(\theta) \equiv \max_{x \in [0,1]} u(x,\theta) - p(x), \text{ for all } \theta \in \Theta. \tag{4}$$

Define $\mathbb{P}$ as the implied joint distribution over $[0,1] \times \Theta$ induced by the prior distribution over types and the allocation, and denote the expectation with respect to this measure as $\mathbb{E}[\cdot]$. The expected profit maximization of each firm has a solution only if no contract generates positive profits. Additionally, any contract that is indeed offered by a firm must make zero expected profits, i.e.,

$$0 = p(x) - x\mathbb{E}[\widetilde{\mu} \mid t(\widetilde{\mu},\widetilde{\rho}) = x] \geq p(\widehat{x}) - x\mathbb{E}[\widetilde{\mu} \mid t(\widetilde{\mu},\widetilde{\rho}) = \widehat{x}], \tag{5}$$

for all $x, \widehat{x} \in t(\Theta)$.

In resume, the equilibrium definition that follows requires equilibrium price and allocation to satisfy consumer optimality and price consistency, which is equivalent to say that beliefs must be consistent with Bayesian updating. Perfect competition among firms then implies zero profits. However, these requirements on beliefs are too weak as they allow for an infinite number of equilibria even in the benchmark case $\delta = 0$. This occurs since no constraint is imposed on beliefs regarding contracts that are not traded in equilibrium. We follow Dubey and Geanakoplos (2002); Bisin and Gottardi (2006); Guerrieri et al. (2010) in using an additional restriction on prices that imposes restrictions on non-traded contracts in the tradition of the belief refinement literature (see also Azevedo and Gottlieb (2017)).

Definition 1. A triple of price, allocation and belief $(p, t, \mathbb{P})$ is an equilibrium if it satisfies:
(1) firm's optimality: the condition (5) holds;
(2) agent's optimality: $t(\theta)$ solves (4);
(3) price consistency for non-traded contracts: if $p(x) > 0$, then

$$p(x) = \max\{u(x;\theta) - U(\theta) \mid \theta \in \Theta\}, \tag{6}$$

[7] With some abuse of notation, we use p to denote both the price realization and the price function.

where $U(\theta)$ is defined in (4) and[8,9]

$$\frac{p(x)}{x} \leq \inf\left\{\mu \in [\mu_L, \mu_H] \mid (\mu, \rho) \in \theta^+(x)\right\}, \tag{7}$$

where $\theta^+(x) \equiv \{\theta \in \Theta \mid U(\theta) = u(x, \theta) - p(x)\}$ is the set of types indifferent between to trade their chosen contracts and contract x, where the posterior conditional distribution $\mathbb{P}$ is supported on $\theta^+(x)$.

Conditions (1) and (2) are fairly standard. Condition (3), however, is more subtle as it also applies to non-traded contracts. This condition rules out trivial equilibria in which any subset of contracts remains non-traded as their prices are set at an arbitrarily high level. The right-hand side of equation (6) represents the highest price such that some agent types are indifferent between trading such contract and the one they choose in equilibrium. On the other hand, condition (7) states that at such price if firms were to deviate and offer the contract they would make negative or at most zero profits. This condition ensures that markets for contracts that are not traded clear with a zero level of trades, so that no firm wants to deviate and offer such contracts. The beliefs used by firms to evaluate profits for these contracts attribute positive weight only to the consumers with types in $\theta^+(x)$, which are the ones most willing to buy such contracts (see Guerrieri et al. (2010); Azevedo and Gottlieb (2017)).

3 Equilibrium characterization

In this section, we characterize equilibrium allocation and prices, starting from the benchmark one-dimensional case ($\delta = 0$) and extending it subsequently to the environment with preference heterogeneity $\delta > 0$.

One-dimensional types

Suppose that there is no risk aversion heterogeneity (i.e., $\delta = 0$). In this case, a unique equilibrium exists and it only depends on the support of the risk distribution (for detailed proofs of this case, see Dubey and Geanakoplos (2002); Azevedo and Gottlieb (2017)). It

[8] We use the following the convention $\sup \emptyset = 0$.

[9] If the infimum in condition (7) were substituted by an supreme, the set of equilibria would be potentially larger. In our setting, however, this change would make no difference as the set $\theta^+(\cdot)$ is a singleton for almost all non-traded contracts.

is easy to show that equilibria are necessarily separating. We refer to the risk level of the unique type purchasing an traded coverage $x \in [0,1]$ as the type assignment function $m_0(x)$.

Conditions (1) and (2) in Definition 1 imply the following:

$$\frac{p(x)}{x} = m_0(x) \text{ and } \dot{p}(x) = \frac{\partial u}{\partial x}(x,\mu,0)\bigg|_{\mu=m_0(x)} = m_0(x) + \rho_0(1-x), \tag{8}$$

where from now on we will use a dot to denote the derivative of functions that depend on just one variable. The first equation is simply a zero-profit condition, while the second equation corresponds to the consumers' local optimality condition. Plug the first into the second equation, we can derive a first-order ordinary differential equation (ODE) on the type assignment function $m_0(\cdot)$. Moreover, the riskiest type in the support must buy full coverage. Hence, the unique equilibrium involves a type assignment equal to the solution to this ODE with final condition $m_0(1) = \mu_H$ given by

$$m_0(x) = \rho_0(1 - x + \ln x) + \mu_H, \tag{9}$$

for all $x \in [x_L, 1]$, where x_L is lowest traded coverage satisfying $m_0(x_L) = \mu_L$. The equilibrium allocation $t_0(\cdot)$ is the inverse of $m_0(\cdot)$ and the price function is given by $p_0(x) = x m_0(x)$.

Two-dimensional types

Let us turn back to the case with two levels of risk aversion. Types $\theta = (\mu, i)$ are distributed over Θ according to probability distribution described by a pair of strictly positive and twice continuously differentiable functions $\phi = (\phi_l, \phi_h)$ on $[\mu_L, \mu_H]$, i.e., for any measurable set $A \subset [\mu_L, \mu_H]$

$$\mathbb{P}[(\mu, i) \in A \times \{l, h\}] = \int_{\mu_L}^{\mu_H} \mathbf{1}_{\{\widetilde{\mu} \in A\}} \phi_i(\widetilde{\mu})\, d\widetilde{\mu}.$$

We now proceed to the characterization of equilibria. Accordingly to the one-dimensional case, let us introduce the following useful notation. For a given allocation $t : \Theta \to [0,1]$, define the type assignment correspondence $m_i^t : [0,1] \rightrightarrows [\mu_L, \mu_H]$ as the generalized inverse mapping of the allocation t, i.e., it satisfies $t(\mu, i) = x \iff \mu \in m_i^t(x)$, for any $x \in [0,1]$ and $i \in \{l, h\}$. We say that this correspondence $m_i(\cdot)$ is non-decreasing in an interval I if, for any $x' < x$ with $x', x \in I$ and $m_i(x), m_i(x') \neq \emptyset$, we have that $\sup m_i(x') \leq \inf m_i(x)$. Additionally, we say that $m_l(x) \geq m_h(x)$ if $\sup m_l(x) \geq \sup m_h(x)$ and $\inf m_l(x) \geq \inf m_h(x)$.

Whenever this set is a singleton, we will refer to it as a type assignment function. Also, when the allocation we are referring to is well understood, the dependence on t is omitted. The following lemma provides some important properties of the equilibrium allocation and price functions.

Lemma 1. *If* (m_l, m_h) *are equilibrium type assignments, then:*

(i) $m_i(\cdot)$ *is non-decreasing;*

(ii) for any x *such that* $m_l(x), m_h(x) \neq \emptyset$, $m_l(x) \geq m_h(x)$*;*

(iii) p *is an increasing Lipschitz continuous function.*

Lemma 1 (i) and (ii) hold since preferences satisfy the single-crossing property (SCP) on each dimension (either μ or i) individually, holding the other dimension constant. Conditional on one's level of risk aversion (risk level), an increase in the risk level (risk aversion) increases one's willingness to pay for coverage. However, the two-dimensional type space does not satisfy globally the single-crossing property, i.e., one can find types (μ, i) and (μ', i') with indifference curves that cross twice. Lemma 1 (iii) follows directly from the equi-Lipschitz property of $u(\cdot, \mu, i)$ (for details, see Azevedo and Gottlieb (2017) for a similar result in their more general framework).

No gap equilibria We restrict attention to no-gap equilibria defined as equilibria in which the set of traded contracts is a convex set, i.e., an interval. In Subsection 3.1 (Proposition 2) we show that no-gap equilibria exist when preference heterogeneity is small. Indeed, we show that there are multiple equilibria. In Section 6 we show that comparative static analysis is still possible in the presence of multiplicity.[10] In the benchmark one dimensional model (i.e., $\delta = 0$) only no-gap equilibria exist.

The next result provides a complete characterization of equilibria under small preference heterogeneity. The set of traded coverages is an interval and can be divided into three open sub-intervals and two coverage levels in the transition between them. The two extreme intervals feature separation, while the intermediate coverage levels feature pooling. The highest (lowest) levels of coverage traded in equilibrium feature separation, i.e., each of these contracts is purchased by a single type with high (low) risk-aversion and levels of risk on the right (left) tail of the risk distribution. Coverage levels in the middle open interval are purchased by two types: one with low risk aversion and a higher risk level, and another

[10]Equilibria with gaps in the set of traded-contract have a similar structure to no-gap equilibria characterized in Proposition 1, i.e., the same pattern of pooling and separation of types. The only different is that a set of non-traded coverage levels may exist in the transition between separation and pooling regions.

with high risk aversion and lower risk. We refer to this type of pooling discrete pooling. Finally, the two intermediate coverage levels that determine the boundary between these intervals are purchased by a continuum of types with positive measure, which we refer to as continuous pooling.

Before proving the existence of the equilibrium, let us first show that all equilibria might have the shape we have just described. The focus on small δ in the following proposition comes from our focus on the approximation approach we will use to derive our main results.

Proposition 1. *Suppose that $\delta > 0$ is sufficiently small. In any equilibrium the following properties must hold:*

(i) (separation) there are separating intervals $[x_L, x_d)$ and $(x_u, 1]$ where only low and high risk averse types trade, with $x_d \leq x_u$;

(ii) (continuous pooling) there are non-degenerated intervals of risk types trading at x_d and at x_u;

(iii) (discrete pooling) there are exactly a low and a high risk averse types trading at x, for each $x \in (x_d, x_u)$.

From Proposition 1, there are three possible regions we have to characterize in every no-gap equilibrium: separation, continuous pooling and discrete pooling. According to our equilibrium consistency requirement (3), the belief is determined by the type most willing to trade (i.e., type (μ_L, ρ_l)) and the price of these non-traded contracts (i.e., the interval $[0, x_L)$) makes this type indifferent between trading and not trading such contracts.

Separation

If a contract is traded in a separating region, there exists a unique pair of risk μ and a risk aversion level ρ assigned to it. For the separating region, the equilibrium characterization is analogous to the one-dimensional case. That is, the equilibrium condition is given by (8). Once we establish the initial conditions of the ODE, we can solve for the type assignment and price functions in such region. However, these initial conditions (except for the top interval) are determined by the whole structure of the equilibrium, which depends on the equilibrium type assignment and prices in the other regions.

Continuous pooling

Suppose that contract (p, x) is purchased by a positive mass of types. Lemma 1 and Proposition 1 imply that, for some $i \in \{l, h\}$, coverage x is chosen by all types with risk aversion ρ_i

and risk level in a positive-measure interval $I_i \subset [\mu_L, \mu_H]$. Zero profit condition (5) implies that the price of coverage x is determined by the product of coverage x and the average risk in the pool. Moreover, any type in $I_i \times \{\rho_i\}$ must find it optimal to choose coverage x, and hence their marginal utility of coverage must be between the left and right-derivatives of the price function, i.e., must be in $[\dot{p}(x^-), \dot{p}(x^+)]$.[11] This implies $\dot{p}(x^-) < \dot{p}(x^+)$ and hence the price function must have a kink at point x. Additionally, if the coverage level x is in the interior of the set of traded contracts, the left and right-hand side marginal prices must match exactly the minimum and maximum willingness to pay for coverage among the types pooled at coverage x.[12] Formally, this means:

$$\frac{p(x)}{x} = \sum_{i=l,h} \mathbb{E}\left[\widetilde{\mu} | \widetilde{\mu} \in I_i, \widetilde{i} = i\right] \text{ and } [\dot{p}(x^-), \dot{p}(x^+)] = \{\mu + \rho_i(1-x) | (\mu, i) \in \bigcup_{i\in\{l,h\}} I_i \times \{i\}\}.$$

The presence of continuous pooling is necessary to insure the continuity of prices in the transition between discrete pooling regions, which involve both high and low risk-aversion types, to regions of separation, which only involve types with a single risk-aversion level.[13]

Discrete pooling

Suppose that each type assignment correspondence is single-valued at coverage $x \in [0,1]$. This means that contract (p, x) is traded by only two types: a low risk averse type $(m_l(x), l)$ and a high risk averse type $(m_h(x), h)$. In order to describe the price consistency equilibrium condition (1) we need to characterize the posterior probability of type $(m_l(x), l)$ conditional on coverage x being chosen, which is referred to as the weight function $w(x) \in [0,1]$. As we will see in what follows, this function does not coincide with the marginal prior distribution, plays an important role in the characterization of the equilibrium and determines the effect of the prior distribution on the equilibrium price.

The equilibrium conditions (1) and (2) in Definition 1 imply the price per unit of coverage is the average type in the pool and the marginal price is equal to the marginal utility of the

[11]We use the following convention for the left and right hand side limit of a function $f(x)$: $f(x^-)$ and $f(x^+)$, respectively.

[12]Notice that, for some $i \in \{l, h\}$, there must be a sequence of types $(\mu_n, i)_n$ and coverage levels $(x_n)_n$ such that x_n is optimal for type (μ_n, i), $x_n \searrow x$ and $\mu \equiv \lim \mu_n \in I_i$. Hence $[p(x_n) - p(x)](x_n - x)^{-1} \leq [u(x_n, \mu_n, i) - u(x, \mu_n, i)](x_n - x)^{-1}$. Taking the limit $n \to \infty$ we have that $u_x(x, \mu, i) \geq \dot{p}(x^+)$ and hence this inequality must hold as an equality. An analogous argument holds for coverages below x.

[13]Chang (2018) also obtains continuous pooling as an equilibrium property, even though her model maintains the single-crossing property dropped in our case.

types that are pooling at x:

$$\frac{p(x)}{x} = w(x)m_l(x) + (1 - w(x))m_h(x) \text{ and } \dot{p}(x) = m_i(x) + \rho_i\,(1 - x)\,, \text{ for } i = l, h. \quad (10)$$

The following lemma characterizes the differential equation that the type assignment and weight functions must satisfy to ensure price consistency for traded contracts. It shows that Bayesian updating rule on the discrete pooling region is determined by two components: the prior distribution and the relative curvature of the demand for coverage of both types consuming coverage x. This will play an important role to explain the comparative statics and the signal disclosure exercises in Sections 6 and 5.

Lemma 2. *In a discrete pooling interval, the price consistency condition is equivalent to*

$$\frac{w(x)}{1 - w(x)} = \frac{\phi_l(m_l\,(x))\dot{m}_l\,(x)}{\phi_h(m_h\,(x))\dot{m}_h\,(x)}, \quad (11)$$

for all x where m_l and m_h are differentiable at x.

3.1 Equilibrium existence and multiplicity

We now state our existence result. We show that for sufficiently small dispersion of risk aversion, there exist a continuum of equilibria characterized by Proposition 1.

Proposition 2. *For sufficiently small $\delta > 0$ there exists an equilibrium characterized by Proposition 1. Indeed, there exist a continuum of equilibria.*

The proof[14] of Proposition 2 is based on a constructive argument, whose building blocks are the solutions of ordinary differential equations that describe the equilibrium price and allocations on each separating and discrete pooling regions. The key step in the proof is to guarantee that these solutions are continuous, in particular, across regions. To this end, the transition between separating and discrete pooling regions is given by continuous pooling

[14] Azevedo and Gottlieb (2017) provide an existence result for general multidimensional adverse selection models which encompasses our model. However, their result does not provide any explicit characterization of the equilibrium price and allocations. Their proof is based first on the use of a fixed-point argument to establish the existence of equilibrium prices for a discretized version of the contract space (with the presence of noise traders who have demand for all contracts and for which insurance companies have zero cost of provision). Second the limit is considered as the discretization becomes finer and the noise traders become negligible, so that the equilibria of the perturbed model converge to equilibria of the original one-dimensional model.

intervals constructed so as to satisfy this continuity criterion of the price function. Hence, the price function must have a kink in these transition. Although this continuity principle pins down a solution for the equilibrium construction, it does not deliver a unique solution. Indeed, there is one degree of freedom of how to determine the transition point, whose size depends on the incremental risk aversion parameter (δ).

Figure 1 below illustrates an equilibrium type assignment functions and price computed for the uniform distribution of risk $U([1,5])$ and independent risk aversion with $\rho_0 = 2$ and $\delta = 1$. The parameters $\underline{a}_i, \underline{b}_i, \overline{a}_i, \overline{b}_i$, for $i = d, u$, in the vertical axis determine the bounds of the continuous pooling intervals of the equilibrium; x_d and x_u in horizontal axis are the continuous pooling coverage levels (see the proof of Proposition 2 for more details). It conveys the idea that the unit price per coverage level x is the endogenous average of the low and high risk averse types assigned to x. At the very high and low coverages, the equilibrium is separating and, therefore, coincides with the high and low risk assignment functions. The transition between the separating and the discrete pooling intervals is continuous, but with a kink at the transitions points.

4 Approximation

While the one-dimensional model reviewed in Section 3, is quite tractable, closed formula solutions for two-dimensional models are elusive even in the presence of strong parametric assumptions on preferences and type distributions. In addition, the problem of equilibrium multiplicity established in Proposition 2 makes a meaningful comparative static exercise difficult.[15] Our approach to analyze the properties of equilibria in the two-dimensional model is to look at the one-dimensional model as a limit case, and study the behavior of equilibria in its neighborhood. The advantage of such approach is the ability to use special properties of the one-dimensional model (uniqueness and distribution independence), while still highlighting novel results of the two-dimensional model (including the role played by the type distribution). An important feature of this approach is that it allows to avoid the issues of the multiplicity of equilibria. We show that Taylor approximations of equilibrium price maps, with respect to the parameter $\delta > 0$, are uniquely pinned down in a small neighborhood of $\delta = 0$ and, hence, can be used for comparative statics for $\delta > 0$ sufficiently small. Moreover, we can exploit the closed form solution of the limiting one-dimensional

[15] In Azevedo and Gottlieb (2017), the equilibrium concept studied there does not imply uniqueness. No uniqueness result is provided for the multidimensional model studied numerically.

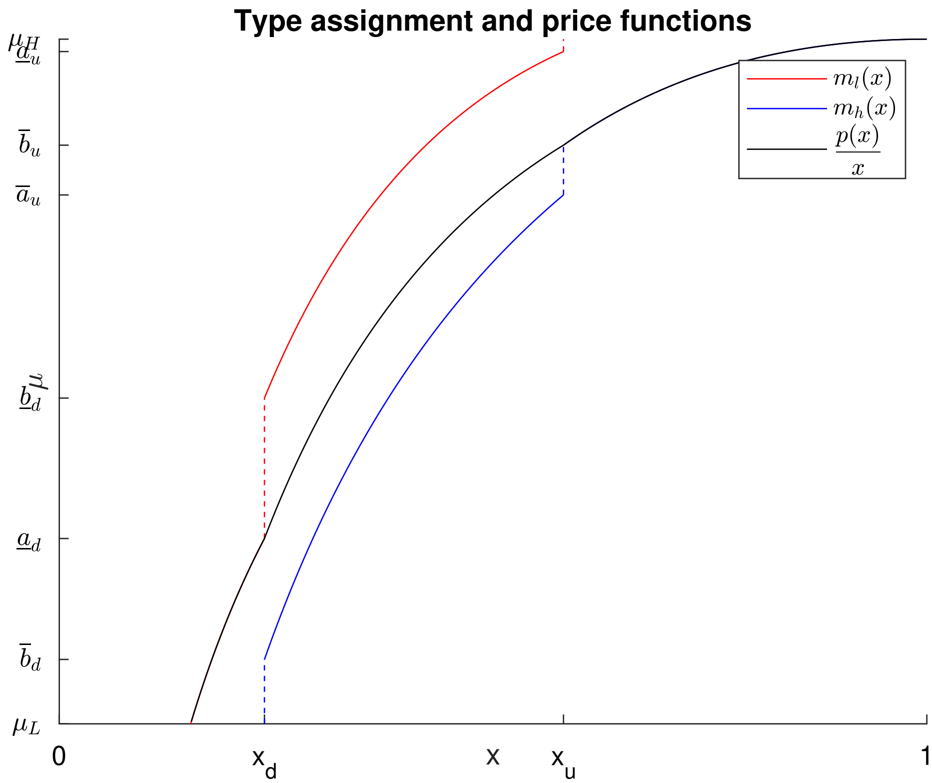

Figure 1: The case of uniform distribution

model to obtain closed form expressions for the Taylor approximation coefficients, which are useful for comparative static exercises. With this trick we are able to overcome the multiplicity issue, which allows us to access the distribution effects not present in the one-dimensional model and without ambiguity.

More precisely, our goal is to study the behavior of the equilibrium outcomes for arbitrarily small δ. Though we will show later (see Proposition 3) how the equilibrium outcomes vary as we modify the level of preference heterogeneity, our main objective is to characterize the effect of changes of the type distribution on market prices and allocations. As discussed in Section 3, these distribution effects are completely absent in the one-dimensional model and novel to the best of our knowledge.

In what follows we prove the convergence and differentiability of any equilibrium selection at $\delta = 0$, formalize the approximation argument described above and apply these results to study the effects of two policy-relevant questions: the effects of changes in the type distribution and of the release of informative signals. We make the dependence on parameter

δ explicit by referring to risk aversion $\rho_i(\delta)$, for $i = l, h$, and equilibrium prices $p(x;\delta)$, for $x \in [0,1]$.

4.1 The heuristic approximation

We start by presenting a heuristic derivation, where we postulate that equilibrium prices are continuously differentiable in the preference heterogeneity parameter, for $\delta > 0$ sufficiently small. This assumption is restrictive. In fact, the equilibrium multiplicity result in Proposition 2 implies that infinitely many non-differentiable equilibrium selections exist. In Subsection 4.2 we show that any equilibrium selection is differentiable with respect to δ at $\delta = 0$, even if not differentiable for $\delta > 0$, and that the approximation terms presented here are valid independently of the equilibrium selection. In practical terms, the heuristic approximation consists in taking directly the total derivative of the equilibrium objects with respect to δ and evaluate them at $\delta = 0$.

Let us consider the case of independently distributed risk and risk aversion, i.e.,

$$\phi_l(\mu) = \omega_0 \phi(\mu) \text{ and } \phi_h(\mu) = (1-\omega_0)\phi(\mu),$$

for density $\phi : [\mu_L, \mu_H] \mapsto \mathbb{R}_{++}$ and some $\omega_0 \in (0,1)$.

Another implicit assumption of the heuristic approach is that as δ goes to zero the discrete pooling region overcomes the whole space, i.e., the separating and continuous pooling regions vanish to zero at a smaller rate than δ. Again, in Subsection 4.2 we show that this assumption is without loss of generality. Hence, plugging the first equation of (10) into the second one, after averaging out over both risk aversion levels, we obtain the following ordinary differential equation on prices:

$$\dot{p}(x;\delta) = \frac{p(x,\delta)}{x} + \overline{\rho}(x;\delta)(1-x),$$

where

$$\overline{\rho}(x;\delta) \equiv w(x;\delta)(\rho_0 - \delta/2) + (1 - w(x;\delta))(\rho_0 + \delta/2).$$

The solution of this ODE with the final condition $p(1;\delta) = \mu_H$ is[16]

$$\frac{p(x;\delta)}{x} = \mu_H - \int_x^1 \left(\frac{1}{z} - 1\right) \overline{\rho}(z;\delta) dz.$$

Notice that the equilibrium price at coverage x depends on a cumulative weighted average

[16]Notice that it corresponds to the one-dimensional equilibrium price (9) when $\delta = 0$, i.e., $\overline{\rho} = \rho_0$.

of the values of risk aversion for all coverage levels above x.

We denoting the first and second order derivatives of prices with respect to δ as $p_\delta(x;\delta)$ and $p_{\delta\delta}(x;\delta)$ respectively, and use the same convention for other equilibrium objects. Direct differentiation then implies

$$\frac{p_\delta(x;\delta)}{x} = \int_x^1 \left(\frac{1}{z} - 1\right)\left[w(z;\delta) - \frac{1}{2} + w_\delta(z;\delta)\delta\right] dz.$$

Therefore, considering the limit $\delta = 0$, we get an expression for the first-order approximation of prices:

$$\frac{p_\delta(x;0)}{x} = \int_x^1 \left(\frac{1}{z} - 1\right)\left(\omega_0 - \frac{1}{2}\right) dz,$$

where, under the assumed independence between risk and risk aversion, $w(x,0) = \omega_0$ is the prior probability of low risk aversion. Hence, the first-order price approximation depends on the relative distribution of risk aversion. A higher relative share of low-risk-aversion agents leads to higher prices: agents with high risk aversion demand higher coverage even when having low risk level, which drives down the average cost/price of such contracts.

Now taking the second-order derivative at $\delta = 0$ we have

$$\frac{p_{\delta\delta}(x;\delta)}{x} = 2\int_x^1 \left(\frac{1}{z} - 1\right) w_\delta(z;0) dz. \tag{12}$$

To compute $w_\delta(z,0)$ we can differentiate equation (11), which leads to (using the notation $\partial_\delta \equiv \frac{\partial}{\partial\delta}$)

$$\begin{aligned}\frac{w_\delta(x;0)}{\omega_0(1-\omega_0)} &= \partial_\delta\left(\log\frac{w(x;\delta)}{1-w(x;\delta)}\right)\\ &= \frac{\dot\phi(m_0(x))}{\phi(m_0(x))}\left[\partial_\delta\left(m_l(x;0) - m_h(x;0)\right)\right] + \frac{\partial_\delta\left(\dot m_l(x;0) - \dot m_h(x;0)\right)}{\dot m_0(x)}.\end{aligned}$$

Using then the fact that $m_l(x;\delta) - m_h(x;\delta) = \delta(1-x)$ (from equation (10)), we get

$$w_\delta(x;\delta) = \omega_0(1-\omega_0)\left[\frac{\dot\phi(m_0(x))}{\phi(m_0(x))}(1-x) - \frac{1}{\dot m_0(x)}\right].$$

Finally, since $\dot{m}_0(x) = \rho_0 \frac{1-x}{x}$, equation (12) can be rewritten as

$$\frac{p_{\delta\delta}(x;\delta)}{x} = 2\rho_0^{-1}\omega_0(1-\omega_0)\left[\int_{m_0(x)}^{\mu_H} \frac{\dot{\phi}(\mu)}{\phi(\mu)}(1-t_0(\mu))\,d\mu - (1-x)\right].$$

The second-order approximation term allows us to study how the risk distribution affects equilibrium prices. With a positive yet small amount of preference heterogeneity, each coverage level is consumed by types with similar, yet distinct, risk levels. The relative frequency of the higher-risk-lower-aversion types to the lower-risk-higher-aversion ones is represented by the rate of growth of the risk density, $\frac{\dot{\phi}(\cdot)}{\phi(\cdot)}$. A higher density rate of growth means that, for any given risk level, there are relatively more types in the population with slightly higher risks. As these two "similar" types are pooled into a single contract, the overall effect on prices is positive. This second-order term is zero in the extreme cases $\omega_0 = 1$ or $\omega_0 = 0$, as the relative share of different risk types is irrelevant in the one-dimensional case.

These results are formally established in Subsection 4.3 (see Corollary 1). First Subsection 4.2 provides the formal derivation of the price approximation terms, relaxing the requirement of continuous differentiability and the assumption of independence of risk and risk preferences. A reader interested purely in the comparative statics exercises obtained from these formulas can skip the next subsection and go directly to Sections 6 and 5 which show how to use these approximation results to evaluate the welfare impact of changes in risk distribution and of signal disclosures.

4.2 Approximation results

In this section, we extend the results in subsection 4.1 to the case of non-independent types and arbitrary (potentially non-differentiable) equilibrium selections. Moreover, we do not assume that the separating and continuous pooling regions vanish, but instead our approximation results imply it. In order to study the equilibrium prices, it is necessary to study the limiting behavior of the endogenous weight function $w(x)$, which represents the share of low risk-aversion agents choosing coverage x. In the limit economy without preference heterogeneity, all consumers purchasing the same coverage level share the same risk level. When preference heterogeneity is positive but small, any coverage with discrete pooling features cross-subsidization between types with almost identical level of risk, i.e., the limiting share of low-risk-preference types within a pool is determined by the ratio:

$$\omega_0(x) \equiv \frac{\phi_l(m_0(x))}{\phi_l(m_0(x)) + \phi_h(m_0(x))},$$

for any $x \in [x_L, 1]$. In the limit $\delta = 0$, agents consuming the same coverage level x have all a risk level equal to $m_0(x)$. And the limiting share of agents with low risk aversion[17], among the ones with risk level $m_0(x)$ is exactly $\omega_0(x)$.

Proposition 2 guarantees the existence of (multiple) equilibria, for all $\delta > 0$ sufficiently small. We now consider an arbitrary equilibrium selection and define the following limits, if they exist,

$$p_\delta(x) \equiv \lim_{\delta \to 0} \frac{p(x;\delta) - p(x;0)}{\delta}$$

and

$$p_{\delta\delta}(x) \equiv 2 \lim_{\delta \to 0} \frac{\frac{p(x;\delta) - p(x;0)}{\delta} - p_\delta(x)}{\delta}.$$

Proposition 3. *Consider an arbitrary equilibrium selection and $\delta > 0$ sufficiently small. For all $x \in (x_L, 1)$, the price and weight functions exhibit the following limiting behavior:*

(a) Convergence: the function $p(x;\cdot)$ is continuous at zero,

$$\lim_{\delta \to 0} \frac{p(x;\delta)}{x} = m_0(x) \ \text{ and } \ \lim_{\delta \to 0} w(x;\delta) = \omega_0(x);$$

(b) Differentiability: the limit $p_\delta(x)$ exists and is given by

$$\frac{p_\delta(x)}{x} = \int_x^1 \left(\omega_0(z) - \frac{1}{2}\right)\left(\frac{1}{z} - 1\right) dz; \tag{13}$$

(c) Second-order differentiability: the limit $p_{\delta\delta}(x_0)$ exists and is given by

$$\frac{p_{\delta\delta}(x)}{x} = 2\int_x^1 w_\delta(z)\left(\frac{1}{z} - 1\right) dz, \tag{14}$$

[17]Although in the limit there is just one risk-preference type, the interpretation is that for each risk type there exactly risk-preference type with the same risk aversion.

where

$$w_\delta(x) = \frac{\dot{\omega}_0(x)}{x\dot{m}_0(x)} p_\delta(x) - \frac{x\omega_0(x)[1-\omega_0(x)]}{\rho_0(1-x)} + (1-x)[1-\omega_0(x)]\omega_0(x)\left\{[1-\omega_0(x)]\frac{\dot{\phi}_l(m_0(x))}{\phi_l(m_0(x))} + \omega_0(x)\frac{\dot{\phi}_h(m_0(x))}{\phi_h(m_0(x))}\right\}. \tag{15}$$

Additionally, all convergence results hold uniformly on any compact subset of $(x_L, 1)$.

Proof. The proof is a combination of the results in Lemmas 17, 18, 19, 22 and 24 in Appendix C. □

The proof of this proposition shows in particular that the approximation of the one-dimensional equilibrium is determined by the approximation on the discrete pooling region of the equilibrium selection, i.e., as δ approaches zero the separating and continuous pooling regions of the equilibrium are of order $o(\delta)$. In the previous section, this property was taken for granted The first part of Proposition 3 shows that prices converge pointwise to its one-dimensional counterpart. This occurs as, when the difference of risk-aversion among agents becomes small, the heterogeneity among buyers choosing the same coverage level disappears. Hence, in the limit, the equilibrium allocation and prices fully separate agents in terms of their risk level. It also determines the limit of the endogenous weight function $w(\cdot)$.

The second part of Proposition 3 is slightly more subtle. It shows that the sign of the first-order approximation of prices does depend on the type distribution through the difference between $\omega_0(x)$ and $\frac{1}{2}$. The reason is that with $\delta > 0$ small, except for very low or very high coverages, all other consumption levels occur in the discrete pooling region where each contract is purchased not by a single risk-type, but by a pair of types: one with low risk and high risk aversion, $(m_h(x), \rho_h)$, and one with high risk and low risk aversion, $(m_l(x), \rho_l)$ (equation (10) then implies $m_l(x) > m_h(x)$). Hence, the effect on the equilibrium prices of some heterogeneity in risk levels depends on which type is more prevalent in the limit. If there are more low risk-aversion agents, since they exhibit a higher risk than the high risk aversion agents with whom they pool (i.e., $\omega_0(x) > \frac{1}{2}$) prices are increased; if otherwise (i.e., $\omega_0(x) < \frac{1}{2}$), the effect is reversed.

The third part of Proposition 3 is even more subtle. It captures how the relative weight assigned to the low risk aversion agents, exhibiting higher risk ($w(x;\delta)$), differs from its limit $\omega_0(x)$ for small parameter $\delta > 0$. If $w_\delta(x) > 0$, the introduction of small preference heterogeneity $\delta > 0$ implies that the share of low-risk-aversion-high-risk types consuming each contract is larger, which drives up prices. The way in which the weights vary with the degree of preference heterogeneity δ depends very finely on the type distribution. Its interpretation is more easily understood when both dimensions are independently distributed, as shown in the following subsection.

For the remainder of the paper, we will use this approximation result to study equilibrium outcomes and interventions of our model, and so we always consider sufficiently small preference heterogeneity $\delta > 0$, such that existence is guaranteed from Proposition 2, and an arbitrary equilibrium selection.

4.3 Independent distribution

Corollary 1. *If risk and risk aversion are independently distributed, then*

$$p_{\delta}(x) \quad = x\left(\omega_0 - \tfrac{1}{2}\right)(x - 1 + \ln x),$$

and

$$\frac{p_{\delta\delta}(x)}{x} = 2\omega_0(1-\omega_0)\left[\int_x^1 (1-z)\left(\frac{1}{z}-1\right)\frac{\dot{\phi}(m_0(z))}{\phi(m_0(z))}dz - \frac{(1-x)}{\rho_0}\right].$$

Proof. By direct substitution in Proposition 3. □

Corollary 1 shows that the risk distribution affects equilibrium prices through the derivative of the log of the density (i.e., increasing rate of density function), which is the key element that we explore in what follows to do comparative statics and signal disclosure analysis. This is due to the fact that, when preference heterogeneity is small, agents pooled together have very similar risk levels and, hence, average prices are determined by the relative likelihood of agents with different, but still quite similar, risk levels. This result suggests that the ordering of distributions in terms of the monotone likelihood ratio property (MLRP) has a central role in this model. This issue is further developed in the next section.

5 Signal disclosure

A central policy question in insurance markets regulation is the extent to which companies should be allowed to discriminate consumers based on observable characteristics. For example, demographic characteristics are useful for firms when pricing insurance contracts as long as they are correlated with consumers' risk, even if they are not direct determinants of risk.

However, the use of additional information in pricing increases the heterogeneity of prices offered to consumers with different observables. We analyze this question by considering the disclosure of an informative signal or observable characteristic which can be used in pricing. If firms observe the realization of this signal, their offers are based on the distribution of types conditional on the signal realization and, as a consequence, equilibrium prices also depend on the signal realization. We focus on pure risk signals, i.e., whose realization is independent of risk preferences, conditional on risk. We assume that the disclosure of the signal realization is a result of individual agents or firms' decisions. If a signal is disclosed, its realization connected with each agent is observed by the agent as well as firms. If a signal is not disclosed, firms cannot price based on this information.

The disclosure or access to new information allows firms to better price discriminate and consumers to make better coverage choices. Our analysis focuses on the first aspect of new information even though the latter is also relevant. On firms' sides, the signal is informative as it is correlated to unobservable risk levels. On consumers' side, we assume that the signal has no direct value, but only matters through its effect on prices. In other words, the signal contains no additional predictive power on final losses relative to risk type μ, which is know by consumers. Since the signal has no direct value to consumers, our positive welfare result (Proposition 4) regarding information disclosure is more surprising.

Our welfare analysis takes an interim perspective: for a fixed type, we consider the expected utility gain from the release of a signal, taking expectations over the set of possible signal realizations. We say that a signal structure is interim Pareto improving if it leads to an expected utility improvement for almost all types. This is the adequate measure of consumer's well-being if consumers do not observe the signal realization prior to the intervention. For example, if the signal is the result of an imprecise health test or, as in HHW, if one evaluates interventions from the point of view of a young agent's expected lifetime utility which is affected by the potential use of individual characteristics that evolve stochastically over time. In many relevant applications, however, these signals are observable prior to the intervention, such as gender-based pricing. In this case, our results characterize the aggregate (utilitarian) welfare gain from all consumers (across men and women) sharing the same risk and preference characteristics, and hence the same equilibrium level of coverage. Any Pareto improvement according to this interim criterion also implies an improvement according to an ex ante criterion. Our main result (Proposition 4) provides necessary and sufficient conditions for a signal structure to be interim Pareto improving.

A signal has finitely many realizations $s \in S$. A signal structure is denoted by a function

$\pi(\cdot \mid \cdot) : S \times [\mu_L, \mu_H] \mapsto [0,1]$ such that $\pi(\cdot \mid \mu) \in \Delta(S)$ for any $\mu \in [\mu_L, \mu_H]$. The value $\pi(s \mid \mu)$ denotes the probability distribution over signal realizations, conditional on risk level μ. We assume that the conditional signal distributions have full support and are continuously differentiable in risk μ. The distribution of types conditional on signal s is then given by

$$\phi_i(\mu \mid s) = \frac{\phi_i(\mu)\,\pi(s \mid \mu)}{\Pi(s)}, \tag{16}$$

for $i = l, h$, where $\Pi(s)$ denotes the ex-ante probability of signal realization s, given by

$$\Pi(s) \equiv \int [\phi_l(\mu')\,\pi(s \mid \mu') + \phi_h(\mu')\,\pi(s \mid \mu')]\,d\mu'.$$

We use superscript s to refer to endogenous equilibrium variables under distribution $(\phi_l(\cdot \mid s), \phi_h(\cdot \mid s))$ (e.g., $p^s(x;\delta)$) and superscript 0 when referring to endogenous equilibrium variables under prior distribution (ϕ_l, ϕ_h) (e.g., $p^0(x;\delta)$).

In what follows to easy the notation we refer to type $i \in \{l, h\}$ instead of type ρ and use it as a subscription. We define the equilibrium payoffs of an agent with type $(\mu, i) \in [\mu_L, \mu_H] \times \{l, h\}$ with preference heterogeneity $\delta > 0$ as, for $k \in \{0\} \cup S$,

$$V_i^k(\mu;\delta) \equiv v\left(t_i^k(\mu;\delta), p^k\left(t_i^k(\mu;\delta);\delta\right); \mu, \rho_i(\delta)\right), \tag{17}$$

where $v(\cdot)$ is defined by (1), and $p^k(\cdot)$ and $t^k(\cdot)$ represent equilibrium prices and allocation respectively.

We say that a signal structure is interim Pareto improving for a given type distribution (ϕ_l, ϕ_h) and equilibrium selection if, for every $\epsilon > 0$, there exists a $\bar{\delta} > 0$ such that

$$\mathbb{P}\left[(\mu, i) \mid \sum_{s \in S} \pi_i(s \mid \mu)\,V_i^s(\mu;\delta) > V_i^0(\mu;\delta)\right] > 1 - \epsilon,$$

for all $\delta < \bar{\delta}$. A signal structure is interim Pareto improving if it is interim Pareto improving for any prior type distributions with full support and equilibrium selection.[18]

Section 5.1 shows that the welfare impact of signal disclosure follows directly from its expected effect on prices. Sections uses this result to provide necessary and sufficient conditions for a signal structure to be interim Pareto improving.

[18]The reason for the ϵ qualification in the definition comes from the fact that equilibrium prices in the transition between the discrete pooling and separating regions cannot be identified. However, the mass of types trading in this region becomes negligible as $\delta \to 0$.

5.1 Welfare and price effects

The disclosure of a signal affects the utility of any given agent as it changes the equilibrium price function and, as a consequence, also affects their choice of coverage. The main result in this section (Lemma 3) shows that the interim expected utility impact of signal disclosure is determined directly by the interim expected price change generated by the signal.

This result is surprising since agents preferences, represented in (1), are risk averse and the disclosure of a signal introduces a new source of uncertainty coming from its realization and its impact on equilibrium prices.

Different realizations of the signal will affect both the price $p\left(t\left(\mu,\rho_i;\delta\right);\delta\right)$ (price effect) and the equilibrium quantity $t\left(\mu,\rho_i;\delta\right)$ consumed by a particular type (allocation effect). Intuitively, Lemma 3 follows from two observations: (i) that the price effect of signal disclosure dominates the allocation effect, and (ii) that the price effect of a signal is determined solely by the expected price effect of the disclosure of the signal.

Observation (i) follows from an envelope-type argument, which implies that the price impact of shock is of higher order of magnitude than the allocation effect, since agents are optimizing over their coverage amount.

Observation (ii) is more subtle. The disclosure of the signal introduces a lottery into agents' final net consumption. Since equilibrium outcomes in the case of no preference heterogeneity ($\delta = 0$) do not depend on the disclosure of the signal, the impact of signal disclosure is *small* for small preference heterogeneity δ. And since the signal is assumed to not provide additional payoff relevant information to consumers, the realization of the signal is *independent* of their risky losses, conditional on their type. But an agent with a risky consumption level evaluates small lotteries with realizations independent from their initial consumption according to their expected value.[19]

Additionally, Proposition 3 shows that the limit and first order approximation of equilibrium prices as $\delta \to 0$ are independent of the risk distribution, which means that for any $s \in S$,

$$\lim_{\delta\to 0} p^s\left(x;\delta\right) = \lim_{\delta\to 0} p^0\left(x;\delta\right),$$

and

$$p^s_\delta\left(x\right) = p^0_\delta\left(x\right).$$

This means that the expected price impact of signal disclosure can be approximated by its

[19]For any differentiable Bernoulli $\upsilon : \mathbb{R} \mapsto \mathbb{R}$, independent random variables x and y, and $\varepsilon > 0$ $\mathbb{E}\left[\upsilon\left(x+\varepsilon y\right)\right] = \mathbb{E}\left[\upsilon\left(x\right)\right] + \varepsilon\mathbb{E}\left[\upsilon'\left(x\right)\right]\mathbb{E}\left[y\right] + o\left(\varepsilon\right)$.

second order approximation, i.e.,

$$\sum_{s\in S}\mu\left(s\mid\mu\right)p^{s}\left(t_{i}^{s}\left(\mu;\delta\right);\delta\right)-p^{0}\left(t_{i}^{0}\left(\mu;\delta\right);\delta\right)=\frac{\delta^{2}}{2}\Delta E\left[p\left(t_{0}\left(\mu\right)\right)\right]+o\left(\delta^{2}\right),$$

where

$$\Delta E\left[p\left(x\right)\right]\equiv\sum_{s\in S}\pi\left(s\mid m_{0}\left(x\right)\right)\left[p_{\delta\delta}^{s}\left(x\right)-p_{\delta\delta}^{0}\left(x\right)\right] \tag{18}$$

is the expected price change of coverage x.

The following result combines these arguments formally.

Lemma 3. *The welfare effect of signal disclosure on an agent with type* (μ, i) *satisfies*

$$\sum_{s\in S}\pi\left(s\mid\mu\right)V_{i}^{s}\left(\mu;\delta\right)-V_{i}^{0}\left(\mu;\delta\right)=$$
$$\frac{\delta^{2}}{2}\frac{\partial v}{\partial p}\left(t_{0}\left(\mu\right),p_{0}\left(t_{0}\left(\mu\right)\right),\mu,\rho_{0}\right)\Delta E\left[p\left(t_{0}\left(\mu\right)\right)\right]+o_{i}\left(\delta^{2};\mu\right), \tag{19}$$

where $\delta^{-2}\left|o_{i}\left(\delta^{2};\cdot\right)\right|$ *converges uniformly to* 0 *on* M*, for any compact set* $M\subset\left(\mu_{L},\mu_{H}\right)$*.*

Proof. See Appendix C. □

5.2 Welfare improving signals

The main result of this section shows that signals are interim Pareto improving if, and only if they satisfy an informativeness condition referred to as monotonicity. The information content of a signal structure is determined by how the signal distribution varies with changes in risk level. Intuitively, monotonicity means that consumers with more distant risk levels must generate more "distinct" signal distributions. Hence, to define monotonicity we need to specify a notion of comparison between two distributions over signal realizations. For any two distributions $\pi,\widetilde{\pi}\in\Delta\left(S\right)$, the Kullback–Leibler (KL) divergence of $\widetilde{\pi}$ from π is defined as

$$D_{KL}\left(\pi||\widetilde{\pi}\right)\equiv\sum_{s}\pi\left(s\right)\ln\left(\frac{\pi\left(s\right)}{\widetilde{\pi}\left(s\right)}\right).$$

This measure is also referred to as relative entropy and is an asymmetric measure of distribution discrepancy.[20] It becomes standard entropy when $\widetilde{\pi}$ is the uniform distribution over S. It is always non-negative and equals to zero if and only if $\pi=\widetilde{\pi}$.

[20]Information theory (see, for instance, Kraft (1949) and McMillan (1956)) defines this measure as the expected number of extra bits that would be required to code the information if one were to use $\widetilde{\pi}$ instead of

We say that a signal is monotonic if the gap between the signal distributions, as measured by the KL divergence, changes monotonically with the risk level of consumers. Formally,

Definition 2. A signal structure $(\pi(s|\cdot))_{s\in S}$ is monotonic if, for any $\mu \in [\mu_L, \mu_H]$,

$$D_{KL}(\pi(\cdot \mid \mu) \,||\, \pi(\cdot \mid \widetilde{\mu}))$$

is strictly increasing in $\widetilde{\mu}$ for $\tilde{\mu} > \mu$.

Alternatively, monotonicity means that for any three ordered risk levels $\mu_1, \mu_2, \mu_3 \in [\mu_L, \mu_H]$ such that $\mu_1 < \mu_2 < \mu_3$ we must have

$$0 < D_{KL}(\pi(\cdot \mid \mu_1) \,||\, \pi(\cdot \mid \mu_2)) < D_{KL}(\pi(\cdot \mid \mu_1) \,||\, \pi(\cdot \mid \mu_3)).$$

The first inequality simply states that the signal realization allows one to distinguish between any two risk types μ_1 and μ_2, i.e., the signal is informative. The second inequality states that, since the pair of risk levels (μ_1, μ_3) is more dissimilar, they should generate signal distributions that "diverge" more from each other, when compared to pair (μ_1, μ_2).

Monotonicity is a generalization of common restrictions on the signal structure present in contract theory, such as monotone likelihood ratio ordering, as highlighted in Section 5.4. Figure 2 illustrates a non-monotonic binary signal structure, where one realization of the signal has a higher likelihood for extreme risk levels while the other has higher likelihood for intermediary risk levels. We now state our main result, which relates monotonicity to interim welfare improvements.

Proposition 4. *A signal structure is interim Pareto improving if and only if it is monotonic.*

Proof. From Lemma 3 and equation (14) we have that the expected price change of coverage x (18) satisfies

$$\Delta E[p(x)] \quad = 2\int_x^1 \sum_{s\in S} \pi(s \mid m_0(x))\left[w_\delta^s(z) - w_\delta^0(z)\right]\left(\tfrac{1}{z} - 1\right) dz, \tag{20}$$

where the expressions of $\omega_\delta^s(\cdot)$ and $\omega_\delta^0(\cdot)$ are given by (15).

π. In Economics this notion has been used in several applications. For instance, Cabrales et al. (2017) use it to define the normalized value of an information purchase that induces a complete ranking of information structures. Galichon and Salanié (2010) use it to estimate the socially optimal matching distribution when the available date can be so poor that the econometrician observes only that marginal distributions of the observable types of a man and a woman. Differently from these works, the relative entropy emerges naturally in our analysis, i.e., it is not imposed as a measure of informativeness.

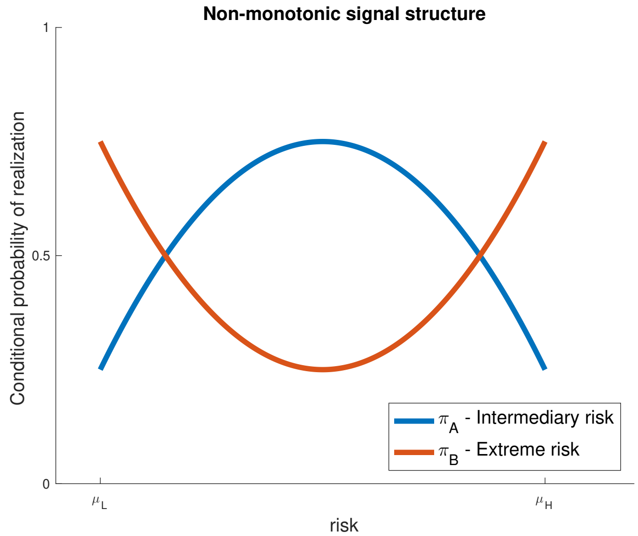

Figure 2: Non-monotonic binary signal structure.

But the expression (20) can be further simplified by using: (i) signal realization has no first order price effect, i.e., $p_\delta^s(\cdot) = p_\delta^0(\cdot)$; (ii) signal structure has no risk preference information, i.e., $\omega^s(\cdot) = \omega^0(\cdot)$; and (iii) conditional type distributions $\phi_i(\mu \mid s)$ satisfy (16). As a consequence, we have that

$$\sum_{s\in S} \pi(s \mid m_0(x)) \left[w_\delta^s(z) - w_\delta^0(z)\right] = (1-z)\,\omega^0(z)\left[1-\omega^0(z)\right] \sum_{s\in S} \pi(s \mid m_0(x)) \frac{\dot{\pi}(s \mid m_0(z))}{\pi(s \mid m_0(z))}$$

where $\dot{\pi}(s \mid \mu) \equiv \frac{\partial}{\partial\mu}\pi(s \mid \mu)$.

But direct substitution in (20) and a change of variables in the integration (to $\mu' = m_0(z)$) gives us

$$\Delta E\left[p\left(t_0(\mu)\right)\right] = 2\int_\mu^{\mu_H} \left(\frac{1-z}{\rho_0}\right) \omega^0\left(t_0(\mu')\right)\left[1-\omega^0\left(t_0(\mu')\right)\right]\left[\partial_2 D_{KL}\left(\pi(\cdot \mid \mu)\,||\pi(\cdot \mid \mu')\right)\right] d\mu'$$

where $\partial_2 D_{KL}(\pi(\cdot \mid \mu)\,||\pi(\cdot \mid \mu')) \equiv \frac{\partial}{\partial\mu'} D_{KL}(\pi(\cdot \mid \mu)\,||\pi(\cdot \mid \mu'))$. This result ties the expected price change of a given contract with monotonicity of the Kullback-Leibler divergence measure.

The rest of the proof is in Appendix D. □

From Proposition (4) it also follows that the disclosure of a monotonic signal structure leads to an aggregate welfare gain (i.e., integrating over all possible types), for $\delta > 0$ sufficiently small. The reverse also holds: the proof of Proposition 4 shows that, for any non-monotonic signal structure, there exists a type distribution such that a positive mass of types is worse off with disclosure. The construction also implies that those types loosing from the signal disclosure are almost full measure and hence imply a negative aggregate welfare loss.

5.3 The statistical content of monotonicity

The abstract nature of the Kullback-Leibler divergence measure makes the exact content of the monotonicity restriction somewhat inscrutable. In this section we show that monotonicity is equivalent to a simple statistical property related to the impact signal realizations have on the expected risk assessment of a small risk pool. Its connection with equilibrium analysis is discussed below.

Consider a set or pool of types with different levels of risk aversion and risk heterogeneity $\varepsilon > 0$, defined by:

$$T\left(\overline{\mu}, \varepsilon\right) \equiv \left\{\left(\overline{\mu}, \rho_h\right), \left(\overline{\mu} + \varepsilon, \rho_l\right)\right\},$$

and denote the expected risk level in this set as its cost C. For a signal structure π, the impact of signal realization $s \in S$ on the cost of pool T is given by

$$\Delta C^s\left(\overline{\mu}, \varepsilon\right) \equiv \mathbb{E}\left[\widetilde{\mu} \mid \left(\widetilde{\mu}, \widetilde{\rho}\right) \in T\left(\overline{\mu}; \varepsilon\right), \tilde{s} = s\right] - \mathbb{E}\left[\widetilde{\mu} \mid \left(\widetilde{\mu}, \widetilde{\rho}\right) \in T\left(\overline{\mu}; \varepsilon\right)\right].$$

The top-down property of equilibrium prices implies that any changes in the cost of a pool T will indirectly affect the prices of lower coverage contracts consumed by agents with lower risk level $\underline{\mu} < \overline{\mu}$. We now focus on the *indirect expected* effect that signal disclosure has on agents with risk level $\underline{\mu}$ *through* its impact on the cost of pool T:

$$\Delta C\left(\overline{\mu}, \underline{\mu}; \varepsilon\right) \equiv \sum_{s \in S} \pi\left(s \mid \underline{\mu}\right) \Delta C^s\left(\overline{\mu}, \varepsilon\right).$$

The result below shows that monotonicity is equivalent to negativity of the indirect cost effect described here.

Proposition 5. *A signal structure $\pi(\cdot)$ is monotonic if, and only if for any full support continuous type distribution (ϕ_l, ϕ_h), almost all $\underline{\mu}, \overline{\mu} \in (\mu_L, \mu_H)$ and $\varepsilon > 0$ sufficiently small,*

$$\Delta C\left(\overline{\mu}, \underline{\mu}, \varepsilon\right) > 0.$$

Proof. See Appendix D. □

While Proposition 5 does not use equilibrium objects, these two are connected since, in equilibrium, prices are determined by the average riskiness of risk pools and the top-down property of prices is represented by the indirect cost assessment introduced here.

5.4 MLRP versus Monotonicity

A common restriction on signals structures in contract theory is that of the monotone likelihood ratio property (MLRP; Mirrlees (1976)). A signal structure π with realizations $S = \{s_1, ..., s_n\}$ satisfies MLRP if

$$\frac{\pi\left(s_{k+1} \mid \mu\right)}{\pi\left(s_k \mid \mu\right)}$$

is strictly increasing in μ, for any $k \in \{1, ..., n-1\}$. Notice that in the case of binary signals, monotonicity is equivalent to MLRP. In general, monotonicity is weaker than MLRP, as shown by Proposition 21. While MLRP requires that signal realizations can be ordered on how much they indicate higher risks, monotonicity does not.

Recalling the definition of the conditional distribution (16), MLRP is equivalent to higher signals indicate higher risks, i.e.,

$$\frac{\phi_i\left(\mu \mid s_{k+1}\right)}{\phi_i\left(\mu \mid s_k\right)}$$

is strictly increasing in μ, for each $i = l, h$.

The next proposition shows that MLRP implies monotonicity.

Proposition 6. *Any signal structure satisfying MLRP is monotonic.*

Proof. See Appendix D. □

Remark. An alternative restriction on informativeness states that the (absolute) likelihood of a particular signal varies monotonically with the risk level, i.e., we say that a signal structure π is strongly monotonic (SM) if, for all $s \in S$, $\pi(s \mid \cdot)$ is strictly increasing or decreasing in μ. This condition means that signals can be classified in two categories: ones that are likely with high risks and the others that are more likely with low risks. Notice that MLRP and SM are not nested in general, but they both coincide in the case of binary signals.[21]

[21] Indeed, suppose that π satisfies SM. Now consider an arbitrary signal $s \in S$ with a strictly increasing (decreasing) function $\pi(s \mid \cdot)$. We then have that $\dot{\pi}(s \mid \cdot) > 0$ a.e. (Lebesgue). Now consider a fixed $\mu \in [\mu_L, \mu_H]$, we have that both elements multiplied in (60) are strictly positive for (almost) all $\widetilde{\mu} \in [\mu, \mu_H]$.

5.5 Independent types

In Subsection 5.1, we showed that the welfare effect of signal disclosure can be characterized by the expected price change expression $\Delta E\left[p\left(x\right)\right]$. If risk and risk aversion are independently distributed (i.e., $\omega_0\left(\cdot\right)$ is constant) =, this expression can be further simplified to:

$$\Delta E\left[p(x)\right] = -2\omega_0\left(1-\omega_0\right)\int_x^1 D_{KL}\left(\pi(\cdot \mid m_0(x))||\pi(\cdot \mid m_0(z))\right)dz,$$

after an integration by parts. This means that the price effect of a signal at coverage x is determined solely by the cumulative divergence $D_{KL}\left(\cdot\right)$ of signal distributions with respect to all coverage levels above x, which is always non-negative. We say that a signal structure is informative if $D_{KL}\left(\pi\left(\cdot \mid \mu\right)||\pi\left(\cdot \mid \widetilde{\mu}\right)\right) > 0$, for any $\mu, \widetilde{\mu} \in [\mu_L, \mu_H]$ with $\mu \neq \widetilde{\mu}$, i.e., the distributions over signal realizations are different for different risk levels. Since $D_{KL}\left(\pi(\cdot \mid m_0(x))||\pi(\cdot \mid m_0(x))\right) = 0$, monotonicity implies informativeness. Under the assumption of independence, a much stronger welfare result holds: informativeness implies that signal disclosure is interim welfare improving.

Corollary 2. *Any informative signal is interim Pareto improving for independent distributions of types.*

This is in contrast with Proposition 4, which states that a signal is interim Pareto improving if and only if it is monotonic. Notice that the definition of an interim Pareto improving signal structure requires that it be interim Pareto improving for all possible distributions of types. Corollary 2suggests that, by using restrictions on the class of type distributions, one may find weaker sufficient conditions for a signal structure to be interim Pareto improving. Corollary 2 can easily be extended to any type distributions such that the mapping $x \to (1-x)\omega_0(x)\left[1-\omega_0(x)\right]$ is non-increasing (a generalization of independence).

6 Comparative statics

In this section we study the effect of changes in the type distribution on prices and consumer utility. A key justification behind multiple policy interventions, such as insurance mandates and subsidies, is its effect on the risk distribution among active buyers in the market. More specifically, that the increase of low-risk consumers can indirectly benefit pre-existing consumers in the market with higher risks. By looking at changes to the market risk distribution, we show that this intuition is only partially true. If we look at distribution shocks that reduce risks in a very strong sense, one can show that almost all pre-existing consumers in the market benefit. However, details matter: we provide an example of a reduction in the distribution of risks in a weaker sense (first order stochastic dominance) that hurts certain types of consumers in the market.

In order to clearly distinguish the source of distribution effects between the two dimensions of consumer heterogeneity, we assume in this section that risk and risk aversion are independently distributed. We compare prices and welfare under type distributions $\left(\omega^A, \phi^A\right)$ and $\left(\omega^B, \phi^B\right)$. We use superscripts to refer to endogenous equilibrium variables under distributions A and B, e.g., $p^A(x;\delta)$ and $p^B(x;\delta)$ for the price of coverage $x \in [0,1]$.

We say that distribution $\left(\omega^A, \phi^A\right)$ welfare interim-dominates distribution $\left(\omega^B, \phi^B\right)$ if for every $\epsilon > 0$ there exists $\bar{\delta} > 0$ such that

$$\mathbb{P}\left[(\mu, i) \mid V_i^A(\mu;\delta) > V_i^B(\mu;\delta)\right] > 1 - \epsilon,$$

for any arbitrary equilibrium selection and any $\delta \leq \bar{\delta}$.[22]

6.1 Risk distribution

We first consider shifts in the risk distribution in the sense of monotone likelihood ratio (MLRP). For any two strictly positive densities ϕ^A and ϕ^B on $[\mu_L, \mu_H]$, we say that ϕ^B MLRP-dominates ϕ^A if

$$\frac{\phi^B(\mu)}{\phi^A(\mu)} \text{ is strictly increasing in } \mu.$$

In derivative terms, if ϕ^A and ϕ^B are continuously differentiable, this is equivalent to

$$\frac{\dot{\phi}^B(\mu)}{\phi^B(\mu)} > \frac{\dot{\phi}^A(\mu)}{\phi^A(\mu)} \tag{21}$$

for almost all risk levels $\mu \in [\mu_L, \mu_H]$. Corollary 1 shows how the rate of increase of the risk density is connected with equilibrium prices, which implies that comparative statics with respect to the risk distribution are tightly connected to MLRP-dominance as shown in the following result.

[22]This notion of interim welfare improvement does not imply aggregate welfare improvement. The latter concept is problematic since a distribution shift may make almost all types strictly better off but also increase the frequency of high-risk types which have lower utility, leading to lower aggregate welfare.

Proposition 7. *(Risk distribution effect) For a fixed $\omega_0 \in (0,1)$, if risk distribution ϕ^B MLRP-dominates ϕ^A, then $\left(\omega_0, \phi^A\right)$ welfare interim-dominates distribution $\left(\omega_0, \phi^B\right)$.*

Proof. Consider $\delta > 0$ sufficiently small and an arbitrary equilibrium selection. Lemma 28, in the appendix, show that the direction of utility effects is determined by price effects. Proposition 3 implies that $p^A_\delta(x) = p^B_\delta(x)$ and hence the direction of comparative statics is determined by the second order effects:

$$V_i^B(\mu;\delta) - V_i^A(\mu;\delta) = \frac{\delta^2}{2}\frac{\partial v}{\partial p}\left(t_0(\mu), p_0(t_0(\mu)), \mu, \rho_0\right)\left[p^A_{\delta\delta}(t_0(\mu)) - p^B_{\delta\delta}(t_0(\mu))\right] + o_i\left(\mu;\delta^2\right),$$

where $\delta^{-2}|o_i(\mu;\delta)|$ converges to zero uniformly on M, for any compact set $M \subset (\mu_L, \mu_H)$. From Corollary 1 we have

$$\frac{p^B_{\delta\delta}(x) - p^A_{\delta\delta}(x)}{x} = 2\omega_0(1-\omega_0)\int_x^1 \frac{(1-z)^2}{z}\left[\frac{\dot{\phi}^B(m_0(z))}{\phi^B(m_0(z))} - \frac{\dot{\phi}^A(m_0(z))}{\phi^A(m_0(z))}\right]dz,$$

which is strictly positive for all $x \in (x_L, 1)$ and implies the result. □

Proposition 7 is fairly intuitive. It shows that if the risk distribution shifts towards a distribution with more mass on higher types according to MLRP order, then the pattern of cross subsidies between risk types pooling on a given coverage level puts more weight on the highest risk type in the pool. This distribution effect increases the average risk in each pool, pushing equilibrium prices up and lowering consumer utility. While Proposition 7 focuses on welfare, its proof implies that a MLRP-increase in risks drives the price of almost all traded coverage levels up.

We should point out that MLRP ordering, although quite demanding, cannot be significantly relaxed. The following example shows that the first order stochastic dominance (FOSD) is too weak to provide such unambiguous comparative statics results. It shows that a FOSD increase in the risk distribution may lead to ambiguous welfare effects, potentially benefiting some consumer types. These results illustrate how distribution effects in competitive screening models can be counter-intuitive and any policy analysis require careful consideration of such subtleties.

Example 1 (FOSD risk increase reducing prices)**.** Suppose that ϕ^A is the uniform distribution on $[\mu_L, \mu_H]$ and ϕ^B is a strictly concave function such that $\dot{\phi}^B(\mu_H) < 0$,

$$\phi^B(\mu_L) < \phi^A(\mu_L) \text{ and } \phi^B(\mu_H) > \phi^A(\mu_H).$$

These properties imply, since ϕ^B is strictly concave, that it crosses ϕ^A only once, which ensures that the FOSD holds. Since $\dot{\phi}^B(\mu_H) < 0$, then there exists $\mu_0 \in (\mu_L, \mu_H)$ such that, for any $\mu \in [\mu_0, \mu_H]$,

$$\frac{\dot{\phi}^B(\mu)}{\phi^B(\mu)} < 0 = \frac{\dot{\phi}^A(\mu)}{\phi^A(\mu)}.$$

Hence, using Lemma 29, types with risk level $\mu \in [\mu_0, \mu_H]$ have strictly higher utility and face strictly lower prices in the market with distribution B when compared to distribution A. In this case, some consumers benefit from being in a riskier market. The intuition for this negative result is that, although distribution B puts more cumulative weight on high risk types than distribution A, at the top risk types distribution B puts relatively less weight than distribution A, as the B density decreases for these types whereas the A density is constant (this comes from the concavity of ϕ^B). As explained in Proposition 7, less weights on top risk types may lead to a lower average risk on the high coverage pools for distribution by the same token, reverting the price and welfare comparison for high coverage levels.

6.2 Preferences distribution

Consumers have two drivers of coverage demand in our model: their risk level, which is the underlying source of adverse selection, and risk aversion. If agents have a high willingness to pay for coverage that is unrelated to their risk level, the problem of adverse selection is alleviated since the purchase of higher coverage may not be a strong signal of higher risks. More precisely, in our equilibrium analysis each contract pool in the discrete pooling region is composed by a ρ_l-type (with higher risk) and a ρ_h-type (with lower risk), and hence an increase in the relative frequency of low-risk-aversion types leads a lower average risk in each pool and, as a consequence, to lower prices. The result below shows that this price reduction leads to an interim welfare improvement.

Proposition 8. *(Risk aversion distribution effect) An increase in the share of consumers with high-risk aversion (i.e., $\omega^B < \omega^A$) leads to a welfare interim-dominating distribution (i.e., $\left(\omega^B, \phi\right)$ welfare-dominates $\left(\omega^A, \phi\right)$).*

Proof. Once again, using Lemma 28, we can connect utility changes to price changes:

$$V_i^B(\mu;\delta) - V_i^A(\mu;\delta) = \delta \frac{\partial v}{\partial p}\left(t_0(\mu), p_0(t_0(\mu)), \mu, \rho_0\right)\left[p_\delta^A(t_0(\mu)) - p_\delta^B(t_0(\mu))\right] + o_i(\mu;\delta),$$

where $\delta^{-1}|o_i(\mu;\delta)|$ converges to zero uniformly on M, for any compact set $M \subset (\mu_L, \mu_H)$.

Now, suppose that $\omega^B < \omega^A$, we then have that

$$\frac{p_\delta^A(x) - p_\delta^B(x)}{x} = x\left(\omega^A - \omega^B\right)(x - 1 + \ln x) > 0.$$

□

This effect is analogous to the one found in one-dimensional models when we compare equilibria across economies where agents exhibit different levels of risk aversion, as presented in section 3.

7 Conclusion

We proposed a new approach to study how the distribution of risk affects equilibrium outcomes in a multidimensional competitive screening environment, and apply our results to study policy-relevant comparative static exercises: the effects of price discrimination based on characteristics of consumers and changes in the type distribution in the market. We depart from the canonical one-dimensional model commonly studied in the literature by introducing a second dimension of heterogeneity in risk preferences. We show existence and characterize competitive equilibria when such preference heterogeneity is small. More importantly, we provide a tractable model with multidimensional types that allows us to carry out important analysis of the distributional effects on equilibrium outcomes. We derive analytical formulas describing how the type distribution affects equilibrium prices, allocation and welfare. In order to accomplish these derivations we introduce a novel approach based on Taylor first- and second-order approximation terms in the second dimension of heterogeneity. Our comparative statics result shows that increase in the risk population distribution, in the MLRP sense, leads to price increase and welfare decrease, while FOSD shifts in the risk distribution can have an unexpected positive effect for some high risk consumers. We fully characterize the disclosure of signals, which are correlated with risk, is interim welfare improving . In the particular experiment we propose, the disclosure of a signal leads to lower prices and higher welfare as long as the signal structure satisfies a certain monotonicity property, which generalizes the MLRP order of binary signal setting; while a non-monotonic signal may be harmful for some or all consumers. Our results provide new insights into insurance markets with multidimensional heterogeneity which do not rely on numerical simulations, but use interpretable closed-formula expressions. There are two possible directions for the future agenda of this project. One is to explore the approximation methodology developed in this paper in other competitive multidimensional settings such as buyer-seller relationship in other market arrangements where sellers have multidimensional characteristics (see, for instance, Guerrieri et al. (2010)). The other is to use the implications derived in this paper into the data (in particular, that monotonicity of the signal is a sufficient condition for interim Pareto improvement).

Appendix A - Equilibrium characterization

We denote, with abuse of notation,

$$\omega(\mu_l, \mu_h) \equiv \frac{\phi_l(\mu_l)}{\phi_l(\mu_l) + \phi_h(\mu_h)},$$

let us define some auxiliary functions that will be used in the proofs that follow:

$$R(\mu, x, \delta) = \omega(\mu, \mu - (1-x)\delta) \tag{22}$$

and

$$e(\mu, \delta) = \mu + \omega(\mu + \delta, \mu)\delta. \tag{23}$$

We will assume the following:

Assumption 1. *If* $0 < \widetilde{\delta} < \delta$, $\mu' > \mu$ *and* $\mu_L \leq \mu < \mu' + \widetilde{\delta} < \mu_H$, *then* $e\left(\mu, \widetilde{\delta}\right) < e\left(\mu', \widetilde{\delta}\right)$.

Notice that this assumption holds if δ is sufficiently small as the following result shows.

Lemma 4. *There exists* $\overline{\delta} > 0$ *such that Assumption 1 holds for all* $\delta \in \left[0, \overline{\delta}\right]$.

Proof. Since densities ϕ_l and ϕ_h are continuously differentiable, we can define their continuously differentiable extensions to interval $[\mu_L - 1, \mu_H + 1]$, bounded away from zero. Suppose for now, with some abuse of notation, that the densities have been extended to domain $[\mu_L - 1, \mu_H + 1]$. Differentiating function $e(\cdot, \delta)$ w.r.t. μ we have that

$$e_\mu(\mu, \delta) \equiv 1 - \Delta \frac{\phi_l(\mu+\delta)\dot{\phi}_h(\mu) - \phi_h(\mu)\dot{\phi}_l(\mu+\delta)}{\left(\phi_l(\mu+\delta) + \phi_h(\mu)\right)^2},$$

which is positive and bounded away from zero in $[\mu_L - 1, \mu_H + 1]$ as long as

$$\Delta < \left[\sup\left\{\frac{\left|\phi_l(\mu+\delta)\dot{\phi}_h(\mu) - \phi_h(\mu)\dot{\phi}_l(\mu+\delta)\right|}{\left(\phi_l(\mu+\delta) + \phi_h(\mu)\right)^2}; \mu \in [\mu_L - 1, \mu_H + 1]\right\}\right]^{-1}.$$

□

Proof of Lemma 1

Properties (i) and (ii) are equivalent to monotonicity of demand in each dimension, i.e., $t(\cdot, i)$ is non-decreasing and $t(\mu, l) \leq t(\mu, h)$. Monotonicity follows from the fact that preferences satisfy the single-crossing property on each dimension, for a given type of the other dimension.

For property (iii): from the agent's optimality and price consistency conditions, we have

$$u(x, m_i(x), i) - u(\widehat{x}, m_i(x), i) \geq p(x) - p(\widehat{x}) \geq u(x, m_i(\widehat{x}), i) - u(\widehat{x}, m_i(\widehat{x}), i)$$

which implies that

$$|p(x) - p(\widehat{x})| \leq L\,|x - \widehat{x}|$$

where $L = \sup \{|u_x(x, \theta)|\,; x \in [0, 1] \text{ and } \theta \in \Theta\}$.

Proof of Proposition 1

Let (m_l, m_h, w) be a non-gap equilibrium allocation and $X \subset [0, 1]$ be its interval of on-the-equilibrium path coverage. From the monotonicity property of the equilibrium in Lemma 1, X is a partition with the following sets:

- X_s^i the separating coverages chosen by agents with risk aversion $i \in \{l, h\}$;
- X_d the discrete pooled coverages chosen by exactly two risk averse types;
- X_c the continuous pooled coverages chosen by intervals of risk types.

The following lemma shows the basic properties of these sets:

Lemma 5. *(a) X_c is a countable set of X;*

(b) If $x \in X_s^h$ and $y \in X \cap [x, 1]$, then $y \in X_s^h$ (i.e., X_s^h is an interval);

(c) If $x \in X_s^l$ and $y \in X \cap [0, x]$, then $y \in X_s^l$ (i.e., X_s^l is an interval).

Proof. (a) This a trivial consequence of the monotonicity of m_i, for $i \in \{l, h\}$.

(b) Let us first show that the set $\left(X_s^l \cup X_d\right)' \cap (x, 1]$ has zero Lebesgue measure[23], where A' is the set of accumulation points of A. Suppose that this is not the case and let $\underline{y} =$

[23]In what follows, when we refer to positive or zero measure sets we mean Lebesgue measure.

$\inf\left\{\left(X_s^l \cup X_d\right)' \cap (x,1]\right\}$. If the interval $[x, \underline{y}]$ is non-degenerated, it must be the union of isolated points in $X_s^l \cup X_d$, points in X_c and points in X_s^h. Since the first two sets are countable, $\underline{y}$ is also the limit of a sequence of points in X_s^h on the left, which is also trivially true in the case $x = \underline{y}$ (i.e., when $[x, \underline{y}]$ is a degenerated interval). Lemma 6 shows that $w(\underline{y}) > 0$. Notice that

$$\frac{p(y)}{y} = w(y)m_l(y) + (1 - w(y))m_h(y),$$

for all $y \in [x, 1]$, with $w(y) = 0$ if and only if $y \in X_s^h$. Since $m_l > m_h$ on $\left(X_s^l \cup X_d\right)' \cap (x, 1]$ and $w(\underline{y}) > 0$, the function $p(y)/y$ would jump at $\underline{y}$, a contradiction. Therefore, $[x, 1] \cap X_s^h$ has full measure on $[x, 1] \cap X$, since $X_c \cup X_s^l \cup X_d$ has also zero measure on this set. Hence, from the equilibrium condition, we have that

$$\frac{p(y)}{y} = m_h(y) \text{ and } w(y) = 0,$$

for almost all $y \in [x, 1] \cap X$. Hence, by the continuity of the function $p(y)/y$ and the monotonicity of m_h, these equalities must hold for all $y \in [x, 1] \cap X$, showing the result.

(c) Let us first show that the set $\left(X_s^h \cup X_d\right)' \cap [0, x)$ has zero measure. Suppose that this is not the case and let $\overline{y} = \sup\left\{\left(X_s^h \cup X_d\right)' \cap [0, x)\right\}$. If the interval $[\overline{y}, x]$ is non-degenerated, it must be the union of isolated points in $X_s^h \cup X_d$, points in X_c and points in X_s^l. Since the first two sets are countable, $\overline{y}$ is also the limit of a sequence of points in X_s^l on the right, which is also trivially true in the case $x = \overline{y}$ (i.e., when $[x, \underline{y}]$ is a degenerated interval). Lemma 2 shows that $\overline{w} = \sup\left\{w(y); y \in \left(X_s^h \cup X_d\right)' \cap [0, x)\right\} < 1$. Notice that

$$\frac{p(y)}{y} = w(y)m_l(y) + (1 - w(y))m_h(y),$$

for all $y \in [0, x]$, with $w(y) = 1$ if and only if $y \in X_s^l$. Since $m_l > m_h$ and $w \le \overline{w} < 1$ on $\left(X_s^h \cup X_d\right)' \cap [0, x)$, the function $p(y)/y$ would jump at $\overline{y}$, which is a contradiction. Therefore, $[0, x] \cap X_s^l$ has full measure on $[0, x] \cap X$, since $X_c \cup X_s^h \cup X_d$ has zero measure on this set. Hence, from the equilibrium condition, we have that

$$\frac{p(y)}{y} = m_l(y) \text{ and } w(y) = 1,$$

for almost all $y \in [0, x] \cap X$. Hence, from the continuity of the function $p(y)/y$ and monotonicity of m_l, these equalities must hold for all $y \in [0, x] \cap X$, showing the result. □

The next lemma is the auxiliary result used in the proof of Lemma 5:

Lemma 6. *(a) If* $\left(X_s^l \cup X_d\right) \cap (x,1]$ *has positive measure for some* x*, then*

$$w\left(\inf\left\{y; y \in \left(X_s^l \cup X_d\right)' \cap (x,1]\right\}\right) > 0;$$

(b) If $X_s^h \cup X_d$ *has positive measure, then* $\sup\left\{w(y); y \in \left(X_s^h \cup X_d\right)'\right\} < 1.$

Proof. (a) Define $\underline{y} = \inf\left\{\left(X_s^l \cup X_d\right)' \cap (x,1]\right\}$. Suppose, by absurd, that $w(\underline{y}) = 0$. We claim that, for every $\epsilon > 0$, $(\underline{y}, \underline{y}+\epsilon) \cap X_d'$ has positive measure. Otherwise, $(\underline{y}, \underline{y}+\epsilon) \cap \left(X_s^l \cup X_s^h\right)'$ has full measure for some $\epsilon > 0$. Hence, $p(y)/y = m_i(y)$ in a full measure subset of $(\underline{y}, \underline{y}+\epsilon)$, for some $i \in \{l,h\}$. Suppose that $(\underline{y}, \underline{y}+\epsilon) \cap X_s^i$ has positive measure for each $i = l,h$ and let $y_0 \in (\underline{y}, \underline{y}+\epsilon)$ be a point in the zero measure complement set. Then, y_0 is a left- and right-hand limit of sequence of points (y_n^i) in X_s^i that satisfy $\dot{p}(y) = m_i(y) + (1-y)\rho_i = \frac{p(y)}{y} + (1-y)\rho_i$, for $i = l$ or $i = h$. Since p is continuous function

$$\frac{p(y_n^i)}{y_n^i} + (1-y_n^i)\rho_i \to \frac{p(y_0)}{y_0} + (1-y_0)\rho_i,$$

for $i = l,h$. Since $\rho_h > \rho_l$, the price function must have a kink at y_0 (i.e., $y_0 \in X_c$). Moreover, $p(y_0)/y_0 = m_h(y_0^-) = m_l(y_0^+)$ or $p(y_0)/y_0 = m_h(y_0^+) = m_l(y_0^-)$. However, in both cases we have contradiction with the kink of the price function at y_0. Finally, since $w(\underline{y}) = 0$, we must have that $(\underline{y}, \underline{y}+\epsilon) \subset X_s^h$, which contradicts the definition of $\underline{y}$.

Since m_i is non-decreasing, it must be differentiable for almost all points in $X_d \cap (\underline{y}, \underline{y}+\epsilon)$, for all $\epsilon > 0$. Notice that

$$\frac{p(x)}{x} = m_h(x) + w(x)(m_l(x) - m_h(x)).$$

Then, (m_l, m_h, w) is the solution to the following ordinary differential equation (ODE) system:

$$\begin{aligned}
\dot{m}_l(x) &= \tfrac{1-R(m_l(x),x,\delta)}{R(m_l(x),x,\delta)-w(x)} w(x)\delta, \\
\dot{w}(x) &= \left(\tfrac{1}{x} - \tfrac{1}{1-x}\right)(1-w(x)) + \tfrac{\rho_l}{x\delta} - \tfrac{\dot{m}_l(x)}{(1-x)\delta}, \\
m_h(x) &= m_l(x) - (1-x)\delta,
\end{aligned}$$

on $X_d \cap (\underline{y}, \underline{y}+\epsilon)$, where $R(\mu, x, \delta)$ is defined in (22). By the same argument above, there are two possible case:

(i) For every $\epsilon > 0$, $(\underline{y}-\epsilon, \underline{y}) \cap X_d$ has positive measure. Since $w(\underline{y}) = 0$, we must have that $\dot{w}(\underline{y}^-) \le 0$ and $\dot{w}(\underline{y}^+) \ge 0$ (otherwise, this would imply that $p(x)/x < m_l(x)$, for some

x close to $\underline{y}$). Using these properties we get

$$\underline{y} = \frac{\delta + \rho_l}{2\delta + \rho_l},$$

which implies that $\dot{w}(\underline{y}) = 0$, and, taking the second derivative, we get

$$\ddot{w}(\underline{y}) = -\left(1 + \frac{\rho_l}{\delta}\right)\frac{1}{\underline{y}^2} - \frac{1}{(1-\underline{y})^2} < 0,$$

which means that $\underline{y}$ is a local maximum of w. Since $w(\underline{y}) = 0$, this is a contradiction.

(ii) There exists $\epsilon > 0$ such that $(\underline{y} - \epsilon, \underline{y}) \subset X_s^h$. Then, for every $z \in (\underline{y} - \epsilon, \underline{y})$, we have that $p(z) = m_h(z)z$ and, consequently,

$$\dot{p}(z) = m_h(z) + (1-z)\rho_h = \dot{m}_h(z)z + \frac{p(z)}{z},$$

which implies that

$$\frac{d}{dy}\left(\frac{p(y)}{y}\right)\bigg|_{y=\underline{y}-} = \dot{m}_h(\underline{y}^-).$$

On the other hand,

$$\dot{p}(z) = m_h(z) + (1-z)\rho_h,$$

for almost all $z \in (\underline{y}, \underline{y}+\epsilon)$. Since $\lim_{z\to\underline{y}} w(z) = 0$, we have that m_h is continuous at $\underline{y}$, which implies that $\dot{p}$ is also continuous at $\underline{y}$. Moreover,

$$\dot{p}(\underline{y}^+) = \left[\dot{m}_h(\underline{y}^+) + \dot{w}(\underline{y}^+)(m_l(\underline{y}) - m_h(\underline{y}))\right]\underline{y} + \frac{p(\underline{y})}{\underline{y}},$$

which implies that

$$\frac{d}{dy}\left(\frac{p(y)}{y}\right)\bigg|_{y=\underline{y}+} \geq \dot{m}_h(\underline{y}^+)$$

where the last inequality is strict if and only if $\dot{w}(\underline{y}^+) > 0$ since $m_l(\underline{y}) - m_h(\underline{y}) = (1-\underline{y})\delta > 0$. By the continuity of $\dot{p}$ at $\underline{y}$, $\dot{w}(\underline{y}^+) = 0$ and the proof is analogous to the case (i).

(b) The proof is analogous to item (a). The only difference is that since $w(x) \leq R(m_l(x), x, \delta)$ on X_d and R is uniformly bounded below 1, and then the last part of the proof in item (a) is unnecessary in this case. □

Lemma 7. *Under Assumption 1, X_d is an interval.*

Proof. Suppose that there exists $x \in X_c \cap X_d$ and not in the extreme points of X_d. Take sequences $(x_n^-)_n$ and $(x_n^+)_n$ in X_d such that $x_n^- < x < x_n^+$ and $\lim_n x_n^- = x = \lim_n x_n^+$. Define $m_i^0 \equiv \lim_n m_i\left(x_n^-\right)$ and $m_i^1 \equiv \lim_n m_i\left(x_n^+\right)$, for $i = l, h$. The optimality condition of types $m_i\left(x_n^-\right)$, for $i = l, h$, implies that $m_l^0 = m_h^0 + \delta\left(1 - x\right)$. The optimality condition for types in X_d implies that $m_l^1 = m_h^1 + 2\delta\left(1 - x\right)$. We then have that $m_h^0 < m_h^1$,

$$m_h\left(x\right) = \left[m_h^0, m_h^1\right]$$

and

$$m_l\left(x\right) = \left[m_h^0 + \delta\left(1 - x\right), m_h^1 + \delta\left(1 - x\right)\right].$$

The zero profit condition at coverage x implies that

$$\frac{p\left(x\right)}{x} = \frac{\int_{m_h^0}^{m_h^1} e\left(z, \delta\left(1 - x\right)\right)\left[\phi_l\left(z + \delta\left(1 - x\right)\right) + \phi_h\left(z\right)\right] dz}{\int_{m_h^0}^{m_h^1}\left[\phi_l\left(z + \delta\left(1 - x\right)\right) + \phi_h\left(z\right)\right] dz},$$

where $e(\cdot)$ is defined in (23). But, for each n, the zero profit condition in X_d implies that

$$\frac{p\left(x_n^-\right)}{x_n^-} = w\left(x_n^-\right) m_l\left(x_n^-\right) + \left(1 - w\left(x_n^-\right)\right) m_h\left(x_n^-\right),$$

which is lower than $e\left(m_l\left(x_n^-\right), \delta\left(1 - x_n^-\right)\right)$, since $w\left(x_n^-\right) \leq \omega\left(m_l\left(x_n^+\right)\right), \left(m_h\left(x_n^+\right)\right)$. Taking limits, we have that

$$\lim_n \frac{p\left(x_n^-\right)}{x_n^-} \leq e\left(m_l^0, \delta\left(1 - x\right)\right)$$

strictly lower than $\frac{p(x)}{x}$ by Assumption 1 and contradicts continuity of $p\left(\cdot\right)$. Therefore, the result follows from Lemma 5. □

We now state and prove the following lemma, which completes the proof and will also be useful in Appendix B.

Lemma 8. *There exist* $0 \leq x_0 < x_d \leq x_u < 1$ *with* $p(x_0) = \mu_L x_0$ *such that:*
(i) $X_s^u = [x_u, 1]$, $m_l\left(x_u\right) = \left[\underline{a}_u, \mu_H\right]$ *and* $m_h\left(x_u\right) = \left[\overline{a}_u, \overline{b}_u\right]$*;*
(ii) $X_s^d = [x_0, x_d]$, $m_l\left(x_d\right) = \left[\underline{a}_d, \underline{b}_d\right]$ *and* $m_h\left(x_d\right) = \left[\mu_L, \overline{b}_d\right]$*;*
(iii) if $x_d < x_u$*,then* $X_d = (x_d, x_u)$.

Proof. From Lemma 5 and doing an extension argument of the solution ODE of the separation part, we can show that $X_s^d = [x_0, x_d]$ and $X_s^u = [x_u, 1]$, where $p(x_0) = \mu_L x_0$.

(i) We claim that $x_u < 1$. Otherwise, by the same argument of the second part of the proof of Lemma 11 we would have a contradiction. From Lemma 1, it is easy to argue that $m_l(x_u)$ and $m_h(x_u)$ should be intervals as stated.

(ii) We claim that $x_d > x_0$. Otherwise, we would have that $m_l(x_0) = \mu_L$ and $m_h(x_0) = \mu_L - (1-x_0)\delta < \mu_L$, which is a contradiction. It is easy to see that $x_d \leq x_u$ belongs to X_c. From Lemma 1, it is easy to argue that $m_l(x_d)$ and $m_h(x_d)$ should be intervals as stated.

(iii) If $x_d < x_u$, then, by Lemma 7, $(x_d, x_u) = X_d$. □

Proof of Lemma 2

Suppose that $I = [x-\delta, x+\delta]$ is an interval of coverage with discrete pooling of an equilibrium (p, m_l, m_h, w) such that m_l and m_h are differentiable at x, with $\delta > 0$ sufficiently small. From the consistency condition of equilibrium definition we must have

$$\Pr\left[x \in I | \theta(x) \in [m_l(x-\delta/2), m_l(x+\delta/2)] \times \{\rho_l\}\right] = \frac{\int_{m_l(x-\delta/2)}^{m_l(x+\delta/2)} \phi_l(z)dz}{\int_{m_l(x-\delta/2)}^{m_l(x+\delta/2)} \phi_l(z)dz + \int_{m_h(x-\delta/2)}^{m_h(x+\delta/2)} \phi_h(z)dz}$$

and, taking the limit $\delta \to 0$, the left hand side must converge to $w(x)$. However, dividing the numerator and denominator of the fraction on the right hand side by δ and taking the limit, we get

$$w(x) = \frac{\phi_l(m_l(x))\dot{m}_l(x)}{\phi_l(m_l(x))\dot{m}_l(x) + \phi_h(m_l(x))\dot{m}_h(x)}$$

which gives the result.

Appendix B - Existence

Proof of Proposition 2

We can re-write the statement of this Proposition as follows:

Proposition. *For sufficiently small $\delta > 0$ there exists a continuum of equilibria characterized by (p, m_l, m_h) and $0 < x_d < x_u < 1$ satisfying the following properties:*

(a) top separation: $p(x) = m_h(x) = \rho_h(1 - x + \ln x) + \mu_H$ and $m_l(x) = \emptyset$, for $x > x_u$;

(b) continuous pooling at x_u: $m_l(x_u) = [\underline{a}_u, \mu_H]$ and $m_h(x_u) = [\overline{a}_u, \overline{b}_u]$ such that:

(b.1) transversality: $\underline{a}_u = m_l(x_u^-)$, $\overline{a}_u = m_h(x_u^-)$ and $\overline{b}_u = m_h(x_u^+)$;

(b.2) smooth pasting: $\mathbb{E}\left[\widetilde{\mu} | \widetilde{\mu} \in [\underline{a}_u, \mu_H], \rho = \rho_l\right] + \mathbb{E}\left[\widetilde{\mu} | \widetilde{\mu} \in [\overline{a}_u, \overline{b}_u], \rho = \rho_h\right] = m_h(x_u^+)$;

(c) discrete pooling in the interval (x_d, x_u)*: there exists* $w : (x_d, x_u) \to [0,1]$ *such that:*

(c.1) zero profit: $\frac{p(x)}{x} = w(x) m_l(x) + (1 - w(x)) m_h(x)$;

(c.2) optimality: $\dot{p}(x) = m_h(x) + (1-x)\rho_h = m_l(x) + (1-x)\rho_l$;

(c.3) consistency of beliefs: $\frac{w(x)}{1-w(x)} = \frac{\phi_l(m_l(x))\dot{m}_l(x)}{\phi_h(m_h(x))\dot{m}_h(x)}$;

(d) continuous pooling at x_d*:* $m_l(x_d) = [\underline{a}_d, \underline{b}_d]$ *and* $m_h(x_d) = [\mu_L, \overline{b}_d]$ *such that:*

(d.1) transversality: $\underline{a}_d = m_l(x_d^-)$, $\underline{b}_d = m_l(x_d^+)$ *and* $\overline{b}_d = m_h(x_d^+)$;

(d.2) smooth pasting: $\mathbb{E}[\widetilde{\mu} | \widetilde{\mu} \in [\underline{a}_d, \underline{b}_d], \rho = \rho_l] + \mathbb{E}[\widetilde{\mu} | \widetilde{\mu} \in [\mu_L, \overline{b}_d], \rho = \rho_h] = m_l(x_d^-)$;

(e) bottom separation: $p(x) = m_l(x) = \rho_l \left(x_d - x + \ln\left(\frac{x}{x_d}\right)\right) + \underline{a}_d$ *and* $m_h(x) = \emptyset$*, for* $x < x_d$.

In order to show the existence of a non-gap equilibrium we divide the proof in three parts corresponding to the construction of the pooling regions: (i) top continuous pooling region; (ii) discrete pooling region; (iii) bottom continuous region. Let us denote

$$\overline{b}(x) := (\rho_0 + \delta/2)[1 - x + \ln x] + \mu_H$$

the top separating equilibrium part.

Top continuous pooling region

For each $\delta > 0$, a top continuous region is characterized by a vector

$$(\underline{a}, \overline{a}, \underline{b}, \overline{b}, x, w) \in [\mu_L, \mu_H]^4 \times [0,1]^2$$

that satisfies

$$\underline{b} = \mu_H \tag{24}$$

$$\text{Top separating region: } \overline{b} = \overline{b}(x) \tag{25}$$

$$u_x(x_u, \underline{a}_u, l) = u_x(x_u, \overline{a}_u, h): \ \underline{a} = \overline{a} + (1-x)\delta \tag{26}$$

$$\text{Price continuity: } w\underline{a} + (1-w)\overline{a} = \overline{b} \Leftrightarrow \overline{a} = \overline{b} - w(1-x)\delta \tag{27}$$

$$\text{Price continuity: } \frac{\int_{\underline{a}}^{\underline{b}} z\phi_l(z)\,dz + \int_{\overline{a}}^{\overline{b}} z\phi_h(z)\,dz}{\int_{\underline{a}}^{\underline{b}} \phi_l(z)\,dz + \int_{\overline{a}}^{\overline{b}} \phi_h(z)\,dz} = \overline{b} \tag{28}$$

$$\text{No deviation for } (\mu_H, \overline{b}): \ \overline{b} + (1-x)\delta \geq \mu_H \tag{29}$$

and

$$\text{Weight feasibility: } w \in [0, R(\underline{a}, x, \delta)], \tag{30}$$

where we are dropping the sub-index u for convenience. Notice that $\left(x, \underline{a}, \overline{a}, \underline{b}, \overline{b}, w\right)$ defines a top continuous pooling region if and only if $x \in (0,1)$ and $w \in [0, R(\overline{b}(x) + (1-w)(1-x)\delta, x, \delta))$ solves the equation

$$G(x,w) := \int_{\overline{b}(x)+(1-w)(1-x)\delta}^{\mu_H} (z - \overline{b}(x))\phi_l(z)dz + \int_{\overline{b}(x)-w(1-x)\delta}^{\overline{b}(x)} (z - \overline{b}(x))\phi_h(z)dz = 0 \tag{31}$$

and $\overline{b}(x) \geq \mu_H - (1-x)\delta$. In this case, $\underline{a} = \overline{b}(x) + (1-w)(1-x)\delta$, $\overline{a} = \overline{b}(x) - w(1-x)\delta$, $\underline{b} = \mu_H$ and $\overline{b} = \overline{b}(x)$. For simplicity, we will refer only to (x,w) in short instead of the whole vector $\left(x, \underline{a}, \overline{a}, \underline{b}, \overline{b}, w\right)$. Define $\underline{x}^\delta \in (0,1)$ as the interior solution of the equation

$$(\rho_0 + \delta/2)\left[1 - x + \ln x\right] + (1-x)\,\delta = 0,$$

which represents the coverage level of risk type with high risk aversion in top separating equilibrium part that pools with risk type μ_h with low risk aversion.

Lemma 9. *(Top continuous pooling region) The top continuous pooling region is parameterized by a non-degenerated interval $[\underline{x}^\delta, \overline{x}^\delta] \subset [0,1]$ satisfying:*

(a) for each $x \in [\underline{x}_\delta, \overline{x}_\delta)$, there exists $w \in [0, R(\overline{b}(x) + (1-w)(1-x)\delta, x, \delta))$ such that (x,w) defines a top continuous region;

(b) if $x = \overline{x}_\delta < 1$, then $w = R(\overline{b}(\overline{x}^\delta) + (1-w)(1-\overline{x}^\delta)\delta, \overline{x}^\delta, \delta)$.

Proof. (a) We have that (x,w) defines a top continuous pooling region if and only if we can find $w \in [0, R(\overline{b}(x) + (1-w)(1-x)\delta, x, \delta))$ that solves the equation (31) and $\overline{b}(x) \geq \mu_H - (1-x)\delta$. It is easy to see that $(\underline{x}^\delta, 0)$ defines the unique top continuous pooling region such that $w = 0$, i.e., $G(\underline{x}^\delta, 0) = 0$ and $\overline{b}(\underline{x}^\delta) = \mu_H - (1-\underline{x}^\delta)\delta$.

The derivative of G w.r.t. w is

$$G_w(x,w) = (1-w)(1-x)^2\delta^2\phi_l(\overline{b}(x) + (1-w)(1-x)\delta) - w(1-x)^2\delta\phi_h(\overline{b}(x) - w(1-x)\delta)$$

which is positive if and only if

$$w < R(\overline{b}(x) + (1-w)(1-x)\delta, x, \delta). \tag{32}$$

The derivative of G w.r.t. x is

$$
\begin{aligned}
G_x(x,w) = \quad & -(1-w)(1-x)\delta\phi_l(\bar{b}(x)+(1-w)(1-x)\delta)\left[\dot{\bar{b}}(x)-(1-w)\delta\right] \\
& +w(1-x)\delta\phi_h(\bar{\bar{b}}(x)-w(1-x)\delta)\left[\dot{\bar{\bar{b}}}(x)+w\delta\right] \\
& -\dot{\bar{b}}(x)\left[\int_{\bar{b}(x)+(1-w)(1-x)\delta}^{\mu_H}\phi_l(z)dz+\int_{\bar{b}(x)-w(1-x)\delta}^{\bar{b}(x)}\phi_h(z)dz\right].
\end{aligned}
$$

At the point $(\underline{x}^\delta, 0)$ we have

$$
G_x(\underline{x}^\delta,0) = -(1-\underline{x}^\delta)\delta\phi_l(\mu_H)\left[\dot{\bar{b}}(\underline{x}^\delta)-\delta\right].
$$

(b) Notice that $\underline{x}^\delta$ is the unique solution in $(0,1)$ of the equation

$$
\bar{b}(x) = \mu_H - (1-x)\delta,
$$

which is the intersection of an increasing strictly concave function with an increasing linear function with the crossing point at μ_H (i.e., the linear function is a secant of the concave function at $\underline{x}^\delta$ and μ_H). Hence, $\dot{\bar{b}}(\underline{x}^\delta)-\delta>0$ and $G_x(\underline{x}^\delta,0)<0$. Therefore, we can apply the implicit function theorem to guarantee the existence of an non-degenerated interval $(\underline{x}^\delta, \overline{x})$ such that, for each $x \in (\underline{x}^\delta, \overline{x})$, there exists $w>0$ that solves the equation (31) and (32). Since $\underline{x}^\delta$ is the unique solution of the equation $\bar{b}(x)=\mu_H-(1-x)\delta$ and $\dot{\bar{b}}(\underline{x}^\delta)-\delta>0$, we have that $\bar{b}(x)>\mu_H-(1-x)\delta$, for all $x\in(\underline{x}^\delta,\overline{x})$. We can now define the supreme $\overline{x}^\delta$ of $\overline{x}\in(\underline{x}^\delta,1]$ such that, for each $x\in(\underline{x}^\delta,\overline{x})$, $\bar{b}(x)>\mu_H-(1-x)\delta$, and there exists $w>0$ such that (31) and (32) hold. This concludes the proof. □

Discrete pooling region

Taking derivatives of (10) and (11) w.r.t. x and manipulating them, we have that (m,w) is the solution to the differential system:

$$
\begin{aligned}
\dot{m}(x) &= \delta\frac{(1-R(m(x),x,\delta))w(x)}{R(m(x),x,\delta)-w(x)} \\
\dot{w}(x) &= \left(\frac{1}{x}-\frac{1}{1-x}\right)(1-w(x))+\delta^{-1}\left(\frac{\rho_0}{x}-\frac{\dot{m}(x)}{1-x}\right)
\end{aligned}
\tag{33}
$$

with initial condition (x_0,μ_0,w_0), where $m=m_l$.

Lemma 10. *Let $\delta>0$ be sufficiently small. For each $x_0\in(0,1)$, $\mu_0\in[\mu_L,\mu_H]$ and $w_0\in[0,R(\mu_0,x_0,\delta)]$, there exists a unique solution (m,w) of (33) with initial condition (x_0,μ_0,w_0) defined in an interval $[x_0,x_1]$ that satisfies:*

(i) $\dot{m}(x) > 0$*;*

(ii) $w(x) \in [0, R(m(x), x, \delta))$*, for all* $x \in (x_0, x_1]$*;*

(iii) $m(x_1) = \mu_H$ *or* $w(x_1) = 0$.

Proof. Let us consider the following cases:

(a) $w_0 \in [0, R(\mu_0, x_0, \delta))$. By standard theorem for existence and uniqueness of a solution for ODE system (see, for instance, Theorem 3.1 (p. 18) of Hale (1969)), there exists a unique local solution (m, w) of the system (33) with initial condition (x_0, μ_0, w_0). For this local solution, we have that $\dot{m}(x) > 0$ and $w(x) \in [0, R(m(x), x, \delta))$. We claim that we can extend the solution until we reach x_1 such that $m(x_1) = \mu_H$ or $w(x_1) = 0$. Otherwise, we must have $w(x_1) = R(m(x_1), x_1, \delta)$. However, rewriting (33) through a change of variable from x to μ, we get with some abuse of notation:

$$\begin{aligned} \dot{x}(\mu) &= \tfrac{R(\mu,x(\mu),\delta)-w(\mu)}{\delta w(\mu)(1-R(\mu,x(\mu),\delta))} \\ \dot{w}(\mu) &= \left(\tfrac{1}{x(\mu)} - \tfrac{1}{1-x(\mu)}\right)(1-w(\mu))\dot{x}(\mu) + \delta^{-1}\left(\tfrac{\rho_0 \dot{x}(\mu)}{x(\mu)} - \tfrac{1}{1-x(\mu)}\right) \end{aligned}$$

with the same initial condition. With some abuse of notation we will consider the same notation for w function in both ODE systems. We have that $\dot{x}(\mu_1) = 0$ where $\mu_1 = m(x_1)$. Then

$$\dot{w}(\mu_1) = -\frac{\delta^{-1}}{1 - x(\mu_1)} \leq -\delta^{-1} < 0.$$

Taking the second derivative of $x(\cdot)$ for $\mu < \mu_1$ and making $\mu \to \mu_1$ we get

$$\ddot{x}(\mu_1) > 0 \text{ if and only if } \frac{\partial R}{\partial \mu}(\mu_1, x(\mu_1), \delta) - \dot{w}(\mu_1) > 0.$$

Since $\frac{\partial R}{\partial \mu}(\cdot, \cdot, \delta)$ uniformly converges to $\frac{\partial R}{\partial \mu}(\cdot, \cdot, 0)$ and $\dot{w}(\mu_1) \to -\infty$ when $\delta \to 0$, $\left|\frac{\partial R}{\partial \mu}(\mu_1, x(\mu_1), \delta)\right| < -\dot{w}(\mu_1)$ for sufficiently small δ, which implies that the function $x(\cdot)$ is locally convex around μ_1 and with zero derivative at μ_1. This contradicts the fact that $\dot{x}(\mu) > 0$, for all $\mu \in [\mu_0, \mu_1)$.

(b) $w_0 = R(\mu_0, x_0, \delta)$. Rewriting (33) through a change of variable from x to μ, by standard theorem for existence and uniqueness of a solution of ODE system, there exists a unique local solution (x, w) of the system (33) with initial condition (x_0, μ_0, w_0). By the same argument as in case (i), this local solution can be extended to a maximal interval $[\mu_0, \mu_1]$ with $\dot{x}(\mu) > 0$ and $w(\mu) \in [0, R(\mu, x(\mu), \delta))$, for all $\mu \in (\mu_0, \mu_1]$, and $\mu_1 = \mu_H$ or $w(\mu_1) = 0$. We can change μ back to x and get the solution of the original system. □

Lemma 11. *For each $\delta \in (0,1)$, there exists a solution (w, m) of (33) with final condition $(x_1, \underline{a}_1, w_1)$ such that:*

(i) $x_1 \in (\underline{x}^\delta, \overline{x}^\delta)$ and $w_1 \in (0, R(\underline{a}_1, x_1, \delta))$ define a top continuous pooling region;

(ii) the maximal interval is $[x_0, x_1]$ such that $x_0 \leq \delta$ or $m(x_0) = \mu_L + (1 - x_0)\delta$.

Proof. (i) By Lemma 9, for each $x_1 \in [\underline{x}^\delta, \overline{x}^\delta]$, there exists $\left(x_1, \underline{a}_1, \overline{a}_1, \underline{b}_1, \overline{b}_1, w_1\right)$ which defines a top continuous pooling region. By Lemma 10, the solution (μ, w) of (33) with final condition $(x_1, \underline{a}_1, w_1)$ and maximal interval $[x_0, x_1]$ is such that: (a) $m(x_0) \geq \mu_L$; (b) $\dot{m}(x) > 0$, for $x \in (x_0, x_1]$; (c) $0 \leq w(x_0) \leq R(m(x_0), x_0, \delta)$.

(ii) Suppose that the result is not true, i.e., $x_0 > \delta$ and $m(x_0) > \mu_L + (1 - x_0)\delta$, for all possible $x_1 \in [\underline{x}^\delta, \overline{x}^\delta)$. There are three possibilities: (a) $w(x_0) = 0$; (b) $w(x_0) = R(m(x_0), x_0, \delta)$; (c) $0 < w(x_0) < R(m(x_0), x_0, \delta)$. Applying Lemma 10, case (c) cannot hold since $[x_0, x_1]$ is the maximal interval. Therefore, let A (resp. B) be the subset of $x_1 \in [\underline{x}^\delta, \overline{x}^\delta]$ such that $m(x_0) > \mu_L + (1 - x_0)\delta$; $\dot{m}(x) > 0$, for $x \in (x_0, x_1]$; and $w(x_0) = 0$ (resp. $w(x_0) = R(m(x_0), x_0, \delta)$).

If $\overline{x}^\delta < 1$, $\underline{x}^\delta \in A$ and $\overline{x}^\delta \in B$, and $A \cup B = [\underline{x}^\delta, \overline{x}^\delta]$. We claim that A and B are closed sets.

To show that A is closed, let (x_1^n) be a sequence in A that converges to $x_1 \in [\underline{x}^\delta, \overline{x}^\delta]$. By the definition of A and Lemma 10, we know that there exists a solution (m^n, w^n) of (33) with initial condition $(x_0^n, \mu_0^n, 0)$ in maximal interval $[x_0^n, x_1^n]$ which satisfies: $\dot{m}^n(x) > 0$; $w^n(x) \in [0, R(m^n(x), x, \delta))$, for all $x \in [x_0, x_1]$; and $m^n(x_1^n) = \mu_H$ or $w^n(x_1^n) = 0$. By continuity and regularity of (33) we have that there exists a solution (m, w) of (33) with initial condition $(x_0, \mu_0, 0)$ in maximal interval $[x_0, x_1]$ which satisfies: $\dot{m}(x) > 0$; $w(x) \in [0, R(m(x), x, \delta))$, for all $x \in [x_0, x_1]$; and $m(x_1) = \mu_H$ or $w(x_1) = 0$. Hence, $x_1 \in A$.

To show that B is closed, let (x_1^n) be a sequence in B that converges to $x_1 \in [\underline{x}^\delta, \overline{x}^\delta]$. By the definition of B and Lemma 10, we know that there exists a solution (m^n, w^n) of (33) with initial condition (x_0^n, μ_0^n, w_0^n) in maximal interval $[x_0^n, x_1^n]$ which satisfies: $w_0^n = R(\mu_0^n, x_0^n, \delta)$; $\dot{m}^n(x) > 0$; $w^n(x) \in [0, R(m^n(x), x, \delta))$, for all $x \in (x_0^n, x_1^n]$; and $m^n(x_1^n) = \mu_H$ or $w^n(x_1^n) = 0$. By continuity and regularity of (33) there exists a solution (μ, w) of (33) with initial condition (x_0, μ_0, w_0) in a maximal interval $[x_0, x_1]$ which satisfies: $w_0 = R(\mu_0, x_0, \delta)$; $\dot{\mu}(x) > 0$; $w(x) \in [0, R(m(x), x, \delta))$, for all $x \in (x_0, x_1]$; and $\mu(x_1) = \mu_H$ or $w(x_1) = 0$. Hence, $x_1 \in B$. Therefore, we must have $x_1 \in A \cap B$, which leads to a contradiction by the uniqueness of the ODE solution.

If $\overline{x}^\delta = 1$ and $B = \emptyset$, then take a sequence (x_0^n) in A such that the corresponding sequence of (x_1^n) of the upper bound of the maximal interval which converges to 1 and the solution

(w^n, μ^n) of (33). Notice that, by Lemma 13 below, $x_0^n \le x^\delta < 1$. Using the second equation of (33), we have that

$$(1-\overline{w})\int_{a_n}^{x_1^n}\left(\frac{1}{z}-\frac{1}{1-z}\right)dz + \delta^{-1}\rho_0\int_{a_n}^{x_1^n}\frac{1}{z}dz \ge w^n(x_1^n) - w^n(a_n)$$

or

$$\left(1-\overline{w}+\delta^{-1}\rho_0\right)\left[\ \ln x_1^n - \ \ln a_n\right] + (1-\overline{w})\left[\ \ln\,(1-x_1^n) - \ \ln\,(1-a_n)\right] \ge w^n(x_1^n) - w^n(a_n),$$

where $\overline{w} = \sup_{\mu,x} R(\mu,x,\delta)$ and $a_n = \max\{1/2, x_0^n\}$. Notice that the left hand side of the above inequality converge to $-\infty$, which implies that $w^n(x_1^n) \to -\infty$, when $n \to \infty$. However, this contradicts Lemma 10 and the definition of x_1^n. Therefore, if $\overline{x}^\delta = 1$, then $B \neq \emptyset$ and the previous argument again applies. □

Lemma 12. *If $\delta > 0$ is sufficiently small, the solution (m,w) of (33) with maximum interval $[x_0,x_1]$ in Lemma 11 satisfies $m(x_0) = \mu_L + (1-x_0)\delta$.*

Proof. Using the notation of Lemma 11, suppose that $x_0 \le \delta$ for $\delta > 0$ is small enough. Integrating the second equation of the ODE system, we have that

$$m(x) = \underline{a}_1 + \rho_0\int_x^{x_1}\left(1-\frac{1}{z}\right)dz + \delta\int_x^{x_1}\left(2-\frac{1}{z}\right)(1-w(z))dz + \delta\int_x^{x_1}(1-z)\dot{w}(z)dz,$$

for all $x \in [\delta, x_1]$. Integrating we have

$$\begin{aligned} m(x) = \ & \underline{a}_1 + \rho_0\left[x_1 - x + \ \ln x - \ \ln x_1\right] + \delta\int_x^{x_1}\left(2-\tfrac{1}{z}\right)(1-w(z))dz \\ & +\delta\left[(1-x_1)w(x_1) - (1-x)w(x) - \int_x^{x_1} w(z)dz\right]. \end{aligned}$$

Since $w(x)$ is uniformly bounded in the interval $[\delta, 1]$, when $\delta \to 0$ we have that $\mu(x)$ uniformly converges to $\underline{a}_1 + \rho_0\left[x_1 - x + \ \ln x - \ \ln x_1\right]$ on the compact interval of $[\delta, x_1]$. Hence, there exist $\delta > 0$ and $\tilde{x} \in [\delta, x_1]$ such that $\mu(\tilde{x}) < \mu_L + (1-x_0)\delta$, which concludes the proof. □

Lemma 13. *Suppose that (m,w) is a solution of (33) and $x \in (0,1)$ is such that $w(x) = 0$. Then, $\dot{w}(x) \gtrless 0$ if and only if $x \lessgtr \frac{\delta+\rho_0}{2\delta+\rho_0}$.*

Proof. We have that $\dot{m}(x) = 0$ and $\dot{w}(x) = \frac{1}{x} - \frac{1}{1-x} + \frac{\rho_0}{\delta x}$. Then, the result follows. □

Bottom continuous pooling region

Fix the equilibrium (m, w) of Lemma 12 on the maximal interval $[x_0, x_1]$. A bottom continuous region is characterized by a vector

$$\left(\underline{a}, \overline{a}, \underline{b}, \overline{b}, x, w\right) \in [\mu_L, \mu_H]^4 \times [0,1]^2$$

that satisfies

$$\underline{b} = \mu(x) \tag{34}$$

$$\overline{b} = \underline{b} - (1-x)\delta \tag{35}$$

$$\overline{a} = \mu_L \tag{36}$$

$$w = w(x) \tag{37}$$

$$\underline{a} = w\underline{b} + (1-w)\,\overline{b} = \underline{b} - (1-w)(1-x)\delta \tag{38}$$

and

$$\frac{\int_{\underline{a}}^{\underline{b}} z\phi_l\,(z)\,dz + \int_{\overline{a}}^{\overline{b}} z\phi_h\,(z)\,dz}{\int_{\underline{a}}^{\underline{b}} \phi_l\,(z)\,dz + \int_{\overline{a}}^{\overline{b}} \phi_h\,(z)\,dz} = \underline{a}, \tag{39}$$

where we are dropping the sub-index d for convenience. Notice that $\left(x, \underline{a}, \overline{a}, \underline{b}, \overline{b}, w\right)$ defines a bottom continuous pooling region if and only if $x \in (0,1)$ solves the equation

$$\begin{aligned} &\textstyle\int_{m(x)-(1-w(x))(1-x)\delta}^{m(x)} (z - m(x) + (1-w(x))(1-x)\delta)\phi_l(z)dz \\ &\quad + \textstyle\int_{\mu_L}^{\mu(x)-(1-x)\delta} (z - m(x) + (1-w(x))(1-x)\delta)\phi_h(z)dz \quad = 0. \end{aligned} \tag{40}$$

It is easy to see that there exists $\overline{x}_0 \in (x_0, 1)$ such that $m(\overline{x}_0) - (1-\overline{x}_0)\delta = \mu_L$ and, therefore, the left hand side of (40) is non-negative at $x = \overline{x}_0$. Moreover, if δ is sufficiently small, the left hand side of (40) at $x = x_1$ is negative. By the intermediate value theorem, there exists $\tilde{x}_0 \in [\overline{x}_0, x_1)$ for which the equation (40) holds at $x = \tilde{x}_0$.

Global deviation

The last step of the proof is to show that the necessary local deviation conditions of the consumer's problem are sufficient for the global deviation conditions. The following lemma states this property.

Lemma 14. *If price and type assignment functions (p, m_l, m_h) satisfy the consumers' local deviation conditions (i.e., FOC and monotonicity), then they also satisfy the global deviation conditions.*

Proof. For each coverage $x \in [0,1]$, let $(m_i(x), \rho_i)$ be a type that chooses coverage x. We have to show that $(m_i(x), \rho_i)$ does not deviate to any contract $\hat{x} \in [0,1]$. Let $(m_j(\hat{x}), \rho_j)$ be a type that is indifferent to choose $\hat{x}$. By continuity, without loss of generality, we can assume that $x, \hat{x} \in [x_L, x_d) \cup (x_d, x_u) \cup (x_u, 1]$, i.e., they do not belong to kinks of the price function and are on-the-path equilibrium allocations. Thus, from the FOC of agent's problem we have that

$$\dot{p}(x) = u_x(x, m_i(x), \rho_i) \text{ and } \dot{p}(\hat{x}) = u_x(\hat{x}, m_j(\hat{x}), \rho_j).$$

Let us consider the following cases:

(a) $i = j$. From the FOC of the agent's problem,

$$u(x, m_i(x), \rho_i) - p(x) \geq u(\hat{x}, m_i(x), \rho_i) - p(\hat{x})$$

if and only if

$$\int_{\hat{x}}^{x} \int_{m_i(\hat{x})}^{m_i(x)} u_{x\mu}(s, t, \rho_i) dt ds \geq 0.$$

Since m_i is a non-decreasing function and $u_{x\mu} > 0$, this is always true.

(b) $i = l$ and $j = h$. Then, $x \in [x_L, x_d) \cup (x_d, x_u)$ and $\hat{x} \in (x_d, x_u) \cup (x_u, 1]$. There are two subcases to consider:

(b.1) $x \leq \hat{x}$. If $\hat{x} \in (x_d, x_u)$, then $u_x(\hat{x}, m_h(\hat{x}), \rho_h) = u_x(\hat{x}, m_l(\hat{x}), \rho_l)$; if $\hat{x} \in (x_u, 1]$, then $m_l(\hat{x}) = \mu_H$ and $u_x(\hat{x}, m_h(\hat{x}), \rho_h) \geq u_x(\hat{x}, m_l(\hat{x}), \rho_l)$.[24] In both cases we have

$$u_x(x, m_l(x), \rho_l) - u_x(\hat{x}, m_h(\hat{x}), \rho_h) \leq u_x(x, m_l(x), \rho_l) - u_x(\hat{x}, m_l(\hat{x}), \rho_l).$$

Since $u_{x\mu} > 0$ and $m_l(x) \leq m_l(\hat{x})$, then

$$\begin{aligned} u_x(x, m_l(x), \rho_l) - u_x(\hat{x}, m_l(\hat{x}), \rho_l) &\leq u_x(x, m_l(x), \rho_l) - u_x(\hat{x}, m_l(x), \rho_l) \\ &= \int_{\hat{x}}^{x} u_x(s, m_l(x), \rho_l) ds. \end{aligned}$$

From the FOC of the agent's problem and combining the above two inequalities, we have

[24] Here we are using the properties of the top continuous pooling in this proof. In particular, $\overline{b}(x) \geq \mu_H - (1-x)\delta$, for all $x \in (x_u, 1]$.

the deviation condition

$$u(\hat{x}, m_l(x), \rho_l) - p(\hat{x}) \leq u(x, m_l(x), \rho_l) - p(x).$$

(b.2) $x > \hat{x}$. Then, $x, \hat{x} \in (x_d, x_u)$ and we can assume $i = j$ without loss of generality. Therefore, the proof is equivalent to case (a).

(c) $i = h$ and $j = l$. The proof is analogous to case (b). □

Equilibrium existence and multiplicity

Using the construction in the proof above, we showed that for each $x \in [\underline{x}_\delta, \overline{x}_\delta)$ defined in Lemma 9, there exists an equilibrium characterized in Proposition 1. This completes the proof of existence and multiplicity of equilibria.

Appendix C - Equilibrium convergence

In this section, we obtain approximation results for equilibrium objects for $\delta > 0$ sufficiently small and an arbitrary equilibrium selection. Since, as $\delta \to 0$, $(x_d, x_u) \to (x_L, 1)$, almost all traded contracts fall in the discrete pooling region eventually. We start by providing results related to the top continuous pooling region, which are then used to obtain approximation results for the discrete pooling region.

Top continuous pooling region

For $\delta > 0$, a top continuous pooling region is defined by coverage level x_u, set of risk levels pooled with low risk aversion, $[\underline{a}_u, \mu_H]$, high risk aversion, $\left[\overline{a}_u, \overline{b}_u\right]$, and the weight function at the top of the discrete pooling region, $w\left(x_u\right)$. In this section, for brevity, we denote these objects as $\left(x_u, \underline{a}_u, \overline{a}_u, \overline{b}_u, w\right)$. They must satisfy equilibrium equations (24)-(30).

We us the notation $\left(x_u\left(\delta\right), \underline{a}_u\left(\delta\right), \overline{a}_u\left(\delta\right), \overline{b}_u\left(\delta\right), w\left(\delta\right)\right)$ in order to make the dependence on parameter δ explicit. We will start by characterizing the convergence of $x_u\left(\cdot\right)$ to coverage one as $\delta \to 0$. We proceed by "sandwiching" $x_u\left(\delta\right)$: we find a lower bound $\underline{x}_u\left(\delta\right)$ such that $\left|\underline{x}_u\left(\delta\right) - 1\right|$ is $o\left(\delta\right)$. We then study the limiting behavior of all other endogenous variables involved in the top continuous pooling region. Finally, we proceed to study the behavior of certain equilibrium objects in the discrete pooling equilibrium region, which includes all types in the limit as $\delta \to 0$.

For each $\delta > 0$, the function $x \mapsto \rho_h(\delta)(1 - x + \ln x) + \delta(1 - x)$ is strictly concave, zero and strictly decreasing at $x = 1$, and strictly negative for x sufficiently small, we know that it has a unique zero in $(0, 1)$, denoted by $\underline{x}_u(\cdot)$. From (29), we must have $x_u(\delta) \in [\underline{x}_u(\delta), 1)$ for $\delta > 0$. Finally, define $\underline{x}_u(0) \equiv 1$.

In the next lemma we show that the lower bound on $x_u(\delta)$ converges to 1 at rate δ, which means that any selection $x_u(\delta)$ also converges to 1 at rate δ or faster.

Lemma 15. *The lower bound on the pooling point, $\underline{x}_u(\delta)$ is continuously differentiable and satisfies $\underline{x}_u(0) = 1$ and $\dot{\underline{x}}_u(0) = -\frac{3}{\rho_0}$.*

Proof. For $\delta > 0$, continuous differentiability of $\underline{x}_u(\cdot)$ follows from the implicit function theorem, while continuity at $\delta = 0$ follows from the fact that $1 - x + \ln x < 0$, for any $x \in (0, 1)$.

Notice that, for $\delta > 0$,

$$\dot{\underline{x}}_u(\delta) = -\frac{1}{2}\frac{\frac{\ln \underline{x}_u(\delta)}{\delta} + 3\left(\frac{1-\underline{x}_u(\delta)}{\delta}\right)}{\frac{\rho_h(\delta)}{\underline{x}_u(\delta)}\frac{1-\underline{x}_u(\delta)}{\delta} - 2}. \tag{41}$$

Which implies that $\dot{\underline{x}}_u(\delta) < 0$. If there exists a sequence $\{\delta_n\}_n$ such that $\delta_n \to 0$ and $\dot{\underline{x}}_u(\delta_n)$ converges, then $\underline{x}_u(\cdot)$ is differentiable at zero and, using (41) satisfies

$$1 = -\frac{1}{\rho_0 \dot{\underline{x}}_u(0) + 2} \implies \dot{\underline{x}}_u(0) = -\frac{3}{\rho_0}.$$

Otherwise, we must have $\lim_{\delta\to 0} \dot{\underline{x}}_u(\delta) = \lim_{\delta\to 0} \frac{\underline{x}_u(\delta)-1}{\delta} = -\infty$, and (41) implies that $\lim_{\delta\to 0} \underline{x}_u(\delta) = -\frac{1}{\rho_0}$, a contradiction. □

Corollary 3. *For any arbitrary selection $(x_u(\delta))_{\delta>0}$, we have that*

$$\lim_{\delta\to 0} x_u(\delta) = 1 \text{ and } \limsup_{\delta\to 0} \frac{1 - x_u(\delta)}{\delta} \le -\dot{\underline{x}}_u(0).$$

Proof. The results follow directly from $1 \le x_u(\delta) \le \underline{x}_u(\delta)$ and Lemma 15. □

This means that, potentially passing to a sub-sequence, we can assume that $\frac{1-x_u(\delta)}{\delta}$ converges. Define $x_u(0) \equiv \lim_{\delta\to 0} x_u(\delta) = 1$ and $\dot{x}_u(0) \equiv \lim_{\delta\to 0} \frac{x_u(\delta)-1}{\delta}$. The same definition will be extended to equilibrium variables $\overline{b}_u(\delta)$, $\overline{a}_u(\delta)$, $\underline{a}_u(\delta)$, $w(x;\delta)$ and $p(x;\delta)$.

We now characterize the changes in the top continuous pooling parameters around $\delta = 0$.

Lemma 16. *For any arbitrary selection we have that:*

(i) Convergence: $\overline{a}_u(0) = \underline{a}_u(0) = \overline{b}_u(0) = \mu_H$

(ii) δ-order convergence: $\dot{\underline{a}}_u(0) = \dot{\overline{a}}_u(0) = \dot{\overline{b}}_u(0) = 0$;

(iii) δ^2-order convergence: $\ddot{\overline{b}}_u(0) = -\rho_0 (\dot{x}_u(0))^2$.

Proof. (i) follows directly from taking limits on equations (24)-(30). (ii) follows from subtracting both sides of equations (25), (26) and (27) by μ_H, dividing by δ and taking limits $\delta \to 0$. To obtain (iii) we rewrite equation (**??**) as follows:

$$\frac{\overline{b}_u - \mu_H}{\delta^2} = \rho_h(\delta) \left(\frac{x_u(\delta) - 1}{\delta} \right)^2 \frac{\left(\frac{\ln x_u(\delta)}{x_u(\delta)-1} - 1 \right)}{x_u(\delta) - 1}.$$

Taking limits we have that $\ddot{\overline{b}}_u(0) = -\rho_0 [\dot{x}_u(0)]^2$. □

Discrete pooling approximation

For clarity, our results in this section are grouped according to the equilibrium object they refer to. Let the discrete pooling prices and relative weights be given by $p(x;\delta)$ and $w(x;\delta)$. In the discrete pooling region of any equilibrium, the price function follows the differential system: from (10) we have

$$\dot{p}(x;\delta) = \frac{p(x;\delta)}{x} + (1-x) \left[\rho_0 + \delta \left(\frac{1}{2} - w(x;\delta) \right) \right], \tag{42}$$

while (10) and (11) imply

$$\dot{w}(x;\delta) = (1 - w(x;\delta)) \frac{1-2x}{x(1-x)} + \frac{\rho_0 - \delta}{\delta x} - \frac{w(x;\delta)\left[1 - \tilde{w}(p(x;\delta), w(x;\delta), x;\delta)\right]}{\left[\tilde{w}(p(x;\delta), w(x;\delta), x;\delta) - w(x;\delta)\right](1-x)}, \tag{43}$$

where we define the following function

$$\tilde{w}(p, w, x;\delta) \equiv \frac{\phi_l\left(\frac{p}{x} + \delta(1-w)(1-x)\right)}{\phi_l\left(\frac{p}{x} + \delta(1-w)(1-x)\right) + \phi_h\left(\frac{p}{x} - \delta w(1-x)\right)}.$$

Notice that the differential equation (42) has following integral form

$$\frac{p(x;\delta)}{x} = \overline{b}_u(\delta) - \int_x^{x_u(\delta)} \left[\rho_0 + \delta \left(\frac{1}{2} - w(z;\delta) \right) \right] \left(\frac{1}{z} - 1 \right) dz. \tag{44}$$

Hence, characterization of the behavior of the weight function is sufficient to study the price function.

The next lemma characterize the limits of the functions $p(x;\delta)$ and $w(x;\delta)$, and their derivatives when $\delta \to 0$.

Price convergence

In this section we will use the fact that the one-dimensional prices and type assignment functions satisfy

$$\left(\frac{p_0(x)}{x}\right)' = m_0' = \rho_0\left(\frac{1}{x} - 1\right). \tag{45}$$

Lemma 17. *The price function is continuous in δ at zero: for any $x \in (x_L, 1)$ $\lim_{\delta\to 0} \frac{p(x;\delta)}{x} = m_0(x)$. This convergence is uniform in any compact set $M \subset (x_L, 1)$.*

Proof. The function $p(x;\delta)$ satisfies (44). From the Lemma 16, we know that $\bar{b}_u(\delta) \to \mu_H$, which, together with (45), imply pointwise convergence. If $z_0 \equiv \inf M$, then for $\delta_0 > 0$ sufficiently small, the family $\left(\frac{p(\cdot;\delta)}{x}\right)_{\delta_0>\delta>0}$ is equi-Lipschitz[25] in M with constant $L \equiv \left(\frac{1}{z_0} - 1\right)(\rho_0 + \delta_0)$, which means that the convergence is uniform. □

Lemma 18. *(First-order price approximation) For any compact $M \subset (x_L, 1)$ and $x \in M$,*

$$\lim_{\delta\to 0} \frac{\frac{p(x;\delta)}{x} - m_0(x)}{\delta} = \frac{p_\delta(x)}{x} = \int_x^1 \left(\omega_0(z) - \frac{1}{2}\right)\left(\frac{1}{z} - 1\right) dz.$$

with the convergence being uniform in M.

Proof. From equations (44) and (45) we have

$$\frac{\frac{p(x;\delta)}{x} - m_0(x)}{\delta} = \frac{\bar{b}_u(\delta) - m_0(x_u(\delta))}{\delta} - \int_x^{x_u(\delta)} \left(\frac{1}{2} - w(z;\delta)\right)\left(\frac{1}{z} - 1\right) dz, \tag{46}$$

which, using $m_0'(1) = 0$ as well as lemmas (16) and (22), implies pointwise convergence of the object of interest. Uniform convergence comes from the fact that the left-hand side of (46) is equi-Lipschitz with constant $L \equiv \left(\frac{1}{z_0} - 1\right)$, where $z_0 \equiv \inf M$. □

[25] For any family of function $g_\delta : A \mapsto \mathbb{R}$, with index $\delta \in E$ and $A \subset \mathbb{R}$, we say that this family is equi-Lipschitz in $B \subset A$ if there exists a constant L such that

$$|g_\delta(x) - g_\delta(x')| \leq L|x - x'|,$$

for any $x, x' \in B$ and $\delta \in E$.

Lemma 19. *(Second-order price approximation) The second order approximation of equilibrium prices is given by, for any* $x \in (x_L, 1)$*,,*

$$p_{\delta\delta}(x) = 2 \int_0^1 w_\delta(z) \left(\frac{1}{z} - 1 \right) dz,$$

with the convergence being uniform for any compact $M \subset (x_L, 1)$*.*

Proof. Consider any compact $M \subset (x_L, 1)$ and $x \in M$. Using (46) and Lemma (18) we have

$$\frac{1}{\delta} \left[\frac{\frac{p(x;\delta)}{x} - m_0(x)}{\delta} - \frac{p_\delta(x)}{x} \right] = \frac{\bar{b}_u(\delta) - m_0(x_u(\delta))}{\delta^2} + \int_x^{x_u(\delta)} \frac{(w(z;\delta) - \omega_0(z))}{\delta} \left(\frac{1}{z} - 1 \right) dz - \frac{1}{\delta} \int_{x_u(\delta)}^1 \left(\omega_0(z) - \frac{1}{2} \right) \left(\frac{1}{z} - 1 \right) dz. \quad (47)$$

Taking the limit $\delta \to 0$ and using Lemmas 3, 16 and 24gives us pointwise convergence.Now, consider a compact set $M \subset (x_L, 1)$. Using Lemma 24, we can find $\delta_0 > 0$ sufficiently small such that

$$B_M \equiv \sup_{\delta \le \delta_0} \sup_{z \in M} \left| \frac{\omega_0(z) - w(z;\delta)}{\delta} \right| < \infty.$$

For $\delta > 0$ smaller than δ_0, the left-hand side of expression (47) is equi-Lipschitz, for $\delta_0 > \delta > 0$ with constant$L \equiv 2 \left(\frac{1}{z_0} - 1 \right) B_M$, given $z_0 \equiv \inf M$, and hence the convergence obtained is uniform on M. □

Weight function convergence

We now obtain some necessary convergence results for weight function $w(\cdot;\delta)$. For convenience, we refer to the partial derivatives of $\tilde{w}$ as $\tilde{w}_p(p, w, x; \delta) = \frac{\partial}{\partial p}(p, w, x; \delta)$, with same notation used for derivatives with respect to w, x and δ; we also denote $\tilde{w}(x;\delta) \equiv \tilde{w}(p(x;\delta), w(x;\delta), x; \delta)$ and the total derivative $\frac{d}{dx}[\tilde{w}(x;\delta)]$ as $d_x \tilde{w}(x;\delta)$. From Lemma 17, it is easy to show that, as $\delta \to 0$, $\tilde{w}(x;\delta) \to \omega_0(x)$ and, using direct differentiation,

$$(\tilde{w}_p(x;\delta), \tilde{w}_w(x;\delta), \tilde{w}_x(x;\delta)) \to \left(\frac{1}{x} \frac{\omega_0'(x)}{m_0'(x)}, 0, -\frac{m_0(x)}{x} \frac{\omega_0'(x)}{m_0'(x)} \right) \quad (48)$$

$$\tilde{w}_\delta(x;\delta) \to (1-x)[1 - \omega_0(x)] \omega_0(x) \left\{ [1 - \omega_0(x)] \frac{\dot{\phi}_l(m_0(x))}{\phi_l(m_0(x))} + \omega_0(x) \frac{\dot{\phi}_h(m_0(x))}{\phi_h(m_0(x))} \right\}.$$

Lemma 20. *For $\delta > 0$ sufficiently small and interval $I \subseteq (x_d(\delta), x_u(\delta))$ of size $D > 0$, there exists $x \in I$*

$$|\dot{w}(x;\delta)| \leq \frac{1}{D}.$$

Proof. Consider an interval $[a,b]$ with $b-a=D$. Since $w(\cdot;\delta)$ has maximal variation of one in $[a,b]$, the mean value theorem implies the result. $\square$

Lemma 21. *For $\delta > 0$ sufficiently small and interval $I \subset (x_d(\delta), x_u(\delta))$ of size $D > 0$, there exists $x \in I$ and bound $B(\delta, D, x)$ such that*

$$\frac{|\tilde{w}(x;\delta) - w(x;\delta)|}{\delta} \leq B(\delta, D, x),$$

where $B(\cdot)$ is strictly positive, continuous and satisfies $B(0, D, x) < \infty$.

Proof. From Lemma 20 we can find $x \in I$ such that

$$\left|(1-w(x;\delta))\frac{1-2x}{x(1-x)} + \frac{1}{\delta}\left[\frac{\rho_0-\delta}{x} - \frac{w(x;\delta)[1-\tilde{w}(x;\delta)]}{\left[\frac{\tilde{w}(x;\delta)-w(x;\delta)}{\delta}\right](1-x)}\right]\right| \leq \frac{1}{D} \tag{49}$$

which implies that, using $\tilde{w}(x;\delta) - w(x;\delta) > 0$ and $\delta > 0$ sufficiently small:

$$\frac{|\tilde{w}(x;\delta) - w(x;\delta)|}{\delta} \leq \frac{xw(x;\delta)[1-\tilde{w}(x;\delta)]}{(1-x)(\rho_0-\delta) + \delta[1-w(x;\delta)](1-2x) - \frac{\delta x(1-x)}{D}}.$$

Now define $B(\delta, D, x)$ as the right-hand side of this last inequality. $\square$

Lemma 22. *(Level convergence of w) For any $x \in (x_L, 1)$ $\limsup_\delta \left[\sup_{x\in M} \frac{|w(x;\delta)-\tilde{w}(x;\delta)|}{\delta}\right] < \infty$, which implies that $w(x;\delta)$ converges to $\omega_0(x)$. This convergence is uniform for any $M \subset (x_L, 1)$.*

Proof. Suppose, by way of contradiction, that there exists sequence $(z_n)_n$ in $M \subset (x_L, 1)$ and $(\delta_n)_n$ such that $\frac{|w(z_n;\delta_n)-\tilde{w}(z_n;\delta_n)|}{\delta_n} \to \infty$. We first show that one can find another convergent sequence $(x_n)_n$ in M such that

$$\frac{|w(x_n;\delta_n) - \tilde{w}(x_n;\delta_n)|}{\delta_n} \to \infty \text{ and } \dot{w}(x_n) = d_x\tilde{w}(x_n;\delta_n).$$

We then show that the existence of such sequence leads to a contradiction.

Denote $z_0 \equiv \inf M$ and $z_1 \equiv \sup M$. Consider $D \in \left(0, \frac{1}{2}\min\{z_0 - x_L, 1 - z_1\}\right)$ and define

$$K \equiv 1 + \sup_{x\in[z_0-D,z_0]\cup[z_1,z_1+D]} B(0, D, x).$$

Continuity of B implies that, for n sufficiently large,

$$\sup_{x\in[z_0-D,z_0]\cup[z_1,z_1+D]} B(\delta_n, D, x) < K.$$

From Lemma 21, there exist sequences $(x_n^-)_n$ in $[z_0 - D, z_0]$ and $(x_n^+)_n$ in $[z_1, z_1 + D]$ such that

$$\max\left\{\frac{|\tilde{w}(x_n^-;\delta_n) - w(x_n^-;\delta_n)|}{\delta_n}, \frac{|\tilde{w}(x_n^+;\delta_n) - w(x_n^+;\delta_n)|}{\delta_n}\right\} < K,$$

while defining $x_n \equiv \arg\max_{x\in\left[x_n^-,x_n^+\right]} |\tilde{w}(x;\delta_n) - w(x;\delta_n)|$, we have that

$$\lim_n \frac{|\tilde{w}(x_n;\delta_n) - w(x_n;\delta_n)|}{\delta_n} \geq \lim_n \frac{|\tilde{w}(z_n;\delta_n) - w(z_n;\delta_n)|}{\delta_n} = \infty.$$

Hence, $x_n \in (z_0, z_1)$ is an interior optimizer and, hence, satisfies the required properties.

We now show that the construction of sequence $(x_n)_n$ leads to a contradiction. The ordinary differential equation:

$$\dot{w}(x;\delta) = (1 - w(x;\delta))\frac{1-2x}{x(1-x)} + \frac{1}{\delta}\left[\frac{\rho_0 - \delta}{x} - \frac{w(x;\delta)[1 - \tilde{w}(x;\delta)]}{\left[\frac{\tilde{w}(x;\delta) - w(x;\delta)}{\delta}\right](1-x)}\right], \tag{50}$$

implies that $\dot{w}(x_n, \delta_n) \to \infty$. Now notice that

$$\begin{aligned}\dot{w}(x_n;\delta_n) - d_x\tilde{w}(x_n;\delta_n) &= [1 - \tilde{w}_w(p(x_n;\delta_n), w(x_n;\delta_n), x_n;\delta_n)]\,\dot{w}(x_n;\delta_n)\\ &\quad - \tilde{w}_p(p(x_n;\delta_n), w(x_n;\delta_n), x_n;\delta_n)\,\dot{p}(x;\delta_n) - \tilde{w}_x(p(x_n;\delta_n), w(x_n;\delta_n), x_n;\delta_n),\end{aligned}$$

which, using (48), diverges as $n \to \infty$ This is a contradiction with the construction of $(x_n)_n$. ∎

Lemma 23. *(Derivative convergence of weight) For any compact $M \subset (x_L, 1)$, $\sup_{x\in M}|\dot{w}(x;\delta) - \dot{\omega}_0(x)|$ converge to zero uniformly on M.*

Proof. Suppose, by way of contradiction, the desired convergence result fails. Since both $w(x;\delta)$ and $\omega_0(x)$ are twice continuously differentiable and $|w(x;\delta) - \omega_0(x)|$ converges uniformly to zero, we can find constant $\gamma > 0$ and sequences $\{x^n, \delta^n\}$ such that $x^n \in M$, $\delta^n \to 0$ and

$$\dot{w}(x^n; \delta^n) - \dot{\omega}_0(x^n) = \gamma, \tag{51}$$

$$\ddot{w}(x^n; \delta^n) - \ddot{\omega}_0(x^n) \leq 0.$$

But, using (43) we have that

$$\begin{aligned}\ddot{w}(x^n; \delta^n) &= -\dot{w}(x;\delta)\frac{1-2x}{x(1-x)} - \frac{d}{dx}\left[\frac{w(x;\delta)(1-\tilde{w}(x;\delta))}{(w(x;\delta)-\tilde{w}(x;\delta))(1-x)}\right] \\ &+ (1-w(x;\delta))\frac{d}{dx}\left[\frac{1-2x}{x(1-x)}\right] + \frac{(\rho_0-\delta)}{\delta}\frac{d}{dx}\left[\frac{1}{x}\right] \\ &= \frac{(\dot{w}(x;\delta) - d_x\tilde{w}(x;\delta))\, w(x;\delta)(1-w(x;\delta)) + o(1)}{(\tilde{w}(x;\delta)-w(x;\delta))^2}\end{aligned} \tag{52}$$

Also notice that (51) implies that $\dot{w}$ is bounded and hence, using (48) we have that

$$\lim_n d_x\tilde{w}(x;\delta) = \dot{\omega}_0(\lim x^n).$$

This, together with (52) implies that $\lim_n \ddot{w}(x^n;\delta^n) = \infty$, contradicting the definition of $\{x^n, \delta^n\}_n$. □

Lemma 24. *(First-order approximation of weight) The weight function $w(x;\delta)$ is differentiable in δ at zero, for any $x \in (x_L, 1)$, and its derivative is given by*

$$\lim_{\delta\to 0}\frac{w(x;\delta)-\omega_0(x)}{\delta} = w_\delta(x),$$

with $w_\delta(\cdot)$ described in (15). This convergence holds uniformly in any compact $M \subset (x_L, 1)$.

Proof. Consider a compact $M \subset (x_L, 1)$. From (43), using Lemma 23, we can see that on M

$$\lim_{\delta\to 0}\frac{1}{\delta}\left[\frac{\rho_0-\delta}{2x} - \frac{w(x;\delta)[1-\tilde{w}(x;\delta)]}{\left[\frac{\tilde{w}(x;\delta)-w(x;\delta)}{\delta}\right](1-x)}\right] = \dot{\omega}_0(x) - (1-\omega_0(x))\frac{1-2x}{x(1-x)}. \tag{53}$$

Using Lemma 22, it is easy to see that on M

$$\lim_{\delta\to 0}\left[\frac{\tilde{w}(x;\delta)-w(x;\delta)}{\delta}\right] = \frac{x\omega_0(x)[1-\omega_0(x)]}{\rho_0(1-x)}. \tag{54}$$

Using the fact that $\omega_0(x) = \tilde{w}(m_0(x)x, w(x;\delta), x; 0)$, continuous differentiability of $\tilde{w}(\cdot)$, Lemma 18 and equations (48), we have that the following holds uniformly on M

$$\begin{aligned}\lim_{\delta\to0}\frac{\tilde{w}(x;\delta)-w(x;\delta)}{\delta} &= \lim_{\delta\to0}\frac{\tilde{w}(p(x;\delta),w(x;\delta),x;\delta)-\tilde{w}(m_0(x)x,w(x;\delta),x;\delta)}{\delta}\\ &+\lim_{\delta\to0}\frac{\tilde{w}(m_0(x)x,w(x;\delta),x;\delta)-\omega_0(x)}{\delta}+\lim_{\delta\to0}\frac{\omega_0(x)-w(x;\delta)}{\delta}\\ &=\frac{\dot{\omega}_0(x)}{x\dot{m}_0(x)}p_\delta(x)+\lim_{\delta\to0}\tilde{w}_\delta(p(x;\delta),w(x;\delta),x;\delta)-w_\delta(x),\end{aligned} \tag{55}$$

and hence (54), (55) and (48) give us the result. □

Welfare approximation

In this section we obtain expressions for the second order approximation of equilibrium utilities, defined in (17). The equilibrium payoffs are related to quasi-linear payoff

$$U_i(\mu;\delta)=u(t_i(\mu;\delta);\mu,\rho_i(\delta))-p(t_i(\mu;\delta);\delta),$$

by

$$V^i(\mu;\delta)=\gamma_i(U_i(\mu;\delta),\delta), \tag{56}$$

with function $\gamma_i\in C^2$ is defined by

$$\gamma_i(U,\delta)\equiv-\exp\{-\rho_i(\delta)[W-\mu+U]\}.$$

So we proceed by first characterizing the approximation terms of equilibrium allocation $t_i(\cdot;\delta)$, then quasi-linear utility $U_i(\cdot;\delta)$ and finally utility level $V^i(\mu;\delta)$.

In order to obtain approximation terms for utility levels, we need to use equilibrium behavior of demand functions $t_i(\mu;\delta)$, which satisfy $t_i(m_i(x;\delta);\delta)=x$. This is obtained through a series of lemmas. Notice that the type assignment functions satisfy, in the pooling region:

$$\begin{aligned}m_l(x;\delta)&=\frac{p(x;\delta)}{x}+\delta(1-w(x;\delta))(1-x),\\ m_h(x;\delta)&=\frac{p(x;\delta)}{x}-\delta w(x;\delta)(1-x).\end{aligned}$$

Our previous approximation results imply that $m_i(x;\delta)$ has a quadratic approximation,

$$m_i(x;\delta) = m_0(x) + \delta m_{i,\delta}(x) + \frac{\delta^2}{2} m_{i,\delta\delta}(x) + o\left(x;\delta^2\right),$$

where $\lim_{\delta\to 0}\sup_{x\in M}\frac{o(x;\delta^2)}{\delta^2}$ for any compact $M\subset(x_L,1)$.

Moreover, the first order approximation coefficients are

$$\begin{aligned} m_{l,\delta}(x) &= \frac{p_\delta(x)}{x} + (1-\omega_0(x))(1-x), \\ m_{h,\delta}(x) &= \frac{p_\delta(x)}{x} - \omega_0(x)(1-x), \end{aligned} \tag{57}$$

which implies that $m_{i,\delta}(\cdot)$ only depends on the distribution of types through $\omega_0(\cdot)$.

We can now state our limiting results regarding the demand functions.

Lemma 25. *The demand functions $t_i(\mu;\delta)$, for $i\in\{l,h\}$, satisfy, for $\mu\in(\mu_L,\mu_H)$,*

(a) continuity:

$$\lim_{\delta\to 0} t_i(\mu;\delta) = t_0(\mu);$$

(b) first-order approximation:

$$t_{i,\delta}(\mu) \equiv \lim_{\delta\to 0}\frac{t_i(\mu;\delta)-t_0(\mu)}{\delta} = -\frac{m_{i,\delta}(t_0(\mu))}{\dot m_0(t_0(\mu))};$$

(c) second-order approximation:

$$\begin{aligned} t_{i,\delta\delta}(\mu) &\equiv 2\lim_{\delta\to 0}\frac{1}{\delta}\left\{\frac{t_i(\mu;\delta)-t_0(\mu)}{\delta} - t_{i,\delta}(\mu)\right\} \\ &= -\frac{1}{\dot m_0(t_0(\mu))}\left\{\ddot m_0(t_0(\mu))\left[t_{i,\delta}(\mu)\right]^2 + 2\dot m_{i,\delta}(t_0(\mu))\, t_{i,\delta}(\mu) + m_{i,\delta\delta}(t_0(\mu))\right\}, \end{aligned}$$

with the convergence being uniform in any compact $C\subset(\mu_L,\mu_H)$. Additionally, $t_{i,\delta}(\mu)$ only depends on the type distribution through $\omega_0(x)$, for any $x\in(x_L,1)$.

Proof. Consider a compact $C\subset(\mu_L,\mu_H)$, $i\in\{l,h\}$ and $\delta>0$ sufficiently small. Let $x_1\equiv t_0(\inf C-\varepsilon)$ and $x_2\equiv t_0(\sup C+\varepsilon)$, for $\varepsilon<\min\{1-\sup C,\inf C-x_L\}$. Continuity of $m_i(\cdot;\delta)$ in δ implies that, as $\delta\to 0$, $m_i(x_1;\delta)\to\inf C-\varepsilon$ and $m_i(x_2;\delta)\to\sup C+\varepsilon$. Hence, monotonicity of m_i and t_i imply that $t_i(m_i(x_1;\delta);\delta) < t_i(\mu;\delta) < t_i(m_i(x_2;\delta);\delta)$ and hence $T_i(\delta)\equiv\{t_i(\mu;\delta)\mid\mu\in C\}$ is a compact set contained in $[x_1,x_2]\subset(x_L,1)$.

Hence, using uniform convergence of $m_i(\cdot;\delta)$ in $[x_1,x_2]$ we have that

$$m_i\left(t_i\left(\mu;\delta\right);\delta\right)-m_0\left(t_i\left(\mu;\delta\right)\right)=\delta m_{i,\delta}\left(t_i\left(\mu;\delta\right)\right)+\frac{\delta^2}{2}m_{i,\delta\delta}\left(t_i\left(\mu;\delta\right)\right)+o\left(t_i\left(\mu;\delta\right);\delta\right), \quad (58)$$

where $\sup_{\mu\in C}\frac{o(t_i(\mu;\delta);\delta)}{\delta}\to 0$.

Finally, using $m_i\left(t_i\left(\mu;\delta\right);\delta\right)=m_0\left(t_0\left(\mu\right)\right)=\mu$ we have

$$m_0\left(t_i\left(\mu;\delta\right)\right)-m_0\left(t_0\left(\mu\right)\right)=-\left[m_i\left(t_i\left(\mu;\delta\right);\delta\right)-m_0\left(t_i\left(\mu;\delta\right)\right)\right],$$

and, since the right hand side satisfies approximation equation (58), we have a quadratic approximation of the left hand side, which holds uniformly in $\mu\in C$. Since $m_0(\cdot)$ is twice continuously differentiable and has strictly positive derivative in $[x_1,x_2]$, direct differentiation gives us the result. □

We are now in position to state the main quasi-linear-payoff approximation result. Define $U^0(\mu)\equiv u\left(t_0\left(\mu\right);\mu,\rho_0\right)-p_0(\mu)$, for $\mu\in[\mu_L,\mu_H]$ and $\dot{p}_\delta(x)\equiv\frac{d}{dx}\left[p_\delta(x)\right]$.

Lemma 26. *For any $\mu\in(\mu_L,\mu_H)$ and $i\in\{l,h\}$, the payoff function has the following limiting behavior, for any compact $C\subset(\mu_L,\mu_H)$:*

(i) Continuity:

$$\lim_{\delta\to 0}U_i\left(\mu;\delta\right)=U^0\left(\mu\right);$$

(ii) First-order approximation:

$$U_{h,\delta}\left(\mu\right)\equiv\lim_{\delta\to 0}\frac{U_h\left(\mu;\delta\right)-U^0\left(\mu\right)}{\delta}=-p_\delta\left(t_0\left(\mu\right)\right)-\frac{1}{4}\left(1-t_0\left(\mu\right)\right)^2,$$

$$U_{l,\delta}\left(\mu\right)\equiv\lim_{\delta\to 0}\frac{U^l\left(\mu;\delta\right)-U^0\left(\mu\right)}{\delta}=-p_\delta\left(t_0\left(\mu\right)\right)+\frac{1}{4}\left(1-t_0\left(\mu\right)\right)^2;$$

(iii) Second-order approximation:

$$U_{h,\delta}\left(\mu\right)\equiv 2\lim_{\delta\to 0}\frac{\frac{U_h(\mu;\delta)-U^0(\mu)}{\delta}-U_{h,\delta}\left(\mu\right)}{\delta}=-\left[\begin{array}{c}p_{\delta\delta}\left(t_0\left(\mu\right)\right)\\+\left[\dot{p}_\delta\left(t_0\left(\mu\right)\right)-\frac{1}{2}\left(1-t_0\left(\mu\right)\right)\right]t_{h,\delta}\left(\mu\right)\end{array}\right],$$

$$U_{l,\delta\delta}\left(\mu\right)\equiv 2\lim_{\delta\to 0}\frac{\frac{U^l(\mu;\delta)-U^0(\mu)}{\delta}-U_{l,\delta}\left(\mu\right)}{\delta}=-\left[\begin{array}{c}p_{\delta\delta}\left(t_0\left(\mu\right)\right)\\+\left[\dot{p}_\delta\left(t_0\left(\mu\right)\right)+\frac{1}{2}\left(1-t_0\left(\mu\right)\right)\right]t_{h,\delta}\left(\mu\right)\end{array}\right],$$

with the convergence being uniform in C.

Proof. Consider compact $C \subset (\mu_L, \mu_H)$, $\delta > 0$ sufficiently small and $i \in \{l, h\}$ (the argument for $i = l$ is analogous). Uniform continuity, in δ, and monotonicity, in x, of $m_i(\cdot; \delta)$ implies that we can find x_1, x_2 such that $t_i(\mu; \delta) \subset [x_1, x_2] \subset (x_L, 1)$ (for details, see proof of Lemma 25). Notice that the function $H : (x, p, \delta) \mapsto u(x; \mu, \rho_i(\delta)) - p$ is twice continuously differentiable, hence we have that

$$\begin{aligned}
U_i(\mu; \delta) - U^0(\mu) = &\left[H\left(t_i(\mu; \delta), p\left(t_i(\mu; \delta); \delta\right), \delta\right) - H\left(t_i(\mu; \delta), p_0\left(t_i(\mu; \delta)\right), \delta\right)\right] \\
&+ \left[H\left(t_i(\mu; \delta), p_0\left(t_i(\mu; \delta)\right), \delta\right) - H\left(t_i(\mu; \delta), p_0\left(t_0(\mu)\right), \delta\right)\right] \\
&+ \left[H\left(t_i(\mu; \delta), p_0\left(t_0(\mu)\right), \delta\right) - H\left(t_0(\mu), p_0\left(t_0(\mu)\right), \delta\right)\right] \\
&+ \left[H\left(t_0(\mu), p_0\left(t_0(\mu)\right), \delta\right) - H\left(t_0(\mu), p_0\left(t_0(\mu)\right), 0\right)\right],
\end{aligned}$$

where twice continuous differentiability of p_0 and Lemmas 18 and 25 imply that each of the four terms in the right hand side have a well-defined quadratic approximation with error term $o(\mu; \delta^2)$ satisfying $\lim_{\delta \to 0} \sup_{\mu \in C} \frac{o(\mu; \delta^2)}{\delta^2}$, implying that both first and second order derivative limits are well-defined. The exact formulas obtained come from direct differentiation. □

Lemma 27.

Comparing distributions

In sections 5 and 6, we compare the utility obtained by each type under different type distributions. Consider any two distributions $\left(\phi_l^k, \phi_h^k\right)$, for $k \in \{A, B\}$, with same support. We make the dependence of equilibrium objects by using superscripts, as in $V_i^k(\mu; \delta)$, for $i \in \{l, h\}$. The superscript is omitted whenever the equilibrium object does not depend on k.

Lemma 28. *For any type with risk $(\mu, i) \in (\mu_L, \mu_H) \times \{l, h\}$ and $k \in \{A, B\}$, the following hold:*

$$V^k(\mu, i) \equiv \lim_{\delta \to 0} V^k(\mu, i; \delta) = \gamma_i\left(U^0(\mu), 0\right),$$

$$V_\delta^k(\mu, i) \equiv \lim_{\delta \to 0} \frac{V^k(\mu, i; \delta) - V^k(\mu, i; 0)}{\delta} = \frac{\partial}{\partial U}\gamma_i\left(U^0(\mu), 0\right) U_{i,\delta}^k(\mu) + \frac{\partial}{\partial \delta}\gamma_i\left(U^0(\mu), 0\right),$$

and

$$V_{\delta\delta}^{k}(\mu,i)\equiv 2\lim_{\delta\to 0}\frac{1}{\delta}\left[\frac{V^{k}(\mu,i;\delta)-V^{k}(\mu,i;0)}{\delta}-\frac{\partial}{\partial U}\gamma_i\left(U^0(\mu),0\right)U_{i,\delta}^{k}(\mu)+\frac{\partial}{\partial\delta}\gamma_i\left(U^0(\mu),0\right)\right]=$$

$$\left[\begin{array}{c}\frac{\partial^2}{\partial U^2}\gamma_i\left(U^0(\mu),0\right)\left[U_{i,\delta}^{k}(\mu)\right]^2+\frac{\partial^2}{\partial U\partial\delta}\gamma_i\left(U^0(\mu),0\right)U_{i,\delta}^{k}(\mu)\\ +\frac{\partial}{\partial U}\gamma_i\left(U^0(\mu),0\right)U_{i,\delta\delta}^{k}(\mu)\\ +\frac{\partial^2}{\partial\delta\partial U}\gamma_i\left(U^0(\mu),0\right)U_{i,\delta}^{k}(\mu)+\frac{\partial^2}{\partial\delta^2}\gamma_i\left(U^0(\mu),0\right)\end{array}\right],\quad (59)$$

with convergence of the limits above being guaranteed uniformly on any compact set $C\subset(\mu_L,\mu_H)$.

Proof. The results follow from definition expression (56), twice differentiability of γ_i and the uniform convergence results in lemma (26). □

Proof of Lemma 3

We now consider distributions indexed by $k\in\{0\}\cup S$, with index $k=0$ represents the prior and $k=s$ represents the conditional type distribution with signal realization $s\in S$. Since the signal considered is a pure risk signal, Lemma (26) and Proposition (3) imply that term $U_\delta(\mu,i)$ and, as a consequence, V_δ^k, do not vary with $k\in\{0,S\}$. So we omit notation k in their expressions.

From Lemma 28 and 26 we have that the signal effect is

$$\sum_{s\in S}\pi(s\mid\mu)V_i^s(\mu;\delta)-V_i^0(\mu;\delta)=\sum_{s\in S}\pi(s\mid\mu)\left[V_i^s(\mu;\delta)-V_i^0(\mu;\delta)\right]$$
$$=-\frac{\delta^2}{2}\frac{\partial}{\partial U}\gamma_i\left(U^0(\mu),0\right)\sum_{s\in S}\left[\pi(s\mid\mu)p_{\delta\delta}^{s}(t_0(\mu))-p_{\delta\delta}^{0}(t_0(\mu))\right]+o\left(\mu;\delta^2\right),$$

with $\lim_{\delta\to 0}\sup_{\mu\in C}\frac{o(\mu;\delta^2)}{\delta^2}=0$. The equality $\frac{\partial}{\partial U}\gamma_i\left(U^0(\mu),0\right)=\frac{\partial v}{\partial p}\left(t_0(\mu),p_0(t_0(\mu)),\mu,\rho_0\right)$ gives us the result.

Appendix D - Signal Disclosure and Comparative Statics

Proof of Proposition 4

(If): Consider an arbitrary $\varepsilon > 0$ and let $M_\varepsilon \subset (\mu_L, \mu_H)$ be compact such that $\int_{M_\varepsilon} \phi_l(\mu') + \phi_h(\mu')\, d\mu' > 1 - \varepsilon$. From Lemma 3, we can find $\bar{\delta} > 0$ such that, for any $0 < \delta < \bar{\delta}$ and $i = l, h$,

$$\sup_{\mu \in C} \frac{|o_i(\delta^2; \mu)|}{\delta^2} < \inf_{\mu \in M_\varepsilon} \frac{1}{2} \frac{\partial v}{\partial p} (t_0(\mu), p_0(t_0(\mu)), \mu, \rho_0)\, \Delta E(p(t_0(\mu))),$$

which implies that all types with risk level in M_ε have a strict interim improvement from the signal disclosure.

(Only if): Now suppose that there exists $\mu_1, \mu_2 \in (\mu_L, \mu_H)$ with $\mu_1 < \mu_2$ satisfying $\partial_2 D_{KL}(\pi(\cdot \mid \mu_1) || \pi(\cdot \mid \mu_2)) < 0$.

Now consider a sequence of absolutely continuous full-support distributions on $[\mu_L, \mu_H]$ that weakly converge to the Dirac measure $\delta_{\{\mu_2\}}$, with continuously differentiable densities $\{f^n(\cdot)\}_n$. It is easy to show that there exists C^1 functions $\omega_0^n \circ t_0 : [\mu_L, \mu_H] \mapsto (0, 1)$ such that $\omega_0^n \circ t_0(\cdot)[1 - \omega_0^n \circ t_0(\cdot)] = \frac{1}{4\bar{f}^n} f^n(\cdot)$, where $\bar{f}^n \equiv \sup f^n$.[26] Hence, for any density ϕ on $[\mu_L, \mu_H]$, consider type distributions

$$(\phi_l^n(\mu), \phi_h^n(\mu)) = (\omega_0^n \circ t_0(\mu)\, \phi(\mu), [1 - \omega_0^n \circ t_0(\mu)]\, \phi(\mu)),$$

for each n. The price effect of disclosing this signal under distribution $(\phi_l^n(\mu), \phi_h^n(\mu))$ on the price of coverage $x = t_0(\mu_1)$ is then given by

$$\Delta E(p^n(x)) = -\frac{1}{2\rho_0 \bar{f}^n} \int_{\mu_1}^{\mu_H} f^n(\mu)(1 - t_0(\mu))\, \partial_2 D_{KL}(\pi(\cdot \mid \mu_1) || \pi(\cdot \mid \mu))\, d\mu,$$

which converges, as $n \to \infty$, to

$$\lim_n \Delta E(p^n(x))\, \bar{f}^n = -\frac{1}{2\rho_0}(1 - t_0(\mu_2))\, \partial_2 D_{KL}(\pi(\cdot \mid \mu_1) || \pi(\cdot \mid \mu_2)) > 0.$$

This implies that the expected price effect on coverage $t_0(\mu_1)$ is strictly positive.

[26] A simple calculation shows that ω_0^n must satisfy $\omega_0^n \circ t_0(\mu) = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 4f^n(\mu)}$, for all $\mu \in [\mu_L, \mu_H]$. In fact, there are always at least two such solutions.

Proof of Proposition 5

For brevity, define $\pi_0(\cdot) \equiv \frac{1}{\#S}$. We then have that, for $k \in S \cap \{0\}$,

$$\Delta C^s(\overline{\mu}, \varepsilon) \equiv \varepsilon \frac{\pi(s \mid \overline{\mu}+\varepsilon)\phi_l(\overline{\mu}+\varepsilon)}{\pi(s \mid \overline{\mu}+\varepsilon)\phi_l(\overline{\mu}+\varepsilon) + \pi(s \mid \overline{\mu})\phi_h(\overline{\mu})},$$

$$\begin{aligned}
\Delta C\left(\overline{\mu}, \underline{\mu}, \varepsilon\right) &= \varepsilon \left[\sum_{s \in S} \pi\left(s \mid \underline{\mu}\right) \frac{\pi(s \mid \overline{\mu}+\varepsilon)\phi_l(\overline{\mu}+\varepsilon)}{\pi(s \mid \overline{\mu}+\varepsilon)\phi_l(\overline{\mu}+\varepsilon) + \pi(s \mid \overline{\mu})\phi_h(\overline{\mu})} - \frac{\phi_l(\overline{\mu}+\varepsilon)}{\phi_l(\overline{\mu}+\varepsilon)+\phi_h(\overline{\mu})}\right] \\
&= \varepsilon \sum_{s \in S} \pi\left(s \mid \underline{\mu}\right) \int_0^{\varepsilon} \frac{\partial}{\partial z}\left[\frac{\phi_l(\overline{\mu}+\varepsilon)}{\phi_l(\overline{\mu}+\varepsilon) + \frac{\pi(s|\overline{\mu})}{\pi(s|\overline{\mu}+z)}\phi_h(\overline{\mu})}\right] dz.
\end{aligned}$$

These imply that

$$\begin{aligned}
\lim_{\varepsilon \to 0} \frac{\Delta C\left(\overline{\mu}, \underline{\mu}, \varepsilon\right)}{\varepsilon^2} &= \sum_{s \in S} \pi\left(s \mid \underline{\mu}\right) \frac{\partial}{\partial z}\left[\frac{\phi_l(\overline{\mu}+\varepsilon)}{\phi_l(\overline{\mu}+\varepsilon) + \frac{\pi(s|\overline{\mu})}{\pi(s|\overline{\mu}+z)}\phi_h(\overline{\mu})}\right]\Bigg|_{z=\varepsilon=0} \\
&= \sum_{s \in S} \pi\left(s \mid \underline{\mu}\right) \frac{\dot{\pi}(s \mid \overline{\mu})}{\pi(s \mid \overline{\mu})} \frac{\phi_l(\overline{\mu})\phi_h(\overline{\mu})}{\left[\phi_l(\overline{\mu}) + \phi_h(\overline{\mu})\right]^2},
\end{aligned}$$

which is strictly negative, for any full support continuous (ϕ_l, ϕ_h) and almost all $\mu_L < \underline{\mu} < \overline{\mu} < \mu_H$ if, and only if $\sum_{s \in S} \pi\left(s \mid \underline{\mu}\right) \frac{\dot{\pi}(s|\overline{\mu})}{\pi(s|\overline{\mu})} < 0$ for almost all $\mu_L < \underline{\mu} < \overline{\mu} < \mu_H$, i.e., monotonicity holds.

Proof of Proposition 6

First notice that

$$\frac{\partial}{\partial \widetilde{\mu}} D_{KL}\left(\pi(\cdot \mid \mu) || \pi(\cdot \mid \widetilde{\mu})\right) = \sum_{s \in S} \frac{\dot{\pi}(s \mid \widetilde{\mu})}{\pi(s \mid \widetilde{\mu})}\left[\pi(s \mid \widetilde{\mu}) - \pi(s \mid \mu)\right], \tag{60}$$

where we have used the fact that, for any $\mu \in [\mu_L, \mu_H]$, $\sum_{s \in S} \dot{\pi}(s \mid \mu) = 0$.

From (60),

$$\frac{\partial}{\partial \widetilde{\mu}} D_{KL}\left(\pi\left(\cdot \mid \mu\right) || \pi\left(\cdot \mid \widetilde{\mu}\right)\right) = -\sum_{s \in S} \pi\left(s \mid \widetilde{\mu}\right) \frac{\dot{\pi}\left(s \mid \widetilde{\mu}\right)}{\pi\left(s \mid \widetilde{\mu}\right)} \left[\frac{\pi\left(s \mid \mu\right)}{\pi\left(s \mid \widetilde{\mu}\right)} - 1\right] \tag{61}$$

$$= -\text{cov}^{\,\widetilde{\mu}} \left(\frac{\dot{\pi}\left(s \mid \widetilde{\mu}\right)}{\pi\left(s \mid \widetilde{\mu}\right)}, \frac{\pi\left(s \mid \mu\right)}{\pi\left(s \mid \widetilde{\mu}\right)}\right), \tag{62}$$

where $\text{cov}^{\,\widetilde{\mu}}$ represents the co variance across different signal realizations w.r.t. the measure described by $\{\pi\left(s \mid \widetilde{\mu}\right)\}_{s \in S}$.

Condition MLRP implies that, for any $\ell \in \{1, ..., n-1\}$,

$$\frac{\pi(s_{\ell+1} \mid \widetilde{\mu})}{\pi(s_{\ell} \mid \widetilde{\mu})} > \frac{\pi(s_{\ell+1} \mid \mu)}{\pi(s_{\ell} \mid \mu)} \iff \frac{\pi(s_{\ell} \mid \mu)}{\pi(s_{\ell} \mid \widetilde{\mu})} > \frac{\pi(s_{\ell+1} \mid \mu)}{\pi(s_{\ell+1} \mid \widetilde{\mu})},$$

and

$$\frac{\dot{\pi}(s_{\ell+1} \mid \mu)}{\pi(s_{\ell+1} \mid \mu)} > \frac{\dot{\pi}(s_{\ell} \mid \mu)}{\pi(s_{\ell} \mid \mu)},$$

for almost all $\mu \in [\mu_L, \mu_H]$. Summing up, the first term in the co variance (61), evaluated at s_ℓ, is increasing in ℓ while the second term is decreasing in ℓ. Hence, the covariance is negative and the expression in (61) is positive.

The following lemma is an auxiliary result and shows that for the welfare comparison it suffices to consider the first- or second-order terms of the price function.

Lemma 29. *Suppose that, for every compact set $M \subseteq (x_L, 1)$, there exists $\delta > 0$ sufficiently small such that*

$$p^A_\delta\left(x\right) < p^B_\delta\left(x\right) \;\; \textit{or} \; p^A_\delta\left(x\right) = p^B_\delta\left(x\right) \;\; \textit{and} \; p^A_{\delta\delta}\left(x\right) < p^B_{\delta\delta}\left(x\right),$$

for all $x \in M$. Then, $\left(\omega^A, \phi^A\right)$ welfare-dominates $\left(\omega^B, \phi^B\right)$.

Proof. Consider a compact set $M \subset (x_L, 1)$. Proposition 3 implies that

$$\frac{p^A\left(x;\delta\right) - p^B\left(x;\delta\right)}{\delta} = \frac{p^A\left(x;\delta\right) - m_0\left(x\right)x}{\delta} - \frac{p^B\left(x;\delta\right) - m_0\left(x\right)x}{\delta},$$

which converges, uniformly in M, to $p^A_\delta\left(x\right) - p^B_\delta\left(x\right)$ and implies the result if $p^A_\delta\left(x\right) < p^B_\delta\left(x\right)$. Alternatively, if $p^A_\delta\left(x\right) = p^B_\delta\left(x\right)$, for all $x \in M$, we have that

$$\frac{p^A\left(x;\delta\right) - p^B\left(x;\delta\right)}{\delta^2} = \frac{\frac{p^A(x;\delta) - m_0(x)x}{\delta} - p^A_\delta\left(x\right)}{\delta} - \frac{\frac{p^B(x;\delta) - m_0(x)x}{\delta} - p^B_\delta\left(x\right)}{\delta},$$

which converges to $\frac{1}{2}\left[p^A_{\delta\delta}(x) - p^B_{\delta\delta}(x)\right]$ uniformly over M and implies again the result. The welfare results follow from Lemma 26. □

Lemma 29 does guarantee uniform dominance with respect to δ.

References

Akerlof, G. (1970): "The market for "lemons": Quality uncertainty and the market mechanism," *The quarterly journal of economics*, 488–500.

Azevedo, E. and D. Gottlieb (2017): "Perfect competition in markets with adverse selection," *Econometrica*, 85, 67–105.

Bisin, A. and P. Gottardi (2006): "Efficient competitive equilibria with adverse selection," *Journal of political Economy*, 114, 485–516.

Cabrales, A., O. Gossner, and R. Serrano (2017): "A normalized value for information purchases," *Journal of Economic Theory*, 170, 266–288.

Chang, B. (2018): "Adverse selection and liquidity distortion," *The Review of Economic Studies*, 85, 275–306.

Cohen, A. and L. Einav (2007): "Estimating Risk Preferences from Deductible Choice," *American Economic Review*, 97, 745–788.

Crocker, K. J. and A. Snow (1986): "The efficiency effects of categorical discrimination in the insurance industry," *Journal of Political Economy*, 94, 321–344.

Deneckere, R. and S. Severinov (2015): "Multi-Dimensional Screening: A Solution to a Class of Problems," *working paper*.

Dubey, P. and J. Geanakoplos (2002): "Competitive pooling: Rothschild-Stiglitz reconsidered," *The Quarterly Journal of Economics*, 117, 1529–1570.

Einav, L., A. Finkelstein, S. P. Ryan, P. Schrimpf, and M. R. Cullen (2013): "Selection on Moral Hazard in Health Insurance," *American Economic Review*, 103, 178–219.

Farinha Luz, V. (2017): "Characterization and uniqueness of equilibrium in competitive insurance," *Theoretical Economics*, 12, 1349–1391.

Finkelstein, A., J. Poterba, and C. Rothschild (2009): "Redistribution by insurance market regulation: Analyzing a ban on gender-based retirement annuities," *Journal of financial economics*, 91, 38–58.

Gale, D. (1992): "A Walrasian theory of markets with adverse selection," *The Review of Economic Studies*, 59, 229–255.

Galichon, A. and B. Salanié (2010): "Matching wint Trade-offs: Revealed Preferences over Competing Characteristics," *CEPR Discussion Papers 7858, C.E.P.R. Discussion Papers*.

Garcia, D. and M. Tsur (2021): "Information design in competitive insurance markets," *Journal of Economic Theory*, 191, 105160.

Guerrieri, V. and R. Shimer (2014): "Markets with multidimensional private information," *National Bureau of Economic Research*.

Guerrieri, V., R. Shimer, and R. Wright (2010): "Adverse selection in competitive search equilibrium," *Econometrica*, 78, 1823–1862.

Hale, J. K. (1969): *Ordinary differential equations*, Dover Publications, INC.

Handel, B., I. Hendel, and M. D. Whinston (2015): "Equilibria in health exchanges: Adverse selection versus reclassification risk," *Econometrica*, 83, 1261–1313.

Holmstrom, B. and P. Milgrom (1987): "Aggregation and Linearity in the Provision of Intertemporal Incentive," *Econometrica*, 55, 303–328.

Kraft, L. G. (1949): "A Device for Quantizing, Grouping, and Coding Amplitude-Modulated Pulses," *MS Thesis, Electrical Engennering Departmnet, MIT*.

Laffont, J.-J., E. Maskin, and J.-C. Rochet (1987): "Optimal Nonlinear Pricing with Two-Dimensional Characteristics," *in: T. Groves, R. Radner, and S. Reiter, eds., Information, Incentives, and Economic Mechanisms, Minneapolis*, 256–266.

McMillan, B. (1956): "Two inequalities implied by unique decipherability," *IEEE Trans. InformationTheory*, 2(4), 115–116.

Mirrlees, J. A. (1976): "The optimal structure of incentives and authority within an organization," *The Bell Journal of Economics*, 105–131.

Miyazaki, H. (1977): “The rat race and internal labor markets,” *The Bell Journal of Economics*, 8(2), 394–418.

Riley, J. G. (1979): “Informational Equilibrium,” *Econometrica*, 47, 331–359.

Rothschild, C. (2011): “The efficiency of categorical discrimination in insurance markets,” *Journal of Risk and Insurance*, 78, 267–285.

Rothschild, M. and J. Stiglitz (1976): “Equilibrium in competitive insurance markets: An essay on the economics of imperfect information,” *The Quarterly Journal of Economics*, 90(4), 629–649.

Smart, M. (2000): “Competitive insurance markets with two unobservables,” *International Economic Review*, 41(1), 153–169.

Wambach, A. (2000): “Introducing heterogeneity in the Rothschild-Stiglitz model,” *Journal of Risk and Insurance*, 67(4), 579–591.