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Relative Scarcity of Commodities with a Long-Term Economic Relationship and the Correlation of Futures Returns*

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Relative Scarcity of Commodities with a Long-Term Economic Relationship and the Correlation of Futures Returns

Abstract

This paper finds that the long-term co-movement of commodity prices is driven by economic relationships, such as production, substitution, and complementary relationships. Such relationships imply that the convenience yield of a given commodity depends on its relative scarcity with respect to associated commodities. The economic linkage between two commodities creates a new source of positive correlation between the futures returns of both commodities. We build an empirical, multi-commodity maximal affine model that allows the convenience yield of a commodity to depend on its relative scarcity. We estimate the model using three commodity pairs: heating oil-crude oil, WTI-Brent crude oil and heating oil-gasoline. Our model allows for a flexible correlation term structure of futures returns that matches the upward-sloping patterns observed in the data. The high long-term correlation implied by an economic relationship reduces the volatility of the spread between commodities, which implies lower spread option prices. An out-of-sample test using short-maturity crack spread options data shows that our model considerably reduces the negative bias generated by traditional models.

Keywords: relative scarcity, correlation term structure, futures returns, long-term economic relationships, convenience yield, feedback effect, multi-commodity maximal affine,spread option.

JEL Classification: C0, G12, G13, D51, D81, E2.

Commodity markets have experienced dramatic up-and-down movements recently within a relatively short period of time. Closest-to-maturity crude-oil futures increased from almost \$50 per barrel in January 2007 to \$147 per barrel in July 2008, the highest level in history since it began trading on the New York Mercantile Exchange (NYMEX). Surprisingly, only five months later, the oil price dropped to nearly \$30 per barrel; currently, however, the oil price is around \$100. The energy, agricultural commodities, and industry metals sectors have all experienced similar patterns. While academics and policy makers are still trying to understand the causes of this behavior, the following stylized facts, among others, have been reinforced after the turmoil: 1) commodity prices are volatile, 2) spot prices and futures prices are mean-reverting, and 3) prices of multiple commodities co-move. These characteristics play a critical role in modeling financial contingent claims on commodities.

Since Keynes (1923), many scholars have studied the stochastic behavior of individual commodities. However, relationships involving multiple commodities have received little attention in theoretical modeling and commodity-related contingent-claim pricing. These cross-commodity relationships imply that two or more commodities share an equilibrium that links prices in the long run. Examples of long-term economic relationships between commodities include production relationships, where upstream commodities and downstream commodities are tied together in a production process, and substitute (or complementary) relationships, where two commodities serve as substitutes for (or complements to) one another in either consumption or production.

This paper proposes that the dynamics of certain commodities depends not only on their own characteristics (i.e., prices, inventories), but also on the fundamentals of other commodities with which they share a long-term economic relationship. In particular, we show that the convenience yield of a commodity, a benefit realized for holding inventories of the asset, depends on its "relative scarcity" with respect to other related commodities. We extend the traditional Theory of Storage of

Kaldor (1939), Working (1948), Brennan (1958) and Telser (1958), which connects the convenience yield of a commodity with its own scarcity level, to a multi-commodity level. In our model, temporary deviation from a long-term relation between commodity prices (because of supply and demand imbalances caused by macro-economic factors, inventory shocks, etc.) will be corrected over the long run. This implies that co-movement exists not only in spot prices but also in futures prices, which are determined by the convenience yield, among other things. Moreover, we show that the economic linkage between two commodities implies a new source of correlation of futures returns.

Take the production relationship of the heating and crude oil pair as an example. As shown in Section 2, the price ratio of heating oil to crude oil is a natural measure for the relative scarcity of heating oil. A high price ratio of heating oil to crude oil (i.e. relative scarcity) increases the heating oil producer demand for crude oil, which increases the production of heating oil. The relative scarcity of heating oil implies higher *expected* inventories and lower *expected* prices for this commodity in the next period. In our theoretical model we connect this decrease in expected prices with a higher heating oil convenience yield today.

The relative scarcity is a key determinant of the joint dynamics of commodity prices and, therefore, also affects the correlation between the futures returns of both commodities. To see this, consider an increase in the relative scarcity of heating oil produced by an increase in the crude oil stocks and a consequent fall in the crude oil price. The absence of arbitrage relationship between spot and futures prices implies a negative change in crude oil futures prices. Also, a higher degree of relative scarcity of heating oil implies a higher convenience yield for this asset, which, again, in the absence of arbitrage, indicates a negative change in heating oil futures prices. This simple mechanism shows that, due to the production relationship between these commodities, a positive change in the crude oil inventories generates a decrease in the futures prices of both commodities.

Hence, the long-term economic relationship is a source of positive correlation between the crude and heating oil futures returns.

Figure 1 shows the correlation term structure of weekly futures returns for the heating-crude oil and the WTI-Brent crude oil pairs from 2007.04 to 2010.09.[1] These commodity pairs are related by a production relationship and a substitution relationship, respectively. The plot shows upward-sloping correlation term structures for both commodity pairs. Prices are tied to the long-term relationship, which translates into higher long-term correlations. Interestingly, traditional commodity pricing models, such as correlated versions of the Gibson and Schwartz (1990) (hereafter GS) and the Casassus and Collin-Dufresne (2005) (hereafter CCD) models, are unable to match this evidence. Since the correlation structure is crucial in the valuation of commodity spread options, the spread option prices implied by the traditional models have strong biases. The model proposed in this paper allows for a flexible correlation structure and matches the pattern observed in the data. We find that, for long-maturity spread options, the prices implied by our model are lower than the ones predicted by the traditional models are, because the higher long-term correlation reduces the volatility of the spread. We show that the opposite is true for short-maturity options. An out-of-sample test using short-maturity crack spread options data shows that our model reduces the negative bias in traditional models.

In econometrics, long-term equilibrium relationships are usually expressed in the format of cointegration or Error Correction Models (ECMs). Engle and Granger (1987) show that an ECM is identical to a cointegration model if the underlying time series are non-stationary. An ECM predicts that the adjustment in a dependent variable depends not only on the explanatory variables but also on the extent to which a particular explanatory variable deviates from the equilibrium (refer to Banerjee, Dolado, Galbraith, and Hendry 1993). Many scholars have empirically studied

[1] Note that, since 2007, the maturity of heating oil can be greater than 24 months.

the cointegration and ECM relationships among commodities. For example, Malliaris and Urrutia (1996), and Girma and Paulson (1999) document long-term cointegration among commodity prices in agricultural commodities and in petroleum markets, respectively. Ai, Chatrath, and Song (2006) document that market-level indicators, such as inventory and harvest size, explain a strikingly large portion of price co-movements. Recently, Paschke and Prokopczuk (2007) and Cortazar, Milla, and Severino (2008) have studied the statistical relationship among commodities in a multi-commodity framework using futures prices. However, none of these models provides an economic foundation on the basis of which to explain which types of assets are involved or why prices of multiple commodities move together through time. To the best of our knowledge, no previous research has looked at patterns of co-movements among multiple commodities under long-term economic relationships.[2]

The rest of the paper is organized as follows. Section 1 identifies three economic equilibrium relations and provides examples of such relationships. Section 2 solves an economic model for the case of two commodities that have a production relationship and generates an endogenous cross-commodity feedback effect. Guided by the economic model, Section 3 develops an empirical model that captures the co-movement among prices (and price dynamics) in a multi-commodity system. We also show that our model is an extension of the "maximal" affine model to a multi-asset case. Section 4 describes the estimation of the model and shows the results. Section 5 presents the valuation of spread options under our multi-commodity framework and shows an out-of-sample comparison of several pricing models. Section 6 concludes.

[2]Our empirical model is part of a growing body of literature on asset pricing that focuses on the dynamics of commodity prices. This literature documents the following stylized facts about single commodities: the existence of a stochastic convenience yield (e.g., GS, and Brennan 1991), mean-reversion in prices (e.g., Bessembinder, Coughenour, Seguin, and Smoller 1995, and Schwartz (1997)), seasonality (e.g., Richter and Sørensen 2002), time-varying risk-premia (e.g., CCD), and stochastic volatility (e.g., Trolle and Schwartz 2009).

1 Long-Term Economic Relationships

The co-movement of commodity prices and the existence of long-term relationships are pervasive in the economy. Examples of economic relationships involving commodities include, but are not restricted to, the following cases:

Production Relationships

One commodity can be produced from another commodity when the former is the output of a production process that uses the other commodity as an input factor. For example, the petroleum refining process "cracks" crude oil into its constituent products, among which heating oil and gasoline are actively traded commodities on the NYMEX along with crude oil. Spread futures and spread options, such as 3:2:1 crack spreads (the purchase of three crude oil futures with the simultaneous sales of two unleaded gasoline futures and one heating oil future), are widely used by refiners and oil investors to lock in profit margins. A similar production relationship can be found in the soybean complex. Soybeans can be crushed into soybean meal and soybean oil. The three commodities in the complex are traded separately on the Chicago Board of Trade. By analogy to the crack spread, the crush spread is also an actively traded derivative. Not all production-linked commodities have spread derivatives established for trading. Aluminum-Aluminum Alloy and corn-ethanol are examples of production-linked relationships that do not involve spread trading.

Substitution Relationships

A substitution relationship exists when two traded commodities are substitutes in consumption. Crude oil and natural gas are commonly viewed as substitute goods. Competition between natural gas and petroleum products occurs principally in the industrial and electric generation sectors. According to the EIA Manufacturing Energy Consumption Survey (Energy Information Administration 2002), approximately 18 percent of natural gas usage can be switched to petroleum products.

Other analysts estimate that up to 20% of power generation capacity is dual-fired. West Texas Intermediate (WTI), a type of crude oil often referenced in North America, and Brent crude oil from the North Sea, are commonly used as benchmarks in oil pricing and the underlying commodity involved in NYMEX oil futures contracts. WTI and Brent crude form a substitution relationship. Recently NYMEX started trading WTI-Brent spread options. Corn and soybean meal serve as substitute cattle feeds.

Complementary Relationships

A complementary relationship exists when two commodities share a balanced supply or are complementary in either consumption or production. Consider the case of gasoline and heating oil. If the gasoline price increases dramatically, and crude oil is cracked to supply gasoline, this process also produces heating oil and may result in a drop in the price of heating oil. On the other hand, since both heating oil and gasoline are produced from crude oil, a demand and supply shock for crude oil will result in co-movements of both heating oil and gasoline. The relationship between these two commodities is one of complementarity. Lead, tin, zinc, and copper are often smelted from paragenesis mineral deposits. The equilibrium assemblage of mineral phases gives those industrial metals a natural relationship in supply. In addition, industrial metals are seldom used in their pure forms. They find most applications in the form of alloys. For example, the principal alloys in tin are bronze (tin and copper), soft solder (tin and lead), and pewter (75% tin and 25% lead). Two-thirds of nickel stocks are used in stainless steel, an alloy of steel. In 1998, 48% of zinc was applied as zinc coatings, jointly used with aluminum.

The three above-mentioned economic relationships can be present simultaneously among commodities. For example, while complementarity exists between gasoline and heating oil, some substitutability is also in effect. In the following section, we present a simple structural model for the production relationship and explain its implications for price dynamics.[3]

2 The Economic Model

Commodity prices link two interconnected markets: the cash (or futures) market and the inventory market. Immediate ownership of a physical commodity offers some benefit or convenience that is not provided by futures ownership. This benefit for holding inventories, in terms of a rate, is called the "convenience yield" (see Brennan 1991, and Schwartz 1997). The "Theory of Storage" of Kaldor (1939), Working (1948), and Telser (1958) predicts that the return on purchasing a commodity and selling it for delivery (using futures) equals the interest forgone less the convenience yield net of storage costs. The convenience yield is attributed to the benefit of protecting regular production from temporary shortages of a particular commodity or by taking advantage of a rise in demand and price without resorting to revising the production schedule.

The traditional presentation of the Theory of Storage proposes that the marginal benefit for holding inventories increases with the scarcity of a commodity (see Pindyck 2001, and Routledge, Seppi, and Spatt 2000). If we consider only the market for any *single* commodity and use the spot price as a proxy for scarcity,[4] the statement indicates: (1) The convenience yield is an increasing function of the spot price, and (2) there is a positive correlation between incremental changes in the spot price and the convenience yield. This paper extends the traditional Theory of Storage by considering a multi-commodity framework and proposes a positive relation between the convenience yield of a commodity and the commodity's degree of relative scarcity, which reflects the relative abundance of one commodity with respect to that of any other with which it is in a long-term

[3]Given the brevity of this paper, we do not present structural models for the substitution or complementary relationships; interested readers can find those models in Appendix B of the earlier version of this paper, which can be found on SSRN.

[4]In a simple model like the one we present later in this section, commodity prices are marginal rates of substitution; therefore, there is an inverse relation between prices and inventories.

economic relationship. Relative scarcity is naturally represented by the price ratio that relates a certain commodity to its associated commodity.[5] This implies a third prediction: (3) A high convenience yield of a particular commodity corresponds to a high price-level ratio between it and any related commodities. Several empirical studies support the first two predictions, which are derived from the traditional Theory of Storage. For example, CCD explicitly models the positive dependence of the convenience yield on the spot price and the instantaneous positive correlation between the spot price and the convenience yield. However, the third prediction, which connects the convenience yield with relative scarcity, has received little attention.

To demonstrate the importance of our prediction, we first present a stylized example in two periods. Consider two commodities that are in a long-term equilibrium production relationship: heating oil (a downstream product) and crude oil (an upstream product). Assume that at time 0 the prices of heating and crude oil are \$20 and \$15, respectively, while at time 1 their prices move to \$22 and \$21, respectively. If we look only at the heating oil market, the Theory of Storage predicts that heating oil will have a higher convenience yield at time 1 than at time 0, since heating oil is more expensive in the second period. However, if we look at both markets, we observe that, at time 0, heating oil is *relatively scarce* compared with crude oil, because of the higher price ratio (heating oil to crude oil).[6] Indeed, since heating oil is refined from crude oil (and not the other way around), a high price ratio between both prices (i.e., high production profit), indicates that the refining capability cannot satisfy the strong demand for heating oil. In contrast to the traditional prediction of the Theory of Storage, we propose that the heating oil convenience yield will be lower at time 1 than at time 0. Indeed, a higher price ratio at time 0 also implies an increase in the cracking of crude oil and, therefore, both an increase in expected heating oil stocks and a drop in the price of heating oil in the next period. Therefore, considered together, the relative scarcity of

[5] In Section 2.1 we derive this representation endogenously from the production relationship between heating and crude oil.

[6] In a recent study, Ahn and Kogan (2011) use the price difference between heating oil and crude oil to decompose oil shocks into demand and supply components.

heating oil and the production relationship between both commodities predicts a higher heating oil convenience yield at time 0. Note that the dependence of the convenience yield of a certain commodity on its relative scarcity is an extension of the traditional Theory of Storage.

2.1 A General Equilibrium Model

In what follows, we use a general equilibrium model to demonstrate the mechanism that connects a commodity's convenience yield with its relative scarcity with respect to related commodities. We propose a model that extends the single-commodity equilibrium models of Routledge, Seppi, and Spatt (2000) and Casassus, Collin-Dufresne, and Routledge (2008).[7] We consider a production economy that has a capital sector (K_t) and two storable commodity sectors that share a long-run equilibrium relation. For simplicity, we again assume they are crude oil (with stocks denoted as $Q_{1,t}$) and heating oil (with stocks denoted as $Q_{2,t}$). There are infinite resources of crude oil, but to make them available, an investment ($I_{1,t}$) is needed. Heating oil is produced from crude oil with commodity input quantity of q_t and capital input of $I_{2,t}$. An infinitely-long-lived representative agent derives utility from consumption of the following goods: the two commodities plus the standard consumption good from the capital sector, which is used as the numeraire. The representative agent maximizes expected utility with respect to consumption of capital, crude oil, and heating oil ($C_{K,t}, C_{1,t}, C_{2,t}$, respectively), demand for crude oil (q_t) and investments in the commodity sectors ($I_{1,t}$ and $I_{2,t}$):

$$\sup_{\{C_{K,t},C_{1,t},C_{2,t},q_t,I_{1,t},I_{2,t}\}\in\mathcal{A}} \mathbb{E}_0^{\mathbb{P}}\left[\int_0^{\infty} e^{-\theta\, t} u\left[C_{K,t}, C_{1,t}, C_{2,t}\right] dt\right] \tag{1}$$

where $\mathcal{A}$ is the set of admissible strategies. The utility function $u[C_K, C_1, C_2]$ satisfies the standard conditions. The optimization problem is subject to the following processes that describe the

[7] Our model is similar in spirit to the cross-commodity model of Routledge, Seppi, and Spatt (2001).

dynamics of capital, crude oil, and heating oil stocks, respectively:

$$dK_t = (\alpha_K K_t - C_{K,t} - I_{1,t} - I_{2,t})dt + \sigma_K K_t dW^{\mathbb{P}}_{K,t} \quad (2)$$

$$dQ_{1,t} = (f_1[I_{1,t}; Q_{1,t}] - q_t - C_{1,t})dt \quad (3)$$

$$dQ_{2,t} = (f_2[I_{2,t}, q_t; Q_{2,t}] - C_{2,t})dt \quad (4)$$

where $f_1[I_1; Q_1]$ is the crude oil production rate, and $f_2[I_2, q; Q_2]$ is the heating oil production rate and both are concave. As mentioned before, the traditional Theory of Storage suggests that agents benefit from commodity inventories. For simplicity, we include this in an ad-hoc way by assuming that the production functions depend positively on their own commodity stocks. In particular, the benefit for the agents manifests insofar as the marginal productivity of the factors is higher when the commodity stock is higher, i.e., $\frac{\partial^2 f_i}{\partial I_i \partial Q_i} > 0$ and $\frac{\partial^2 f_2}{\partial q \partial Q_2} > 0$.[8] We also assume that capital investment and crude oil are complementary inputs for the production of heating oil, i.e., $\frac{\partial^2 f_2}{\partial I_2 \partial q} > 0$.

2.2 The Convenience Yield and Relative Scarcity of the Commodity

The convenience yield of commodity i, $\delta_{i,t}$, is defined as a benefit of holding inventories of that asset; therefore, the price of commodity i at time t, $S_{i,t}$, must satisfy the following equilibrium condition:

$$e^{-\theta\, t} u_{K,t} S_{i,t} = \mathbb{E}^{\mathbb{P}}_t \left[\int_t^T e^{-\theta\, v} u_{K,v} \delta_{i,v} S_{i,v} dv \right] + e^{-\theta\, T} u_{K,T} S_{i,T} \quad \text{for} \quad i = 1, 2 \quad (5)$$

where $u_{K,t} \equiv \frac{\partial u_t}{\partial C_{K,t}}$ is the marginal utility of consumption $C_{K,t}$, and $S_{i,t}$ is the equilibrium commodity price. Since both commodities are also consumption goods, the commodity price is the marginal

[8] Recall that the objective of our model is to inspect the mechanism that connects the convenience yield to a certain commodity with its relative scarcity with respect to other related commodities. This ad hoc assumption, which could be thought of also as a reduction in adjustment costs or economies of scale, does not affect our cross-commodity results and greatly facilitates the solution.

rate of substitution of that commodity for the numeraire, i.e., $S_{i,t} = \frac{u_{i,t}}{u_{K,t}}$ with $u_{i,t} \equiv \frac{\partial u_t}{\partial C_{i,t}}$. Replacing $S_{i,t}$ in the equation above yields a simple equation for δ_i:

$$e^{-\theta\, t} u_{i,t} = \mathbb{E}_t^{\mathbb{P}} \left[\int_t^T e^{-\theta\, v} u_{i,v} \delta_{i,v} dv \right] + e^{-\theta\, T} u_{i,T} \quad \text{for} \quad i = 1, 2 \tag{6}$$

This equation shows the parallelism between the convenience yield and the interest rate: The convenience yield of a certain commodity is the interest rate in an economy in which the commodity is the numeraire. Indeed, using standard arguments from the asset pricing theory (see Cochrane 2005), it is straightforward to show that:

$$\delta_{i,t} dt = -\mathbb{E}_t^{\mathbb{P}} \left[\frac{d\Lambda_{i,t}}{\Lambda_{i,t}} \right] \quad \text{for} \quad i = 1, 2 \tag{7}$$

where $\Lambda_{i,t} \equiv e^{-\theta t} \frac{u_{i,t}}{u_{i,0}}$ is the time-t pricing kernel of an economy in which everything is expressed relative to the price of commodity i. Note that equation (7) implies that the convenience yield of heating oil, $\delta_{2,t}$, decreases with the expected change in its marginal utility, $\mathbb{E}_t^{\mathbb{P}} \left[\frac{du_{2,t}}{u_{2,t}} \right]$. Let $J_t \equiv J[K_t, Q_{1,t}, Q_{2,t}, t]$ be the indirect utility function for the optimization problem in equations (1)-(4). Using the envelope condition for the heating oil consumption good (i.e., $e^{-\theta t} u_{2,t} = \frac{\partial J_t}{\partial Q_{2,t}}$) and recognizing that J_t inherits the concavity of the utility function $u[.]$ (see for example, Benveniste and Scheinkman (1979)), it is straightforward to show that the marginal utility, $u_{2,t}$, decreases with the heating oil inventories, $Q_{2,t}$. Therefore, today's heating oil convenience yield, $\delta_{2,t}$, increases with the growth rate of the heating oil stocks $\mathbb{E}_t^{\mathbb{P}} \left[\frac{dQ_{2,t}}{Q_{2,t}} \right]$.[9] This equilibrium linkage between a commodity's convenience yield and its expected stock growth rate shows that the production of the commodity is a key determinant of its convenience yield. Moreover, it can be verified that an increase in the expected heating oil stock, $\mathbb{E}_t^{\mathbb{P}}[Q_{2,t+dt}]$, increases the expected consumption of heating oil, decreases

[9]This result is also valid for the crude oil convenience yield.

its expected marginal utility, and increases today's heating oil convenience yield, $\delta_{2,t}$.

We use the previous result to show that the relative scarcity of heating oil affects the heating oil convenience yield by two related mechanisms: The demand for crude oil, q_t, and the investment in the heating oil sector, $I_{2,t}$. Consider the first-order condition for the representative agent's problem with respect to crude oil demand, q_t:[10]

$$S_{1,t} = S_{2,t} \frac{\partial f_{2,t}}{\partial q_t} \quad \text{or} \quad \frac{\partial f_{2,t}}{\partial q_t} = \left(\frac{S_{2,t}}{S_{1,t}} \right)^{-1} \tag{8}$$

This equilibrium condition says that the marginal cost of one unit of crude oil (i.e., the spot price $S_{1,t}$) equals the marginal benefit of that unit when used for the production of heating oil (i.e., $S_{2,t} \frac{\partial f_{2,t}}{\partial q_t}$). Since the production functions are concave, equation (8) implies that an increase in today's relative scarcity of heating oil, $\frac{S_{2,t}}{S_{1,t}}$, increases the demand for crude oil, q_t, and, thus, the production of heating oil. This increases the expected heating oil stocks and therefore also the current heating oil convenience yield. Finally, since we assume that investment and crude oil are complementary inputs for the production of heating oil, an increase in this relative scarcity will also imply an increase in investment in the heating oil sector, $I_{2,t}$. This creates a second mechanism that increases the expected stocks and, therefore, the heating oil convenience yield. This confirms the notion that heating oil producers determine their production schedule based not only on the price of heating oil but also on its relative scarcity with respect to crude oil.

The next proposition formalizes these results and presents a closed-form expression for the convenience yield.

[10]Refer to Appendix A for additional details.

Proposition 1 *The convenience yields of crude oil and heating oil are:*

$$\delta_{i,t} = \frac{\partial f_{i,t}}{\partial Q_{i,t}} \quad \textit{for} \quad i = 1, 2 \tag{9}$$

and the heating oil convenience yield increases in its relative scarcity:

$$\frac{\partial \delta_{2,t}}{\partial \left(\frac{S_{2,t}}{S_{1,t}}\right)} > 0 \tag{10}$$

Proof See Appendix A.1. □

Note that we can infer from this model that the convenience yield of crude oil does not depend on the relative scarcity of crude oil with respect to heating oil, i.e., $\frac{\partial \delta_{1,t}}{\partial \left(\frac{S_{1,t}}{S_{2,t}}\right)} = 0$

2.3 An Example of a Cobb-Douglas Economy

Now we assume a standard Cobb-Douglas economy:

$$f_1[I_1; Q_1] = \alpha_1[Q_1] I_1^{\beta_1} \tag{11}$$

$$f_2[I_2, q; Q_2] = \alpha_2[Q_2] I_2^{\beta_2} q^{\gamma} \tag{12}$$

in which $0 < \beta_i, \gamma < 1$ and the total productivity factors, α_i, are positive, increasing and concave in Q_i. Here, $\frac{\partial \alpha_i}{\partial Q_i} > 0$ is an exogenous incentive to hold commodity stocks in line with the prediction of the Theory of Storage.

The next proposition obtains the equilibrium convenience yields for each commodity in terms of the demand for crude oil and the capital investment, and also solely as a function of commodity prices.

Proposition 2 *In a Cobb-Douglas economy, the convenience yields for crude oil and heating oil are*

$$\delta_{1,t} = \alpha_{1,t}{}' I_{1,t}{}^{\beta_1} \tag{13}$$

and

$$\delta_{2,t} = \alpha_{2,t}{}' I_{2,t}{}^{\beta_2} q_t{}^{\gamma} \tag{14}$$

respectively. Moreover, using the first-order conditions for $I_{1,t}$, $I_{2,t}$, and q_t, we can express the convenience yields in terms of the commodity prices

$$\delta_{1,t} = \alpha_{1,t}{}' \left(\alpha_{1,t}\beta_1 S_{1,t}\right)^{\frac{\beta_1}{1-\beta_1}} \tag{15}$$

$$\delta_{2,t} = \alpha_{2,t}{}' \left(\alpha_{2,t}\beta_2 S_{2,t}\right)^{\frac{\beta_2}{1-\beta_2-\gamma}} \left(\alpha_{2,t}\gamma \frac{S_{2,t}}{S_{1,t}}\right)^{\frac{\gamma}{1-\beta_2-\gamma}} \tag{16}$$

Proof See Appendix A.2. □

As expected, equation (14) shows that the heating oil convenience yield increases with the demand for crude oil, q_t. Since this variable depends on the relative scarcity (see equation (8)), equation (16) confirms that the convenience yield of heating oil depends on the ratio $\frac{S_{2,t}}{S_{1,t}}$. The impact of the relative scarcity on this convenience yield increases with the input share of oil for the production of heating oil, γ. Indeed, if this parameter is zero, only absolute scarcity measured by the spot price $S_{2,t}$, matters. On the other hand, equations (13) and (14) show that the convenience yields increase with the investment rates. This implies that the convenience yields $\delta_{i,t}$ increase with the spot prices $S_{i,t}$ (see equations (15) and (16)). The strength of this effect depends on the elasticity, β_i.

2.4 Implications for the Correlation of Futures Returns

From equation (5) it is straightforward to obtain the standard pricing equations for the commodity spot price $\mathbb{E}_t^{\mathbb{P}}\left[d(\Lambda_{K,t}S_{i,t}) + \Lambda_{K,t}S_{i,t}\delta_{i,t}dt\right] = 0$, where $\Lambda_{K,t} \equiv e^{-\theta t}\frac{u_{K,t}}{u_{K,0}}$ is the pricing kernel for the economy. Using Itô's Lemma we obtain the expected commodity spot price return:

$$\mathbb{E}_t^{\mathbb{P}}[dS_{i,t}] = (r_t - \delta_{i,t} + \lambda_{i,t})S_{i,t}dt \tag{17}$$

where $r_t dt = -\mathbb{E}_t^{\mathbb{P}}\left[\frac{d\Lambda_{K,t}}{\Lambda_{K,t}}\right]$ is the instantaneous risk-free rate and $\lambda_{i,t}dt = -\frac{d\Lambda_{K,t}}{\Lambda_{K,t}}\frac{dS_{i,t}}{S_{i,t}}$ is the spot price risk premium. Equation (17) shows that the convenience yield affects spot returns much as the dividend yield affects stock returns.

Our main equilibrium result from the previous section is that the heating oil convenience yield increases with its relative scarcity (i.e. equation (10)). This implies that a decrease in the crude oil price increases the relative scarcity of heating oil, which in turn increases the heating oil convenience yield (see equation (16)) and thus decreases expected heating oil prices. This mechanism, which is present only because of the production linkage between the two commodities, creates a positive relationship between current crude oil prices and expected heating oil prices. We call this relationship the *positive feedback effect* from crude oil to heating oil. Moreover, this mechanism also affects the correlations of futures returns in a positive way. Indeed, the absence of arbitrage implies that a decrease in the crude oil spot price implies a negative change in crude oil futures prices. On the other hand, the increase in relative scarcity of heating oil implies a higher convenience yield for this asset, which has a negative effect on heating oil futures prices. In other words, our model predicts that crude oil price shocks impact both crude oil and heating oil futures prices in the same direction; thus, the economic linkage between these commodities creates a new source of positive correlation of futures returns across commodities. We will show later in an affine reduced-form model, the positive feedback effect from crude oil to heating oil implies an upward-sloping correlation term structure of futures returns. In Section 3, through a multi-commodity affine model, we prove the existence of the positive feedback effect for the crude and heating oil pair.

3 The Empirical Model

Guided by the economic model presented in Section 2, we develop a reduced-form model that is consistent with the stylized facts about economically related commodities (i.e., upward-sloping correlation term structure, stochastic convenience yields, mean-reversion, etc.). Our multi-commodity model is parsimonious in the sense in which "maximal" affine models are.[11] We prefer building a maximal model in order to avoid the risk of model misspecification. Furthermore, we distinguish two sources of co-movement across commodities: 1) a short-term effect associated with the correlation of instantaneous changes in commodity prices, and 2) a long-term feedback effect that is a consequence of a multi-commodity equilibrium economic relationship. The feedback effect manifests insofar as the dynamics of one commodity is a function of the other commodities in the economy. In particular, we choose to represent the convenience yield in such a way that the long-term effect is present, because as shown in the previous section the convenience yield of a particular commodity depends on relative scarcity. For simplicity, we consider an *affine* relationship among the convenience yields and the risk factors.

[11] An affine structure is the standard framework for commodity pricing reduced-form models (see for example, GS, and Schwartz 1997). See Dai and Singleton (2000), and CCD for the definition of "maximal" in this context.

3.1 The Data-generating Processes

Assume there are n commodities in the system, in which the commodities are in long-term economic relationships. Denote

$$x_i = \log(S_i) \quad \text{for} \quad i = 1, \ldots, n \tag{18}$$

where S_i is the spot price of commodity i. Under the physical measure ($\mathbb{P}$), we assume that the log spot prices follow Gaussian processes

$$dx_i = (\mu_i^P - \delta_i)dt + \sigma_i dW_i^{\mathbb{P}} \quad \text{for} \quad i = 1, \ldots, n \tag{19}$$

where δ_i is the convenience yield of commodity i, and μ_i^P and σ_i are constants. Here, $W_i^{\mathbb{P}}$ ($i = 1, \ldots, n$) are correlated Brownian motions. Motivated by our theoretical framework, we propose a specification where the convenience yield of commodity i, δ_i, is a function of its spot price (as shown in CCD) and its relative scarcity. Note that, in order to keep the affine form of our empirical model, we use the log of the price ratio (instead of the price ratio itself) to represent the relative scarcity. Specifically, we use the log-price difference $(x_i - x_j)$ to represent the relative scarcity of the i^{th} commodity with respect to the j^{th} commodity. Note that, in the GS and CCD models, the convenience yield does not depend on the relative scarcity of the commodity. Furthermore, there are n extra latent factors, η_i ($i = 1, \ldots, n$), affecting the n convenience yields. Therefore,

$$\delta_i = h_i^P x_i + \sum_{j=1, j \neq i}^{n} c_j^P (x_i - x_j) + \eta_i - \sum_{j=1, i \neq j}^{n} a_{i,j} \eta_j \tag{20}$$

where c_j^P represents the extent to which the convenience yield depends on the relative scarcity, and h_i^P and $a_{i,j}$ are constants. If setting $b_{i,i}^P \equiv -\left(h_i^P + \sum_{j=1, j \neq i}^{n} c_j^P\right)$ and $b_{i,j(j \neq i)}^P \equiv c_j^P$, equation (20) thus can be rewritten as

$$\delta_i = -\sum_{j=1}^{n} b^P_{i,j} x_j + \eta_i - \sum_{j=1, i\neq j}^{n} a_{i,j}\eta_j \tag{21}$$

The latent factor η's follow mean-reverting processes of the form

$$d\eta_i = (\chi^P_i + \omega_i(t) - \kappa_i \eta_i)dt + \sigma_{n+i} dW^{\mathbb{P}}_{n+i} \quad \text{for} \quad i = 1, \ldots, n \tag{22}$$

where χ^P_i is a constant and $\omega_i(t)$ is a periodical function on t to capture the seasonality of commodity futures prices (if any). Refer to Richter and Sørensen (2002) and Geman and Nguyen (2005) for a similar setup on the seasonality of the convenience yields. Following Harvey (1991) and Durbin and Koopman (2001), we specify $\omega_i(t)$ as:

$$\omega_i(t) = \sum_{l=1}^{L} \left(s^{c,l}_i \cos 2\pi\, l\, t + s^{s,l}_i \sin 2\pi\, l\, t \right) \tag{23}$$

Letting $Y = (x_1, \ldots, x_n, \eta_1, \ldots, \eta_n)^T$ denote the $2n$ factors driving the system of n commodity prices, our model can be rewritten in a vector form,

$$dY = (U^P(t) + \Psi^P Y)dt + d\beta^{\mathbb{P}} \tag{24}$$

where $U^P(t) = (\mu^P_1, \ldots, \mu^P_n, \chi^P_1 + \omega_1(t), \ldots, \chi^P_n + \omega_n(t))^T$ and $\Psi^P = \begin{pmatrix} B^P & A \\ 0 & \mathcal{K} \end{pmatrix}$ with

$$B^P = \begin{pmatrix} b^P_{1,1} & b^P_{1,2} & \ldots & b^P_{1,n} \\ b^P_{2,1} & b^P_{2,2} & \ddots & b^P_{2,n} \\ \vdots & \ddots & \ddots & \vdots \\ b^P_{n,1} & b^P_{n,2} & \ldots & b^P_{n,3} \end{pmatrix}, A = \begin{pmatrix} -1 & a_{1,2} & \ldots & a_{1,n} \\ a_{2,1} & -1 & \ddots & a_{2,n} \\ \vdots & \ddots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \ldots & -1 \end{pmatrix}, \mathcal{K} = \begin{pmatrix} -\kappa_1 & 0 & \ldots & 0 \\ 0 & -\kappa_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & -\kappa_n \end{pmatrix}$$

In equation (24), $\beta^{\mathbb{P}} = \left(\sigma_1 W_1^{\mathbb{P}}, \ldots, \sigma_{2n} W_{2n}^{\mathbb{P}}\right)^T$ is a scaled Brownian motion vector with covariance matrix $\Omega = \{\rho_{i,j}\sigma_i\sigma_j\}$ for $i, j = 1, \ldots, 2n$, where $\rho_{i,j}dt$ is the instantaneous correlation between the Brownian motion increments $dW_i^{\mathbb{P}}$ and $dW_j^{\mathbb{P}}$.

3.2 Correlation Term Structure of Futures Returns

We assume that the risk premium of the x_i factor depends not only on itself but also on other associated commodity prices. Note that by making the risk premium of the x_i factor depend on itself, the CCD model can capture the mean-reversion difference of the physical and risk-neutral measures. In our paper, by assuming that the x_i factor depends on the relative scarcity, we are able to see the difference caused by the feedback effect in the risk-neutral and physical measures.[12] Moreover, we assume a constant risk premium for the η factors. Thus, the risk-neutral process can be expressed as follows:

$$d\beta^{\mathbb{Q}} = \Pi \, dt + d\beta^{\mathbb{P}} \tag{25}$$

where Π is a vector with its i^{th} element Π_i specified as

$$\Pi_i = \begin{cases} \pi_{x,i} + \Sigma_{j=1}^{n}(b_{i,j}^{P} - b_{i,j}^{Q})x_j & \text{for} \quad 1 \leq i \leq n \\ \pi_{\eta,i} & \text{for} \quad n+1 \leq i \leq 2n \end{cases}$$

and $\pi_{x,i}$ and $\pi_{\eta,i}$ are constants. Thus, in the risk-neutral measure, the stochastic behavior of the factors can be expressed as

$$dY = (U(t) + \Psi^{Q} Y)dt + d\beta^{\mathbb{Q}} \tag{26}$$

[12] We thank the editor for pointing this out.

where $\beta^{\mathbb{Q}} = \left(\sigma_1 W_1^{\mathbb{Q}}, \ldots, \sigma_{2n} W_{2n}^{\mathbb{Q}}\right)^T$, $U(t) = (R, L(t))^T$ with $R = \left(r^f - \frac{1}{2}\sigma_1^2, \ldots, r^f - \frac{1}{2}\sigma_n^2\right)^T$, $L(t) = (\chi_1 + \omega_1(t), \ldots, \chi_n + \omega_n(t))^T$, $\chi_i = \chi_i^P - \pi_{\eta,i}$, and $\Psi^Q = \begin{pmatrix} B^Q & A \\ 0 & \mathcal{K} \end{pmatrix}$ with $B^Q =$

$$\begin{pmatrix} b_{1,1}^Q & b_{1,2}^Q & \ldots & b_{1,n}^Q \\ b_{2,1}^Q & b_{2,2}^Q & \ddots & b_{2,n}^Q \\ \vdots & \ddots & \ddots & \vdots \\ b_{n,1}^Q & b_{n,2}^Q & \ldots & b_{n,n}^Q \end{pmatrix}.$$

We assume a constant interest risk-free rate r^f to keep the model simple.[13] The following proposition presents the closed-form expression for the futures price of commodity i:

Proposition 3 *Let $F_{i,t}(Y_t, T)$ be the i^{th} commodity futures price maturing at time T. In the model setup (26), the futures prices are determined by*

$$\log(F_{i,t}(Y_t, T)) = m_i(T-t) + G_i(T-t)Y_t \quad for \quad i = 1, \ldots, n \tag{27}$$

where $m_i(\tau) = \int_0^\tau \left(G_i(u)U + \frac{1}{2}G_i(u)\Omega\, G_i(u)^T\right) du$ and $G_i(\tau)$ denotes the i^{th} row of $G(\tau) \equiv \exp(\Psi^Q \tau)$.

Proof See Appendix B.1. □

Let us define the futures return $r_{i,t,t_1}(T)$ as the log return of a long position on a futures contract that expires at T taken at time t and reversed at time t_1, i.e.,

$$r_{i,t,t_1}(T) = \log\left(\frac{F_{i,t_1}(Y_{t_1}, T)}{F_{i,t}(Y_t, T)}\right) \tag{28}$$

The next proposition defines and presents the closed-form solution for the correlation term structure of futures returns.

[13] It is straightforward to extend our model to consider stochastic interest rates as in Schwartz (1997).

Proposition 4 *Let $\Sigma_{t,t_1}(T)$ be the covariance matrix of futures returns defined as:*

$$\Sigma_{t,t_1}(T) = \mathbb{E}_t^{\mathbb{P}}\left[\left(r_{t,t_1}(T) - \mathbb{E}_t^{\mathbb{P}}[r_{t,t_1}(T)]\right)\left(r_{t,t_1}(T) - \mathbb{E}_t^{\mathbb{P}}[r_{t,t_1}(T)]\right)^T\right] \tag{29}$$

where $r_{i,t,t_1}(T)$ is the $1\times n$ vector of commodity futures returns from equation (28). This matrix has the following solution:

$$\Sigma_{t,t_1}(T) = e^{\Psi^Q(T-t_1)}\mathbb{V}ar_t^{\mathbb{P}}[Y_{t_1}]e^{\Psi^{Q^T}(T-t_1)} \tag{30}$$

where $\mathbb{V}ar_t^{\mathbb{P}}[Y_{t_1}] = \int_t^{t_1} e^{\Psi^P(t_1-v)}\Omega\, e^{(\Psi^P)^T(t_1-v)}dv$ is the conditional covariance matrix of the state variables Y_t under the physical measure. Moreover, the instantaneous covariance matrix of futures returns is

$$\Sigma_t(T) = \lim_{t_1\to t+dt}\Sigma_{t,t_1}(T) = e^{\Psi^Q(T-t)}\Omega\, e^{\Psi^{Q^T}(T-t)}dt \tag{31}$$

Finally, the instantaneous correlation term structure between futures returns of commodities i and j is defined as

$$\rho_{i,j,t}(T) = \frac{\Sigma_t(T)_{i,j}}{\sqrt{\Sigma_t(T)_{i,i}\Sigma_t(T)_{j,j}}} \quad for \quad i = 1,\ldots,n \tag{32}$$

Proof See Appendix B.2. $\square$

Equation (30) shows that not only Ψ^Q but also Ψ^P (hence the risk premium) will influence the correlation term structure of futures returns. From (31), we see that the covariance matrix of instantaneous futures returns depends on the matrix Ψ^Q, as does the correlation matrix of futures returns. The longer the maturity of the futures, the stronger the role the matrix Ψ^Q will play in the instantaneous futures returns correlation. Intuitively, the value of $b_{i,j}^Q$'s in the matrix Ψ^Q have a significant influence on *expected* future spot price co-movements in the risk-neutral measure, since futures prices are *expected* spot prices under the risk-neutral measure. Specifically, in the risk-neutral measure the expected return of x_i is

$$\mathbb{E}_t^{\mathbb{Q}}[dx_i] = \left(r^f - \frac{1}{2}\sigma_i^2 + \sum_{j=1}^{n} b_{i,j}^Q x_j - \eta_i + \sum_{j=1, i \neq j}^{n} a_{i,j}\eta_j \right) dt$$

The $b_{i,j}^Q$'s (for $j \neq i$) represent the long-term source of co-movement.[14] These parameters relate the expected return of commodity i with the price and convenience yield of commodity j. The correlated GS and CCD models set these parameters to zero; therefore, they completely ignore the cross-commodity feedback effect between distinct commodities. According to the sign of the $b_{i,j}^Q$'s, we classify the co-movement between commodity (log) prices x_i and x_j ($j \neq i$) into three classes. That is, if both $b_{i,j}^Q > 0$ and $b_{j,i}^Q > 0$, a positive increment of x_i tends to feed a positive increment back on x_i, which is in turn likely to strengthen x_i by another positive feedback; hence x_i and x_j move together. Note that the positive feedback effect strengthens the co-movement of two commodities in addition to the correlation of the increments of the commodity prices. As shown in Engle and Granger (1987), this effect will become more influential with a longer time horizon. Similarly, if $b_{i,j}^Q < 0$ and $b_{j,i}^Q < 0$, x_i and x_j move in opposite directions. Lastly, we have the mixed cases with $b_{i,j}^Q > 0$, $b_{j,i}^Q < 0$ and $b_{i,j}^Q < 0$, $b_{j,i}^Q > 0$, where it is not easy to identify the co-movement between commodity prices by type. In general, if there is a long-term economic relationship, it will appear in the b's, which in turn affects the long-run matrix Ψ^Q. Therefore, the empirical model presented in this paper makes an important contribution regarding the long-term co-movement between distinct commodities. This long-term source of co-movement is a feedback effect that occurs mainly through the connection between the expected prices of distinct commodities. Note that this cross-commodity feedback effect corresponds to an error correction or cointegration between separate time series in the discrete-time econometric literature.

Figures 3, 5 and 7 demonstrate the term structures of the futures return correlations between

[14] The same logic applies under the physical measure.

distinct commodities. These plots show that the cross-commodity feedback effect due to the economic relationship plays an important role in explaining the co-movement of commodity prices. By neglecting the cross-commodity feedback parameters, the GS and CCD models impose strong restrictions on the pricing structure. Therefore, the cross-commodity feedback effect is important for matching the upward-sloping correlation structure in the data.

Our model nests several other classical models:

1. If $b^P_{i,j} = b^Q_{i,j} = 0$ and $a_{i,j\neq i} = 0$ $(i = 1, \ldots, n;\ j = 1, \ldots, n)$, our model reduces to correlated GS models on commodities.

2. If $b^P_{i,j\neq i} = b^Q_{i,j\neq i} = 0$ and $a_{i,j\neq i} = 0$ $(i = 1, \ldots, n; j = 1, \ldots, n)$, our model reduces to correlated CCD models with constant interest rate on commodities.

The correlated GS and CCD models correspond to the GS and CCD models when the spot prices and convenience yields across commodities have correlated shocks (i.e., are instantaneously correlated). The correlated versions of the these models are more flexible than the original models are and later will be considered as benchmarks for our model.

3.3 "Maximal" Affine Model in a Multi-commodity System

Following Duffie and Kan (1996) and Duffie, Pan, and Singleton (2000), Dai and Singleton (2000) propose a "maximal" canonical form for a Gaussian affine multi-factor model of the form:

$$x_i = \alpha_0^i + \psi^i_{\widehat{Y}} \widehat{Y} \tag{33}$$

where x_i denotes the (log) value of the i^{th} asset, $\psi^i_{\widehat{Y}}$ is a $1 \times m$ constant row vector, and α_0^i is a constant. $\widehat{Y}$ is an $m \times 1$ column vector of latent state variables that follow mean-reverting Gaussian diffusion processes under the risk-neutral measure,

$$d\widehat{Y} = -\varphi \widehat{Y} dt + dW^{\mathbb{Q}}_{\widehat{Y}} \quad (34)$$

where φ is a lower triangular matrix and $W^{\mathbb{Q}}_{\widehat{Y}}$ is a vector of independent Brownian motions. The above-mentioned model is "maximal" in the sense that, conditional on observing the single asset, the model offers the maximum number of identifiable parameters (cf. Dai and Singleton 2000, and CCD).

In order to use this model in a multi-commodity system, we have to extend it in two ways. First, the above maximal model is suitable only for a single asset, and thus we need to extend the model to a canonical affine representation for multiple assets. We hence define the maximal model for multiple assets as follows:

In a system of n assets that are governed by m factors, a model for the system is "maximal" if and only if each asset in the system is modeled by an m-factor maximal model as defined in Dai and Singleton (2000):

$$X = \psi_0 + \psi_{\widehat{Y}} \widehat{Y} \quad (35)$$

where $X = (x_1, \ldots, x_n)^T$ represent the n assets that are governed by $\widehat{Y}$ in equation (35). Here, $\psi_{\widehat{Y}} = (\psi^1_{\widehat{Y}}, \ldots, \psi^n_{\widehat{Y}})^T$ is an $n \times m$ matrix and $\psi_0 = (\psi^1_0, \ldots, \psi^n_0)^T$ is an $n \times 1$ vector.

Thus, a simple combination of maximal models for single commodities does not necessarily form a maximal model for a multi-commodity system. For example, the CCD model is maximal for single commodities, but it is not maximal in a multi-commodity system. The previous section shows that an extended version of the CCD model is nested in our model and hence is not maximal, because this model restricts some parameters in the expected return of the factors to zero. These constraints significantly influence the joint long-run behavior of the commodities.

Second, the above maximal model allows for only a constant ψ_0; however, many commodity prices are subject to seasonal movements. Thus, we need to extend the maximal model by letting ψ_0 be time varying. The extended model for multiple assets is:

$$X = \psi_0(t) + \psi_{\widehat{Y}} \widehat{Y} \tag{36}$$

$$d\widehat{Y} = -\varphi \widehat{Y} dt + dW_{\widehat{Y}}^{\mathbb{Q}} \tag{37}$$

where $\psi_0(t) = (\psi_0^1(t), \ldots, \psi_0^n(t))^T$ is an $n \times 1$ vector, $\psi_0^i(t) = \alpha_0^i + \varpi_0^i(t)$ and where $\varpi_0^i(t)$ is a periodical function.

To address the maximal model for multiple assets in an n-commodity system governed by $2n$ factors, we specify X as the $n \times 1$ vector of log spot commodity prices, φ in (37) as a $2n \times 2n$ lower triangular matrix and $W_{\widehat{Y}}^{\mathbb{Q}}$ as a $2n \times 1$ vector of independent Brownian motions.

Following CCD we now show that, for the multi-commodity maximal model, the convenience yield vector $\Delta = (\delta_1, \ldots, \delta_n)^T$ is an affine function of the state variables $\widehat{Y}$. The absence of arbitrage implies that under the risk-neutral measure ($\mathbb{Q}$) the drift of the spot price of the i^{th} commodity must follow

$$\mathbb{E}_t^{\mathbb{Q}}[dS_i] = (r^f - \delta_i) S_i dt \quad \text{for} \quad i = 1, \ldots, n \tag{38}$$

Applying Itô's Lemma, we obtain the following expression for the maximal convenience yield vector Δ implied by our model,

$$\begin{aligned} \Delta &= r^f 1_n - \frac{\mathbb{E}_t^{\mathbb{Q}}[dV] + \frac{1}{2}\left(\mathbb{V}ar_t^{\mathbb{Q}}[dx_1], ..., \mathbb{V}ar_t^{\mathbb{Q}}[dx_n]\right)^T}{dt} \\ &= r^f 1_n + \psi_{\widehat{Y}} \varphi \widehat{Y} - \frac{1}{2} diag\left(\psi_{\widehat{Y}} \varphi \psi_{\widehat{Y}}^T\right) \end{aligned} \tag{39}$$

where $\mathbb{V}ar_t^{\mathbb{Q}}[.]$ denotes the variance under the risk-neutral measure, and 1_n is an $n \times 1$ column vector with all elements equal to 1.

In order to show that our empirical model introduced at the beginning of this section is indeed maximal, we first introduce an intermediate representation that allows us to show that our model and the one presented in equations (36) and (37) are equivalent. The intermediate representation rotates the state vector $\widehat{Y}$ to state variables that have better economic meaning: the log spot prices and the convenience yields of the n commodities. The canonical form model has m factors, while our empirical model has $2n$ factors; therefore, we set $m = 2n$. Proposition 5 formalizes the intermediate representation.

Proposition 5 *Assume $2n$ factors driving the dynamics of the futures prices of n commodities, as in equations (36) and (37). The maximal model under the risk-neutral measure can be presented equivalently by an affine model where the state variables are the log spot prices x_i and the convenience yields δ_i ($i = 1, \dots, n$). The dynamics of the new state vector $\overline{Y} = (x_1, \dots, x_n, \delta_1, \dots, \delta_n)^T$ is:*

$$d\overline{Y} = (\overline{U}(t) + \overline{\Psi^Q}\,\overline{Y})dt + d\beta^{\mathbb{Q}}_{\overline{Y}} \tag{40}$$

where $\overline{U}(t) = (\overline{R}, \overline{L}(t))^T$, $\overline{\Psi^Q} = \begin{pmatrix} 0 & -I_{n\times n} \\ \overline{A} & \overline{B} \end{pmatrix}$, and $\beta^{\mathbb{Q}}_{\overline{Y}}$ is a scaled Brownian motion vector with covariance matrix $\overline{\Omega}$. The $n\times 1$ vectors $\overline{R}$ and $\overline{L}(t)$ and the $n\times n$ matrices $\overline{A}$, $\overline{B}$ and $\overline{\Omega}$ are specified in Appendix B.3.

Proof By writing equations (36) and (39) together, we have

$$\overline{Y} = \begin{pmatrix} X \\ \Delta \end{pmatrix} = \begin{pmatrix} \psi_0(t) \\ \psi_c \end{pmatrix} + \begin{pmatrix} \psi_{\widehat{Y}} \\ \psi_{\widehat{Y}\varphi} \end{pmatrix} \widehat{Y} \tag{41}$$

where $\psi_c = r^f 1_n - \frac{1}{2} diag(\psi_{\widehat{Y}} \psi_{\widehat{Y}}^T)$. Equation (41) shows that the intermediate representation, $\overline{Y}$, is an invariant transformation of $\widehat{Y}$ (see Dai and Singleton 2000). This transformation rotates the state variables, but all the initial properties of the model are maintained; that is, the resulting model is still a maximal affine $2n$-factor Gaussian model. Furthermore, we apply Itô's Lemma to obtain the specific relationships between the model parameters specified in the proposition and those specified in equations (36) and (37). Appendix B.3 shows the derivation in greater detail. □

An important corollary of Proposition 5 is that, in a maximal model, the drift of the convenience yield of a certain commodity depends on other commodity spot prices. This is consistent with the structural model in Section 2 (see, for example, equation (10)). Now we are ready to show that our model is maximal. The next proposition formalizes this.

Proposition 6 *The maximal model specified in Proposition 5 is equivalent to our model in equation (26).*

Proof Equation (21) shows that the convenience yield vector is $\Delta = -B\,X - A\,\eta$, where $\eta = (\eta_1, \ldots, \eta_n)^T$ is the vector of latent state variables that follow the dynamics in equation (22). Thus, we find the following invariant transform from $\overline{Y}$ to Y:

$$Y = \begin{pmatrix} X \\ \eta \end{pmatrix} = \begin{pmatrix} I_{n\times n} & 0 \\ -A^{-1} \times B & -A^{-1} \end{pmatrix} \overline{Y} \tag{42}$$

Similar to Proposition 5, we apply Itô's Lemma to compare the parameters in (40) and (42) and show that they are identical. Appendix B.4 shows the derivation in detail. □

Propositions 5 and 6 show that our model belongs to the maximal model of a multi-commodity system. We thus name our model the multi-commodity maximal affine (MCMA) model. In the following section, we show the calibration of this model and some results.

4 Estimation

We demonstrate the importance of long-term economic relationships in futures pricing using the heating oil and crude oil production pair from Section 2. Even though our model can be applied to price a system of n commodities jointly, two commodities are enough to highlight the main characteristics of our model and the intuition behind the results.[15] We also estimate the model for two commodities that are substitute goods (WTI crude oil and Brent crude oil) and for two commodities that are complementary goods (heating oil and gasoline).

4.1 Empirical Method

One of the difficulties of calibrating the model is that the state variables are not directly observable. A useful method for Maximum Likelihood Estimation of the model is addressing the model in a state-space form and using the Kalman filter methodology to estimate the latent variables.[16] The state-space form consists of a transition equation and a measurement equation. The transition equation shows the data-generating process. The measurement equation relates a multivariate time series of observable variables (in our case, futures prices at varying maturities) to an unobservable vector of state variables (in our case, the (log) spot prices x_i and η_i ($i = 1, \ldots, n$)). The measure-

[15]The computational loads increase exponentially for the case of more than two commodities. Furthermore, commodity pairs are building blocks of any commodity system. Any multi-commodity system can be decomposed into multiple commodity pairs, e.g., the system with three commodities can be priced using no more than three pairs of commodities.

[16]Hamilton (1994) and Harvey (1991) give a good description of estimation, testing, and model selection of state-space models.

ment equation is obtained using the (log) futures prices in equation (27) by adding uncorrelated noises to take account of the pricing errors.

Suppose that data are sampled in equally separated times t_k, $k = 1, \ldots, K$. Denote $\Delta t = t_{k+1} - t_k$ as the time interval between two subsequent observations. Let Y_k represent the vector of state variables at time t_k. Thus, we can obtain the transition equation,

$$Y_{k+1} = (\Psi^P \Delta t + I)Y_k + U^P(t)\Delta t + w_k \tag{43}$$

where w_k is a $2n \times 1$ random noise vector following zero-mean normal distributions.

For the measurement equation at time t_k, we consider the vector of the log of futures prices $F_k = (F_{1,k}(\tau_1), \ldots, F_{n,k}(\tau_1), \ldots, F_{1,k}(\tau_M), \ldots, F_{n,k}(\tau_M))^T$ where τ_j denotes the times to maturity.[17] The $(nM) \times 1$ vector $\log(F_k)$ can be written as,

$$\log(F_k) = \overline{m} + \overline{G}\, Y_k + \epsilon_k \tag{44}$$

where

$$\begin{aligned} \overline{m} &= (m_1(\tau_1), \ldots, m_n(\tau_1), \ldots, m_1(\tau_M), \ldots, m_n(\tau_M))^T \\ \overline{G} &= (G_1(\tau_1), \ldots, G_n(\tau_1), \ldots, G_1(\tau_M), \ldots, G_n(\tau_M))^T \end{aligned}$$

and ε_k is a $(nM) \times 1$ vector representing the model errors with its variance covariance matrix Υ. In order to reduce the number of parameters to estimate, we assume that the standard errors for all contracts are the same. This also reflects the notion that we want our model to price the n commodities and M contracts equally well. Therefore, we define $\Upsilon = e^2 I_{nM}$, where e is the pricing

[17] Since our model has $2n$ factors we need $M \geq 2$.

error of the log of the futures prices and I_{nM} is the $(nM) \times (nM)$ identity matrix.

4.2 The Data

Our data consist of weekly futures prices of three pairs: 1) the West Texas Intermediate (WTI) crude oil and heating oil pair, 2) the WTI and Brent crude oil pair, and 3) the heating oil and unleaded gasoline pair. The weekly futures in the above pairs are obtained through NYMEX and the London International Petroleum Exchange for the period running from January of 1995 to September of 2010 (821 observations for each commodity). Time to maturity ranges from 1 month to 17 months for these two commodities. We denote Fn as futures contracts with roughly n months to maturity; e.g., F0 denotes the cash spot prices and F12 denotes the futures prices with 12 months to maturity. We use five time series, F1, F5, F9, F13, F17, for the WTI crude and heating oil pair; and F1, F3, F6, F9, F11 for the WTI and Brent crude oil pair and the heating oil and unleaded gasoline pair. Table 1 summarizes the data. Note that, in the calibration, we take the risk-free rate as 0.04, which is the average interest rate during these years.

4.3 Empirical Examination of the Long-term Economic Relationship

In this section, we examine three commodity pairs for three relationships, respectively: the WTI crude oil and heating oil pair (a production relationship), the WTI and Brent crude oil pair (a substitution relationship), and the heating oil and unleaded gasoline pair (a complementary relationship).

WTI Crude and Heating Oil Pair

As mentioned above, since WTI crude oil and heating oil are the input and output of an oil refinery firm, this commodity pair has a production relationship. We arbitrarily define crude oil as commodity 1 and heating oil as commodity 2. From observation of crude and heating oil prices, we find that crude oil prices do not exhibit seasonality, which is consistent with the literature on oil futures, such as Schwartz (1997). However, heating oil exhibits very strong seasonality, which is consistent with Richter and Sørensen (2002). This occurs because demand for heating oil is typically high in the winter, but there are usually not enough available facilities in which to store the heating oil; hence, in the winter, heating oil has relatively higher convenience yield. Therefore, winter-maturing futures princes tend to be higher than are those maturing in summer. Since the seasonality of heating oil is in an annual frequency, by setting $L = 1$, equation (23) reduces to the following:

$$
\begin{aligned}
&\omega_1(t) \equiv 0 \\
&\omega_2(t) = s_2^c \cos 2\pi t + s_2^s \sin 2\pi t
\end{aligned}
\tag{45}
$$

We use the Kalman filter to calibrate our model. Table 2 panel A shows the results. From the model estimation, we see that most parameters are significant.

The coefficient $b_{2,1}^P$ is significantly positive, which shows that the convenience yield of heating oil does depend *positively* on the relative scarcity of heating oil to crude oil (see also equations (20) and (21)). This is consistent with Proposition 1 (equation (10)) in our economic model. It also implies that the convenience yield of heating oil also depends negatively on the (log) price of crude oil. Our theoretical model in Section 2 indicates that $b_{1,2}^P$ should be zero; however, although $b_{1,2}^P$ is significant, its magnitude is much smaller than $b_{2,1}^P$ (about one-fourth). In this sense, the estimation results are *consistent* with the proposition of the theoretical model. From the econometrics point of view, the positive signs of $b_{1,2}^P(b_{1,2}^Q)$ and $b_{2,1}^P(b_{2,1}^Q)$ indicate a positive feedback from the two commodities in both the risk-neutral and physical measures, i.e., a positive increment of crude oil will in turn imply a positive movement of heating oil, and vice versa. Hence, the time series of crude and heating oil tend to move together, which is consistent with historical observations of crude and heating oil prices.

Figure 2 shows the convenience yields for both WTI crude oil and heating oil that are implied by the MCMA model. Figure 3 shows the correlation structure for the correlated GS model, the correlated CCD model, and the MCMA model. We bootstrap the model parameters by assuming that each parameter estimate has a normal distribution and obtain the 95% confidence level of the correlation. The plot shows that, for the MCMA model, the correlation curve is upward sloping and the errors of correlations diminish when the correlation approaches one. However, for the GS and CCD models, the correlations begin to go down when the futures time to maturity is longer than two years; in the meantime the error of the correlation also becomes larger. Hence, only the MCMA model is able to generate the upward-sloping correlation curve present in the data—mainly because of the significant *positive* value of $b^Q_{1,2}$ and $b^Q_{2,1}$, which, as mentioned in Section 3.2, links the two commodities by a positive feedback effect. It is easy to understand that the positive value of $b^Q_{1,2}$ and $b^Q_{2,1}$ has a significant influence on the *expected* future spot prices in the risk-neutral measure; and, since futures prices are the *expected* spot prices in the risk-neutral measure, $b^Q_{1,2}$ and $b^Q_{2,1}$ both play an important role in determining the correlation term structure of futures returns on the crude and heating oil pair (see also equation (31)). Furthermore, the longer the futures time to maturity, the stronger the role this positive feedback effect will play and hence the larger the correlation. This thus results in an upward-sloping correlation term structure in the MCMA model.

In the short run, we see that the correlation is smaller in the MCMA model than it is in the correlated GS and CCD models. This occurs because the MCMA model is more flexible when capturing the co-movement between two futures prices, which allows us to disentangle the various sources of co-movement (i.e., the correlation and the long-term effects).[18] Indeed, the correlated versions of the GS and CCD models, which do not consider long-term relationships, are forced to include some existing mid-term correlations in the short-term component of co-movement. In the long run, the MCMA model allows for a greater correlation than the other two models do, which is consistent with the significance of the cross-commodity relationship.

In order to test whether the MCMA model is better than the correlated versions of the GS and CCD models at fitting the futures prices, we run a likelihood ratio test on the three models. Table 3 shows that, in terms of fitting the futures curves, the MCMA model is significantly better than either the correlated GS model or the correlated CCD model. This result suggests that a maximal specification is indispensable when jointly modeling multiple commodities.

WTI and Brent Crude Oil Pair

As mentioned, since WTI and Brent crude oils have very similar quality and thus similar usage, the relationship between WTI and Brent crude oil belongs to the substitution relationship. We arbitrarily define WTI crude oil as commodity 1 and Brent crude oil as commodity 2. Neither WTI nor Brent crude oil exhibits seasonal behavior. We thus set

$$\begin{aligned} \omega_1(t) &\equiv 0 \\ \omega_2(t) &\equiv 0 \end{aligned} \tag{46}$$

We use the Kalman filter to calibrate the MCMA model. Table 2 panel B shows the results. Both $b^P_{1,2}$ ($b^Q_{1,2}$) and $b^P_{2,1}$ ($b^Q_{2,1}$) are highly significant. This indicates that the convenience yield of WTI crude oil depends positively on the relative scarcity of WTI crude oil with respect to Brent crude oil, and vice versa. This is easy to understand since these two commodities are substitutes for

[18] Note that the functional form in the MCMA model does not impose the upward-sloping correlation structure, it is a result. We thank the referee for pointing this out.

one another; if one commodity has a higher price (larger relative scarcity), people tend to switch to the other commodity. Hence, the expected future scarcity tends to decrease, and thus the current convenience yield is higher compared with the future convenience yield. From an econometrics point of view, positive $b^P_{1,2}$ ($b^Q_{1,2}$) and $b^P_{2,1}$ ($b^Q_{2,1}$) also indicate a positive feedback effect between the two commodities, and hence they co-move.

Figure 4 shows the convenience yield of WTI and Brent crude oils implied by the MCMA model. Figure 5 shows the correlation term-structure for the correlated GS model, the correlated CCD model, and the MCMA model. As is true regarding the heating and crude oil pair, the MCMA model indicates an upward-sloping returns correlation term structure, and the errors of correlations diminish when the correlation approaches one. However, for the GS and CCD models, the correlations begin to go down when the futures time to maturity is longer than one year; in the meantime the error of the correlation also becomes larger. Hence, only the MCMA model is able to generate the upward-sloping correlation curve present in the data. This is largely caused by the positive $b^Q_{1,2}$ and $b^Q_{2,1}$.

From the likelihood ratio tests in Table 3, we again see that the MCMA model is significantly better than either the CCD model or the GS model in fitting the futures prices.

Heating oil and unleaded gasoline pair

The heating oil and unleaded gasoline pair is a good example of commodities having a complementary relationship, because both share a balanced supply as products of crude oil. In the model estimation, we arbitrarily set heating oil as commodity 1 and gasoline as commodity 2.

From observations on the futures term structures, we see that both heating oil and unleaded gasoline exhibit seasonality; we hence set

$$\omega_1(t) \equiv s_1^c \cos 2\pi t + s_1^s \sin 2\pi t$$
$$\omega_2(t) = s_2^c \cos 2\pi t + s_2^s \sin 2\pi t \qquad (47)$$

Table 2 panel C shows the results. As we saw in the cases of the crude and heating oil pair and the WTI and Brent oil pair, $b^P_{1,2}$ ($b^Q_{1,2}$) and $b^P_{2,1}$ ($b^Q_{2,1}$) are positive and significant. This indicates that the two commodities tend to move in the same direction in both physical and risk-neutral measures. Note that with the complementary relationship there are two scenarios regarding movements of the two commodities. First, if the demand and supply shocks are from one *output* commodity (e.g., heating oil), the two output commodities tend to move in opposite directions. For example, if heating oil is experiencing a high demand shock (but gasoline is not), then more crude oil will be refined to produce heating oil. However, since gasoline is the byproduct of this refinery process, the gasoline price will be suppressed. On the other hand, if the demand and supply shocks are from the input commodity (e.g., crude oil), the two output commodities tend to move in the same direction.

Figure 6 shows the convenience yield for both heating oil and unleaded gasoline that is implied by the MCMA model. Figure 7 shows the correlation term-structure for the correlated GS, correlated CCD, and MCMA models. Again, the MCMA model shows an upward-sloping correlation term structure that the other two models do not have. From the likelihood ratio tests in Table 3, we again see that our model is better than either the CCD model or the GS model in fitting the futures prices.

In the next section, we show that a well-behaved empirical model can guide investors in correctly pricing financial contingent claims.

5 Spread Option Valuation

Spread options are based on the difference between two commodity prices. This difference can be, for example, between the price of an input and the price of the output of a production process (processing spread). NYMEX offers tradable options on the crack spread: the heating oil-crude oil and gasoline-crude oil spread options (introduced in 1994); and the substitute spread (or location spread) between the WTI and Brent crude oil (introduced in March 2008). Also, many firms face "real options" on spreads. For example, manufacturing firms possess an option of transferring the raw material to products at a certain cost, because they can choose not to produce. This option is on the spread between input and output prices and the strike price corresponds to the production cost. The spread option is of great importance to both commodity market participants and real production firms.

Since the spread is determined by the difference between the two asset prices, it is natural to model the spread by modeling each asset separately. This is the main characteristic of the so-called two-price model, where the short-term correlation is the driver of most of the action in the spread (as in the correlated GS and CCD models). Nearly all researchers have used the two-price model for pricing spread options (see Margrabe 1978, and Carmona and Durrleman 2003). However, the two-price model ignores the long-term co-movement component implied by our model. Thus, the two-price models might be awed especially for the long run. Mbanefo (1997) and Dempster, Medova, and Tang (2008), among others, have documented that the traditional two-price model suffers a problem of overpricing the spread option. Therefore, spread option pricing can be regarded as an out-of-sample test for our theoretical model.

At current time t, the pricing of call and put spread options, $c_t(T, M)$ and $p_t(T, M)$, with strike

K on two commodities with futures prices $F_{1,t}(M)$ and $F_{2,t}(M)$, are specified as:

$$c_t(T, M) = e^{-r^f(T-t)}\mathbb{E}_t^{\mathbb{Q}}\left[\max(F_{2,t}(M) - F_{1,t}(M) - K, 0)\right] \tag{48}$$

$$p_t(T, M) = e^{-r^f(T-t)}\mathbb{E}_t^{\mathbb{Q}}\left[\max(K - F_{2,t}(M) - F_{1,t}(M), 0)\right] \tag{49}$$

where the time to maturity for the spread options is T. To the best of our knowledge, the analytical solution for spread options is not available if $K \neq 0$. Thus, to price the options, we use Monte Carlo simulation. In this section, we simulate the futures prices using three models-the MCMA, the correlated CCD, and the correlated GS models. The futures price dynamics under the risk-neutral measure are specified as,

$$\frac{dF_{i,t}(M)}{F_{i,t}(M)} = G_i(M-t)d\beta^{\mathbb{Q}} \quad \text{for} \quad i = 1, 2 \tag{50}$$

We choose two spread options: the crack spread option —the spread between heating oil and WTI crude oil— and the substitute spread option —the spread between WTI crude oil and Brent crude oil. For the crack spread, we assume crude and heating oil prices as $F_{1,t}(M) = 100$ (crude oil) and $F_{2,t}(M) = 105$ (heating oil), respectively; and for Brent and WTI crude oil, we use $F_{1,t}(M) = 100$ (Brent crude) and $F_{2,t}(M) = 102$ (WTI crude), respectively.

We focus on spread options of varying maturities to understand the effect of the correlation structure implied by the models. We choose $T = 3$ months for short-maturity options and $T = 5$ years for long-maturity options. Also, for both crack and substitution spreads we choose the same maturity on futures and options, which is the convention of the spread option specification on NYMEX. We use the estimates from the crude-heating oil and WTI-Brent oil pairs to conduct our simulations, where 2000 paths are simulated for the three models. In order to make the simulation accurate, we use anti-variate techniques in generating random variables and use the same random seed for all three models. The risk-free rate r^f is 0.04 in the simulation. Table 4 shows the option values with various strikes for both call and put options of the crack spread and the substitutive spread, respectively. The tables show that both short-term and long-term effects are important determinants of spread option prices. The results indicate that, for long-maturity options ($T = 5$ years), the MCMA model implies lower call and put spread option prices than the correlated GS and CCD models do. Our finding is consistent with the evidence of Mbanefo (1997) that the two-price models tend to overprice the spread option by ignoring the equilibrium relationship, especially for long-maturity options. This is a consequence of the higher long-term correlations implied by the MCMA model. Intuitively, the feedback-effect (positive $b^Q_{1,2}$ and $b^Q_{2,1}$) restricts commodity prices from large deviations from their equilibrium, and thus makes the spread of the prices relatively smaller and less volatile than it is in models without this feature. The lower volatility of the spread traduces into lower options values.

The opposite occurs for short-maturity options ($T = 3$ month). The results suggest that the two-price model may under-price short-maturity option values. The short-term correlation in the CCD and GS models is contaminated because these models are misspecified.[19] Indeed, these models cannot capture the long-term source of co-movement; they tend therefore to accommodate long-term effects in the short end of the correlation structure. This creates important biases in spread option prices.

Table 5 presents an out-of-sample test for short-maturity heating oil-crude oil (1:1) crack spread options for the MCMA model and for the correlated GS and CCD models. The results show that the MCMA model does considerably better than the others do in matching real data. The other two

[19]Figures 3 and 5 show that the cross-commodity feedback effect in our model implies a lower short-term correlation and a larger long-term correlation than is found in the correlated GS and CCD models.

models tend to under-price both the call and put options. The lower option values are consistent with higher short-term correlation estimates as predicted by our previous analysis. However, the MCMA model reduces the mean pricing error to approximately one-third the size of the error in the CCD model. The root mean square error columns also show that the MCMA model outperforms the benchmark models. Long-maturity options data are not available so we are unable to test the long-term predictions implied by the MCMA model.

6 Conclusion

We study the determinants of co-movement among commodity prices in a multi-asset framework. We find that a long-term source of co-movement is driven by economic relations, such as production, substitution, or complementary relationships. Such relationships imply that the convenience yield of a certain commodity depends on its relative scarcity with respect to associated commodities, which is represented by the (log) price ratio. This notion is an extension of the traditional "Theory of Storage." Using an equilibrium model for the crude and heating oil pair, we show that the production relationship of the two commodities indicates that the heating oil convenience yield depends positively on the relative scarcity of heating oil with respect to crude oil. The production relationship also implies a positive feedback effect from crude oil to heating oil (and hence an upward-sloping futures term structure), which is a new source of positive correlation the between futures returns of both commodities.

Empirically, we propose a reduced-form model with a multi-commodity maximal affine (MCMA) framework that nests the GS and CCD models. We explicitly consider the interdependence of convenience yields on the relative scarcity. Our model allows us to disentangle the two sources of co-movement and implies a flexible correlation term structure that matches the upward-sloping shape observed in the data for related commodities. We find that traditional commodity pricing models, such as the GS and CCD models, impose strong restrictions on the correlation structure. These models account for only a short-term source of co-movement; therefore, the estimation accommodates this component to match the higher long-term correlation in the data, and hence cannot generate an upward-sloping correlation term structure. We estimate the model for three commodity pairs: heating oil-crude oil, WTI-Brent crude oil, and heating oil-gasoline. The result is consistent with our economic model, where the convenience yield of heating oil does depend positively on the relative scarcity in the heating oil-crude oil pair. Likelihood-ratio tests show that our model is significantly better at fitting futures prices than the correlated versions of the GS and CCD models are, which proves the importance of modeling cross-commodity relationships.

The MCMA model is then used to price spread options because spread options depend largely on the equilibrium relationship between the two underlying commodities. For long-maturity options, the MCMA model predicts lower prices than those predicted by the correlated GS and CCD models. This occurs because the MCMA model correctly accounts for an upward-sloping correlation structure. The long-run relationship ties both commodity prices together, reducing the volatility of the spread and yielding lower spread option values. Our results also show that the short-term correlation in the MCMA model is lower than the correlations in the GS and CCD models. This implies higher prices for short-maturity spread options. An out-of-sample test shows that the MCMA model does a much better job of fitting short-maturity crack spread options than do the benchmark GS and CCD models.

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Appendix

A Proofs for the Economic Model

A.1 Proof of Proposition 1

Let us denote by $J[K_t, Q_{1,t}, Q_{2,t}, t]$ the value function associated with the representative agent's problem in equations (1)-(4), and by $j[K_t, Q_{1,t}, Q_{2,t}] = e^{\theta\, t} J[K_t, Q_{1,t}, Q_{2,t}, t]$ the "current" value function for the same problem. Therefore,

$$j[K_t, Q_{1,t}, Q_{2,t}] = \sup_{\{C_{K,v}, C_{1,v}, C_{2,v}, q_v, I_{1,v}, I_{2,v}\} \in \mathcal{A}} \mathbb{E}_t^{\mathbb{P}} \left[\int_t^{\infty} e^{-\theta(v-t)} u[C_{K,v}, C_{1,v}, C_{2,v}] dv \right] \tag{A1}$$

Note that given the set-up of the model, the value function $j[\cdot]$ is not a function of time. The solution of the our problem is determined by the following Hamilton-Jacobi-Bellman (HJB) equation:[20]

$$\sup_{\{C_K, C_1, C_2, q, I_1, I_2\} \in \mathcal{A}} \{u[C_K, C_1, C_2] + \mathcal{D}j - \theta j\} = 0 \tag{A2}$$

where $\mathcal{D}$ is the Itô operator

$$\mathcal{D}j = (\alpha_K K - C_K - I_1 - I_2)\frac{\partial j}{\partial K} + (f_1[I_1; Q_1] - q - C_1)\frac{\partial j}{\partial Q_1} + (f_2[I_2, q; Q_2] - C_2)\frac{\partial j}{\partial Q_2} + \frac{1}{2}\sigma_K^2 K^2 \frac{\partial^2 j}{\partial K^2} \tag{A3}$$

with $\frac{\partial j}{\partial K}$, $\frac{\partial j}{\partial Q_1}$ and $\frac{\partial j}{\partial Q_2}$ representing the marginal value of an additional unit of numeraire good, crude oil and heating oil, respectively. $\frac{\partial^2 j}{\partial K^2}$ is the second derivative of the current value function with respect to K. The first-order conditions with respect to the consumption of capital, heating oil and crude oil are:

$$u_K = \frac{\partial j}{\partial K}, \quad u_1 = \frac{\partial j}{\partial Q_1} \quad \text{and} \quad u_2 = \frac{\partial j}{\partial Q_2} \tag{A4}$$

where u_i for $i \in \{K, Q_1, Q_2\}$ are the marginal utilities of consumption of capital, heating oil and crude oil, respectively. The first-order conditions with respect to the demand of crude oil and the investment in the commodity sectors in terms of the marginal utilities are:

$$\frac{\partial f_2}{\partial q} = \frac{u_1}{u_2}, \quad \frac{\partial f_1}{\partial I_1} = \frac{u_K}{u_1} \quad \text{and} \quad \frac{\partial f_2}{\partial I_2} = \frac{u_K}{u_2} \tag{A5}$$

We define the relative commodity prices as the shadow prices that solve $j[K, Q_1, Q_2] = j[K + S_1\,\epsilon, Q_1 - \epsilon, Q_2] = j[K + S_2\,\epsilon, Q_1, Q_2 - \epsilon]$ when $\epsilon \to 0$. These imply that $S_i = \left(\frac{\partial j}{\partial K}\right)^{-1} \frac{\partial j}{\partial Q_i}$ for $i \in \{1, 2\}$. Moreover, using the first order conditions in (A4) we obtain the following result:

$$S_1 = \frac{u_1}{u_K} \quad \text{and} \quad S_2 = \frac{u_2}{u_K} \tag{A6}$$

Section 2.2 shows that the convenience yield of commodity i is:[21]

$$\delta_{i,t} dt = -\mathbb{E}_t^{\mathbb{P}}\left[\frac{d\Lambda_{i,t}}{\Lambda_{i,t}}\right] \quad \text{with} \quad \Lambda_{i,t} \equiv e^{-\theta t}\frac{u_{i,t}}{u_{i,0}} \quad \text{for} \quad i = 1, 2 \tag{A7}$$

[20]The following variables are all time dependent. Hereafter, we drop this dependance to simplify the notation.

[21]The convenience yield can also be obtained from the expected commodity return under the risk-neutral measure, which is derived after applying Itô's Lemma to the prices in equation (A6). The difference between the interest rate and the expected return (under the risk-neutral measure) is the convenience yield.

To obtain the result in Proposition 1, we apply Itô's Lemma to $\Lambda_{i,t}$ and replace $d\Lambda_{i,t}$ in equation (A7). We also differentiate the HJB equation in (A2) with respect to Q_i and replace the high-order partial derivatives in the dynamics of $d\Lambda_{i,t}$. This yields equation (9).

For the proof of second part of the proposition, we first note that equation (8) and the concavity of $f_2[.]$ imply that an increase in the relative scarcity of heating oil, $\frac{S_{2,t}}{S_{1,t}}$, increases the demand for crude oil, q_t. Using that $\frac{\partial^2 f_2}{\partial q \partial Q_2} > 0$ (see footnote 8) and the definition of the heating oil convenience yield in (9), we obtain equation (10).

A.2 Proof of Proposition 2

The convenience yields for crude and heating oil in equations (13) and (14) are obtained directly by replacing the Cobb-Douglas production functions from equation (11) and (12) into (9). To get equations (15) and (16) we obtain the optimal crude oil demand and investment rates from the first order condition in (A5) and use equation (A6) to express the convenience yields in term of the commodity prices.

B Proofs for the Empirical Model

B.1 Proof of Proposition 3

Under the risk-neutral measure, the i^{th} futures prices $F_{i,t}(Y_t, T)$ need to satisfy,

$$F_{i,t}(Y_t, T) = \mathbb{E}_t^{\mathbb{Q}}[S_{i,T}] \quad \text{for} \quad i = 1, \ldots, n \tag{B1}$$

Let $\tau = T - t$. The futures price $F_{i,t}(Y_t, t+\tau)$ should satisfy the following vector-based Feynman-Kac equation,

$$-\frac{\partial F_i}{\partial \tau} + \frac{\partial F_i}{\partial Y}^{\top}(U + \Psi^Q Y) + \frac{1}{2}\text{Tr}\left(\frac{\partial^2 F_i}{\partial Y^2}\Omega\right) = 0 \tag{B2}$$

with boundary condition $F_{i,t}(Y_t, t) = \exp(x_{i,t})$.

Assume that

$$\log(F_{i,t}(Y_t, t+\tau)) = m_i(\tau) + G_i(\tau) Y_t \tag{B3}$$

where $m_i(\tau)$ is the i^{th} element of the $m(\tau)$ vector, and $G_i(\tau)$ is the i^{th} row of the $G(\tau)$ matrix. By plugging (B3) into (B2), we have two ordinary differential equations

$$\begin{aligned} -\frac{\partial m_i}{\partial \tau} + G_i U + \frac{1}{2} G_i(\tau)\,\Omega\, G_i(\tau)^{\top} &= 0 \\ \frac{\partial G_i}{\partial \tau} - G_i(\tau)\Psi^Q &= 0 \end{aligned} \tag{B4}$$

with boundary condition

$$\begin{aligned} m_i(0) &= 0 \\ G_{i,i}(\tau) &= 1 \\ G_{j,i}(\tau) &= 0 \ (i \neq j) \end{aligned}$$

Thus, the solution for (B2) is

$$\begin{aligned} m_i(\tau) &= \int_0^{\tau} \left(G_i(u)\, U + \frac{1}{2} G_i(u)\, \Omega\, G_i(u)^{\top} \right) du \\ G(\tau) &= \exp(\Psi^Q\, \tau) \end{aligned} \tag{B5}$$

$G_i(\tau)$ denotes the i^{th} row of the $G(\tau)$ matrix. When Ψ^Q is diagnosable,

$$G(\tau) = \Xi \, \text{diag}(\exp(\lambda_1 \tau), \ldots, \exp(\lambda_{2n}\tau))\Xi^{-1}$$

where Ξ is the matrix composed of eigenvectors of Ψ^Q and λ_k $(k = 1, \ldots, 2n)$ are the eigenvalues of Ψ^Q; otherwise $G(\tau)$ can be calculated by Taylor expansion, i.e. $G(\tau) = I + \frac{1}{2}(\Psi^Q \tau)^2 + \frac{1}{6}(\Psi^Q \tau)^3 \ldots$

Grouping the elements m_i's from equation (B5) yields the solution in Proposition 3.

B.2 Proof of Proposition 4

First we obtain the first two conditional moments for the $2n$ state variables. The solution of the Gaussian system of SDEs in equation (24) that drives the dynamics of the state vector Y_t is given by:

$$Y_T = e^{\Psi^P(T-t)}Y_t + \int_t^T e^{\Psi^P(T-v)}U^P(v)dv + \int_t^T e^{\Psi^P(T-v)}d\beta_v^{\mathbb{P}} \tag{B6}$$

Hence, the conditional moments for Y_T are:

$$\mathbb{E}_t^{\mathbb{P}}[Y_T] = e^{\Psi^P(T-t)}Y_t + \int_t^T e^{\Psi^P(T-v)}U^P(v)dv \tag{B7}$$

and

$$\begin{aligned}
\mathbb{V}ar_t^{\mathbb{P}}[Y_T] &= \mathbb{E}_t^{\mathbb{P}}[(Y_T - \mathbb{E}_t^{\mathbb{P}}[Y_T])(Y_T - \mathbb{E}_t^{\mathbb{P}}[Y_T])^T] && \text{(B8)}\\
&= \mathbb{E}_t^{\mathbb{P}}\left[\left(\int_t^T e^{\Psi^P(T-v)}d\beta_v^{\mathbb{P}}\right)\left(\int_t^T e^{\Psi^P(T-v)}d\beta_v^{\mathbb{P}}\right)^T\right] && \text{(B9)}\\
&= \int_t^T e^{\Psi^P(T-v)}\Omega\, e^{(\Psi^P)^T(T-v)}dv && \text{(B10)}
\end{aligned}$$

To obtain equation (30) in Proposition 4, we replace equation (28) in the covariance between the futures returns of commodities i and j and use the closed-form expression for the futures from Proposition 3:

$$\begin{aligned}
\Sigma_{i,j,t,t_1}(T) &= \mathbb{E}_t^{\mathbb{P}}\left[\left(r_{i,t,t_1}(T) - \mathbb{E}_t^{\mathbb{P}}[r_{i,t,t_1}(T)]\right)\left(r_{j,t,t_1}(T) - \mathbb{E}_t^{\mathbb{P}}[r_{j,t,t_1}(T)]\right)^T\right] && \text{(B11)}\\
&= \mathbb{E}_t^{\mathbb{P}}[(G_i(T-t_1)Y_{t_1} - \mathbb{E}_t^{\mathbb{P}}[G_i(T-t_1)Y_{t_1}])(G_j(T-t_1)Y_{t_1} - \mathbb{E}_t^{\mathbb{P}}[G_j(T-t_1)Y_{t_1}])^T] && \text{(B12)}\\
&= G_i(T-t_1)\mathbb{V}ar_t^{\mathbb{P}}[Y_{t_1}]G_j(T-t_1)^T && \text{(B13)}
\end{aligned}$$

The elements of the covariance matrix of futures returns in equation (30), $\Sigma_{t,t_1}(T)$, are given by the above equation.

B.3 Proof of Proposition 5

Equation (41) specifies a unique transformation from the latent variables $\widehat{Y}$ to $\overline{Y}$. Thus the $\overline{Y}$ processes in (41) preserves the maximal specification of the model. Letting $\Gamma_0(t) = \begin{pmatrix} \psi_0(t) \\ \psi_c \end{pmatrix}$, $\Gamma_{\widehat{Y}} = \begin{pmatrix} \psi_{\widehat{Y}} \\ \psi_{\widehat{Y}}\varphi \end{pmatrix}$ and applying Itô's Lemma to (41) we see that

$$d\overline{Y} = \Gamma_{\widehat{Y}}\varphi\Gamma_{\widehat{Y}}^{-1}(\Gamma_0(t) - \overline{Y})dt + \Gamma_{\widehat{Y}}d\beta_{\widehat{Y}}^{\mathbb{Q}}, \tag{B14}$$

Denoting $\psi_{\widehat{Y}} = \begin{pmatrix} \psi_1 & \psi_2 \end{pmatrix}$ and $\varphi = \begin{pmatrix} \varphi_1 & 0 \\ \varphi_2 & \varphi_3 \end{pmatrix}$ where $\psi_1, \psi_2, \varphi_1, \varphi_2, \varphi_3$ are all $n \times n$ matrixes and, comparing this with (40), we have,

$$\begin{aligned}
\overline{\Omega} &= \psi_{\widehat{Y}}^\top \psi_{\widehat{Y}} + K^\top \psi_{\widehat{Y}}^\top \psi_{\widehat{Y}} \varphi \\
\overline{B} &= (\psi_2\varphi_3\psi_2^{-1}\psi_1 - \psi_1\varphi_1 - \psi_2\varphi_2)^{-1}(\psi_1\varphi_1^2 + \psi_2\varphi_2\varphi_1 + \psi_2\varphi_3\varphi_2 - \psi_2\varphi_3^2\psi_2^{-1}\psi_1) \qquad \text{(B15)} \\
\overline{A} &= (\psi_1\varphi_1 + \psi_2\varphi_2 - \psi_2\varphi_3\psi_2^{-1}\psi_1)^{-1}(\psi_1\varphi_1^2 + \psi_2\varphi_2\varphi_1 + \psi_2\varphi_3\varphi_2 - \psi_2\varphi_3^2\psi_2^{-1}\psi_1)\psi_2\varphi_3\psi_2^{-1} - \psi_2\varphi_3^2\psi_2^{-1}
\end{aligned}$$

There is a one-to-one relationship from $(\varphi, \psi_{\widehat{Y}})$ to $(\overline{\Omega}, \overline{A}, \overline{B})$. Note that there are, in total, $n+2n^2$ parameters in φ and $2n^2$ in $\psi_{\widehat{Y}}$. Also, there are, in total $n + 2n^2$ parameters in $\overline{\Omega}$ and $2n^2$ in $\overline{A}$ and $\overline{B}$.

Given $\overline{B}$ and $\overline{A}$, $\overline{R}$ can be determined easily from $\overline{\Omega}$ and has the form $\overline{R} = (r^f - \frac{1}{2}\overline{\sigma}_1^2, \ldots, r^f - \frac{1}{2}\overline{\sigma}_n^2)^\top$. The other mean vector has the form $\overline{L}(t) = (\overline{\theta}_1(t), \ldots, \overline{\theta}_n(t))^\top$ with $\overline{\theta}_i(t) = \overline{\chi}_i + \overline{\omega}_i(t)$ and it can be determined by

$$\overline{L}(t) = -(\overline{A}\,\psi_0(t) + \overline{B}\,\psi_c), \qquad \text{(B16)}$$

Specifically, $\overline{\chi}_i = \sum_{k=1}^n \overline{A}_{i,k}\alpha_k + \sum_{k=1}^n \overline{B}_{i,k}(\psi_c)_k$, and $\overline{\omega}_i(t) = \sum_{k=1}^n \overline{A}_{i,k}\varpi_k(t)$.

Therefore, equation (41) is identical to the maximal specification under the risk-neutral measure.

B.4 Proof of Proposition 6

Equation (42) specifies a unique linear transformation from the latent variables $\overline{Y}$ to Y. In the following, for simpleness of presentation, we withdraw the superscript Q from B^Q. Denote $\Gamma_{\overline{Y}} = \begin{pmatrix} I_{n\times n} & 0 \\ -A^{-1}B & -A^{-1} \end{pmatrix}$. Performing Itô's Lemma on (42) we have

$$dY = \Gamma_{\overline{Y}}(\overline{U} + \overline{\Psi^Q}\,\Gamma_{\overline{Y}}^{-1} Y)dt + \Gamma_{\overline{Y}}\, d\beta_{\overline{Y}}^{\mathbb{Q}}. \qquad \text{(B17)}$$

By comparing the parameters in (B17) and those in (26), we find that if the following equations hold, the two models are identical:

$$\begin{aligned}
0 &= B^2 - \overline{B}B + \overline{A} \qquad &\text{(B18)} \\
\overline{B} - B &= A\,\mathcal{K}\,A^{-1} \qquad &\text{(B19)} \\
\Omega &= (\Gamma_{\overline{Y}})^\top \overline{\Omega}\,\Gamma_{\overline{Y}} \qquad &\text{(B20)}
\end{aligned}$$

Equation (B18) is a quadratic matrix equation, which has been studied quite often (e.g., Smith, Singh, and Sorensen (1995)). In most of the cases, there is no analytical solution for the quadratic matrix equation, but it can be solved by numerical methods such as the Newton method (*c.f.* Higham and Kim (2001)). After obtaining B, we can solve (B19). Since A and $\mathcal{K}$ can be seen as the Eigenvalue and Eigenmatrix of $(\overline{B} - B)$, we can first obtain $\mathcal{K}$ by calculating the eigenvalues of $(\overline{B} - B)$, then we normalize the i^{th} eigenvector to make its i^{th} element equal to one. A is just the collection of the those eigenvectors. After obtaining A and B, we can easily obtain Ω by equation (B20). Note that there are, in total, $2n^2$ parameters in $\overline{A}$ and $\overline{B}$, and also $2n^2$ parameters in A, B and $\mathcal{K}$. Thus, (B18) and (B19) provide a mapping from $(\overline{A}, \overline{B})$ to $(A, B, \mathcal{K})$. Also, it is easy to show that $R = \overline{R}$, and

$$L = -(A^{-1}B\overline{R} + A^{-1}\overline{L}). \qquad \text{(B21)}$$

Specifically, $\overline{\chi}_i = \sum_{k=1}^n (A^{-1}B)_{i,k}\overline{R}_k + \sum_{k=1}^n (A^{-1})_{i,k}(\overline{\theta})_k$, and $\omega_i(t) = \sum_{k=1}^n (A^{-1})_{i,k}\overline{\omega}_k(t)$.

Table 1: Data Summary for three pairs

All data consist of weekly futures prices from 1995.01 to 2010.09. Fn is denoted as futures contracts with roughly n months to maturity. The mean and standard deviation of returns are in annual terms. The unit for all futures prices are \$/bbl. While heating oil and unleaded gasoline futures prices are originally in cents/gallon, we have converted the figures to \$/bbl.

Panel A: The heating and WTI crude oil pair

Contracts	Mean Price	Std of Price	Mean Return(Annualized)	Std of Price
Panel A: WTI crude oil				
F1	42.25	26.75	0.17	0.38
F5	42.40	27.60	0.14	0.29
F9	42.19	28.07	0.13	0.25
F13	41.96	28.36	0.12	0.23
F17	41.77	28.53	0.12	0.21
Panel B: Heating oil				
F1	49.02	31.58	0.16	0.37
F5	49.46	32.79	0.14	0.29
F9	49.44	33.35	0.12	0.25
F13	49.28	33.46	0.12	0.23
F17	49.27	33.94	0.12	0.22

Panel B: The WTI and Brent oil Pair

Contracts	Mean Price	Std of Price	Mean Return(Annualized)	Std of Price
WTI crude oil				
F1	42.25	26.75	0.17	0.38
F3	42.44	27.24	0.15	0.32
F6	42.36	27.74	0.13	0.28
F9	42.19	28.07	0.13	0.25
F11	42.07	28.23	0.13	0.24
Heating oil				
F1	41.09	27.10	0.16	0.35
F3	41.24	27.64	0.15	0.31
F6	41.26	28.24	0.14	0.27
F9	41.18	28.65	0.13	0.25
F11	41.10	28.85	0.13	0.24

Panel C: Heating oil and Unleaded Gasoline Pair

Contracts	Mean Price	Std of Price	Mean Return(Annualized)	Std of Price
WTI crude oil				
F1	41.09	27.10	0.16	0.35
F3	41.24	27.64	0.15	0.31
F6	41.26	28.24	0.14	0.27
F9	41.18	28.65	0.13	0.25
F11	41.10	28.85	0.13	0.24
Heating oil				
F1	50.04	29.12	0.17	0.42
F3	49.60	29.06	0.14	0.35
F6	49.01	29.01	0.12	0.28
F9	48.87	29.77	0.12	0.26
F11	48.79	30.19	0.12	0.25

Table 2: Parameter estimation for three pairs

The data consist of weekly futures prices of three pairs from 1995.01 to 2010.09 (821 observations). The estimates correspond to the 4-factor multi-commodity maximal affine model.

Panel A: The WTI crude and heating oil pair

Parameter	Estimate	Std. Err.	Parameter	Estimate	Std. Err.
$b_{1,1}^P$	-0.472	(0.031)	$\rho_{2,4}$	-0.179	(0.056)
$b_{1,1}^Q$	-0.663	(0.020)	$\rho_{3,4}$	0.004	(0.083)
$b_{1,2}^P$	0.669	(0.021)	σ_1	0.367	(0.010)
$b_{1,2}^Q$	0.609	(0.035)	σ_2	0.352	(0.010)
$b_{2,1}^P$	2.546	(0.111)	σ_3	0.372	(0.014)
$b_{2,1}^Q$	1.921	(0.335)	σ_4	0.175	(0.013)
$b_{2,2}^P$	-2.188	(0.317)	χ_1	0.134	(0.012)
$b_{2,2}^Q$	-2.888	(0.092)	χ_2	-1.811	(0.123)
$a_{1,2}$	0.010	(0.009)	χ_1^P	0.288	(0.081)
$a_{2,2}$	-0.327	(0.033)	χ_2^P	-2.798	(0.242)
κ_1	1.324	(0.026)	μ_1^P	-0.212	(0.326)
κ_2	0.240	(0.012)	μ_2^P	-0.072	(0.406)
$\rho_{1,2}$	0.772	(0.032)	s_1^c	0	–
$\rho_{1,3}$	0.837	(0.027)	s_1^s	0	–
$\rho_{1,4}$	0.092	(0.070)	s_2^c	4.896	(0.235)
$\rho_{2,3}$	0.680	(0.039)	s_2^s	2.998	(0.194)
ϵ	0.013	(0.002)			
Log-likelihood	20,687				

Panel B: The WTI and Brent oil pair

Parameter	Estimate	Std. Err.	Parameter	Estimate	Std. Err.
$b_{1,1}^P$	-0.555	(0.031)	$\rho_{2,4}$	0.564	(0.058)
$b_{1,1}^Q$	-0.851	(0.041)	$\rho_{3,4}$	-0.049	(0.067)
$b_{1,2}^P$	0.532	(0.138)	σ_1	0.376	(0.010)
$b_{1,2}^Q$	0.764	(0.035)	σ_2	0.338	(0.008)
$b_{2,1}^P$	0.455	(0.120)	σ_3	0.129	(0.001)
$b_{2,1}^Q$	0.456	(0.034)	σ_4	0.369	(0.012)
$b_{2,2}^P$	-0.382	(0.032)	χ_1	-0.503	(0.062)
$b_{2,2}^Q$	-0.450	(0.032)	χ_2	0.228	(0.036)
$a_{1,2}$	-0.964	(0.041)	χ_1^P	-0.548	(0.062)
$a_{2,2}$	-0.253	(0.051)	χ_2^P	0.203	(0.063)
κ_1	1.074	(0.095)	μ_1^P	-0.153	(0.449)
κ_2	1.225	(0.020)	μ_2^P	-0.117	(0.408)
$\rho_{1,2}$	0.929	(0.017)	s_1^c	0	–
$\rho_{1,3}$	0.225	(0.091)	s_1^s	0	–
$\rho_{1,4}$	0.594	(0.046)	s_2^c	0	–
$\rho_{2,3}$	-0.057	(0.090)	s_2^s	0	–
ϵ	0.008	(0.001)			
Log-likelihood	25,663				

Panel C: the heating oil and gasoline pair

Parameter	Estimate	Std. Err.	Parameter	Estimate	Std. Err.
$b_{1,1}^P$	-0.154	(0.031)	$\rho_{2,4}$	0.657	(0.045)
$b_{1,1}^Q$	-0.245	(0.049)	$\rho_{3,4}$	-0.028	(0.079)
$b_{1,2}^P$	0.286	(0.054)	σ_1	0.394	(0.012)
$b_{1,2}^Q$	0.264	(0.059)	σ_2	0.423	(0.013)
$b_{2,1}^P$	1.805	(0.360)	σ_3	0.089	(0.006)
$b_{2,1}^Q$	1.910	(0.037)	σ_4	0.483	(0.033)
$b_{2,2}^P$	-1.919	(0.144)	χ_1	0.377	(0.114)
$b_{2,2}^Q$	-2.065	(0.042)	χ_2	-0.168	(0.019)
$a_{1,2}$	-1.585	(0.113)	χ_1^P	0.344	(0.153)
$a_{2,2}$	1.193	(0.062)	χ_2^P	-0.144	(0.048)
κ_1	0.279	(0.064)	μ_1^P	-0.288	(0.546)
κ_2	2.729	(0.066)	μ_2^P	0.031	(0.539)
$\rho_{1,2}$	0.820	(0.031)	s_1^c	4.647	(1.088)
$\rho_{1,3}$	0.225	(0.091)	s_1^s	-0.917	(0.273)
$\rho_{1,4}$	0.851	(0.028)	s_2^c	-0.195	(0.011)
$\rho_{2,3}$	0.073	(0.092)	s_2^s	-0.310	(0.016)
ϵ	0.019	(0.003)			
Log-likelihood	18,166				

Table 3: Likelihood ratio tests for three pairs

This table compares the MCMA model with the correlated CCD and GS models. The parameters used in the calculation are from Table 2. Correlated CCD and GS models correspond to the cases $b^P_{1,2} = b^Q_{1,2} = b^P_{2,1} = b^Q_{2,1} = a_{1,2} = a_{2,1} = 0$ and $b^P_{1,1} = b^Q_{1,1} = b^P_{1,2} = b^Q_{1,2} = b^P_{2,1} = b^Q_{2,1} = b^P_{2,2} = b^Q_{2,2} = a_{1,2} = a_{2,1} = 0$ respectively. The 1% significant levels are 16.81, 23.2 and 13.28, respectively for MCMA vs. correlated CCD, MCMA vs. correlated GS, and correlated CCD vs. correlated GS models. The statistics that are significant at the 1% level are marked with an asterisk.

Panel A: WTI crude and heating oil pair

	log-likelihood		LR statistic
Our model	20,687	Our model vs. CCD	438 (*)
CCD	20,518	Our model vs. GS	550 (*)
GS	20,462	CCD vs. GS	112 (*)

Panel B: WTI and Brent oil pair

	log-likelihood		LR statistic
Our model	25,663	Our model vs. CCD	3,532 (*)
CCD	23,897	Our model vs. GS	3,760 (*)
GS	23,783	CCD vs. GS	228 (*)

Panel C: Heating oil and Gasoline pair

	log-likelihood		LR statistic
Our model	18,166	Our model vs. CCD	200 (*)
CCD	18,066	Our model vs. GS	330 (*)
GS	18,001	CCD vs. GS	130 (*)

Table 4: Values for two spread options

The table shows the crack spread option prices between heating and WTI crude oil prices and between WTI and Brent oil prices for different strikes. Panel A presents the call option values, while Panel B presents the put option values. The options and the underlying futures have the same maturity. The parameters used in the calculation are from Table 2.

Panel A: The heating oil-crude oil crack spread option

Call Options						
	Time to maturity = 3 months			Time to maturity = 5 years		
Strike	Our model	CCD	GS	Our model	CCD	GS
1	6.097	5.560	5.609	7.222	8.635	8.336
3	4.856	4.268	4.327	6.003	7.424	7.162
5	3.776	3.162	3.228	4.957	6.341	6.120
7	2.864	2.261	2.335	4.068	5.391	5.208
9	2.120	1.551	1.625	3.324	4.567	4.419

Put Options						
	Time to maturity = 3 months			Time to maturity = 5 years		
Strike	Our model	CCD	GS	Our model	CCD	GS
1	2.090	1.551	1.601	3.055	4.542	4.239
3	2.849	2.259	2.319	3.837	5.331	5.065
5	3.769	3.152	3.220	4.791	6.248	6.023
7	4.857	4.252	4.327	5.901	7.298	7.111
9	6.113	5.542	5.617	7.157	8.473	8.322

Panel B: the WTI - Brent oil substitution (or location) spread option

Call Options						
	Time to maturity = 3 months			Time to maturity = 5 years		
Strike	Our model	CCD	GS	Our model	CCD	GS
0	4.058	3.592	3.656	4.786	5.718	6.078
1	3.465	2.991	3.056	4.229	5.163	5.500
2	2.928	2.449	2.513	3.724	4.653	4.964
3	2.442	1.968	2.030	3.278	4.187	4.469
4	2.012	1.553	1.611	2.880	3.766	4.017

Put Options						
	Time to maturity = 3 months			Time to maturity = 5 years		
Strike	Our model	CCD	GS	Our model	CCD	GS
0	2.055	1.586	1.649	2.740	3.728	4.103
1	2.462	1.985	2.050	3.183	4.173	4.525
2	2.925	2.442	2.506	3.678	4.663	4.989
3	3.439	2.962	3.023	4.232	5.197	5.495
4	4.009	3.547	3.604	4.834	5.776	6.042

Table 5: Out-of-sample comparison of heating-crude oil crack spread options

The table shows the results of the out-of-sample tests using short-maturity heating oil-crude oil (1:1) crack spread options data. The market data consists of 2,594 calls and 2,786 puts from January 2000 to December 2006 with maturities between 3 and 12 months and moneyness between 0.6 to 1.4 (strike/spot). The parameters used in the calculation are from Table 2.

	Mean Pricing Error (calls)	Root Mean Squared Error (calls)	Mean Pricing Error (puts)	Root Mean Squared Error (puts)
Our Model	-0.101	0.276	-0.042	0.209
CCD	-0.321	0.493	-0.260	0.390
Schwartz	-0.287	0.466	-0.229	0.368

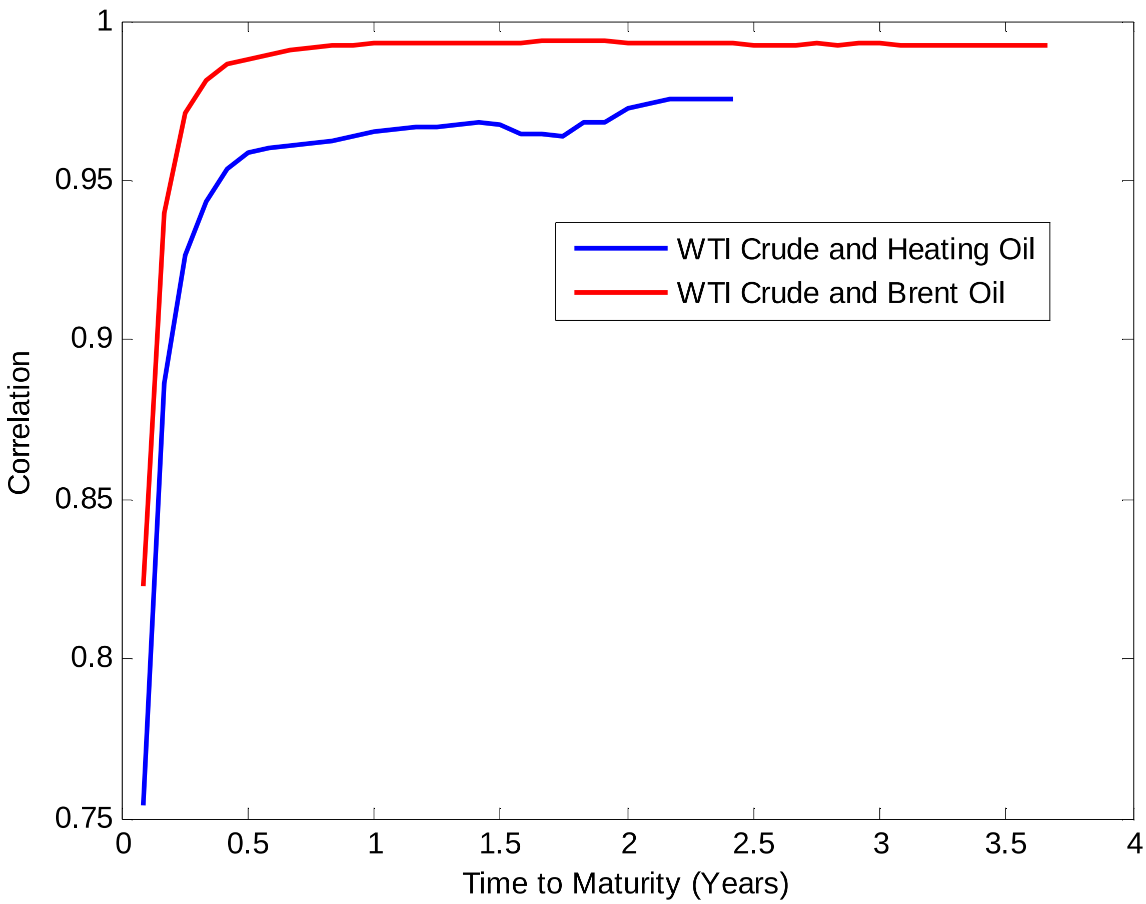

Figure 1: Correlation term structure for the heating oil-crude oil and WTI-Brent crude oil pairs. The figure plots the correlation between weekly futures returns for various maturity futures.

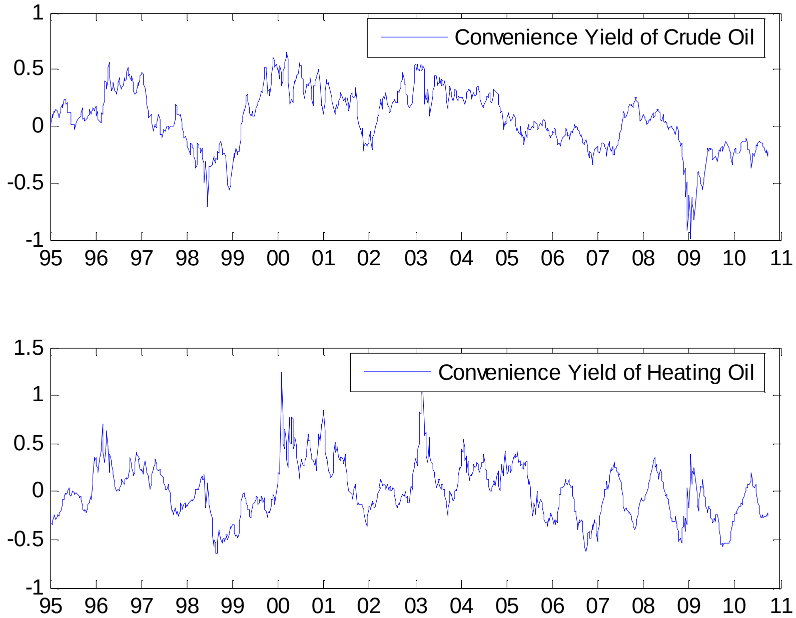

Figure 2: The implied convenience yield for the WTI crude and heating oil pair. The implied convenience yield is from the MCMA model with the parameters obtained from Table 2.

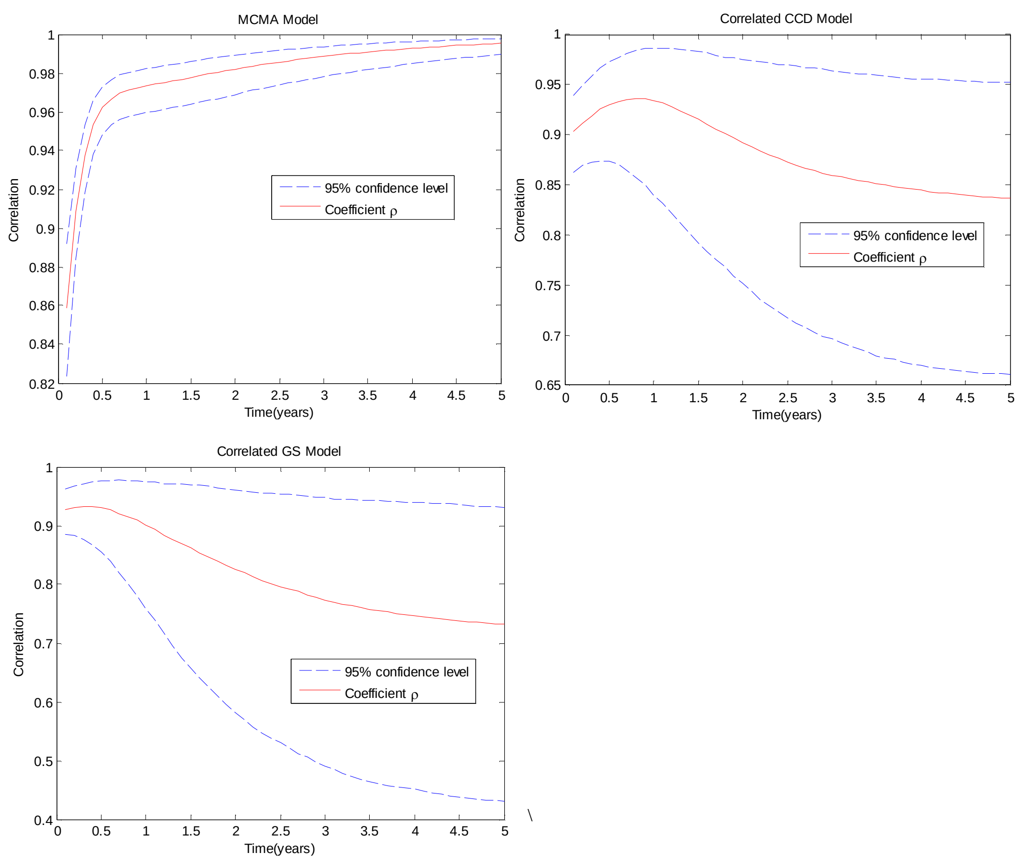

Figure 3: Correlation term structures (instantaneous futures correlation vs. time to maturity) of the WTI crude and heating oil pair for the MCMA, correlated CCD, and correlated GS models. The 95% confidence levels are obtained by bootstrapping the model parameters.

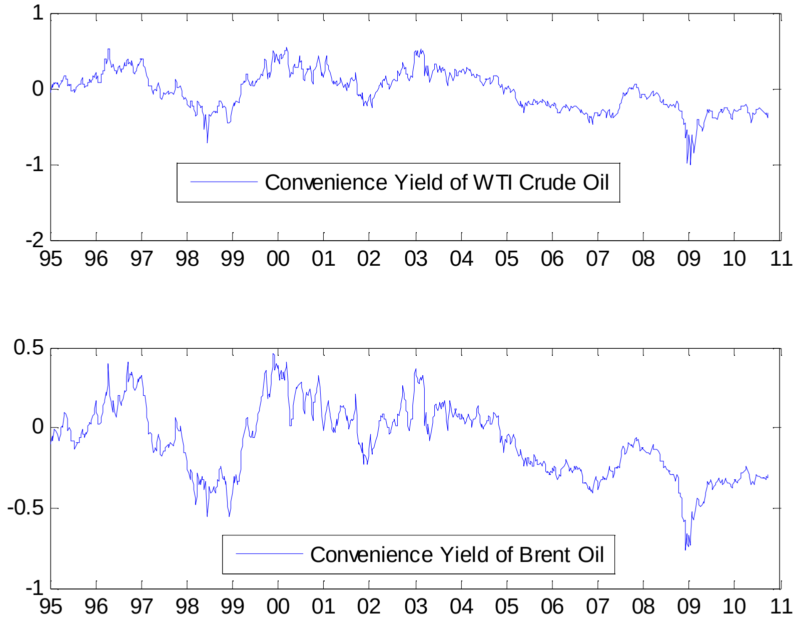

Figure 4: The implied convenience yield for the WTI and Brent crude oil pair. The implied convenience yield is from the MCMA model with the parameters obtained from Table 2.

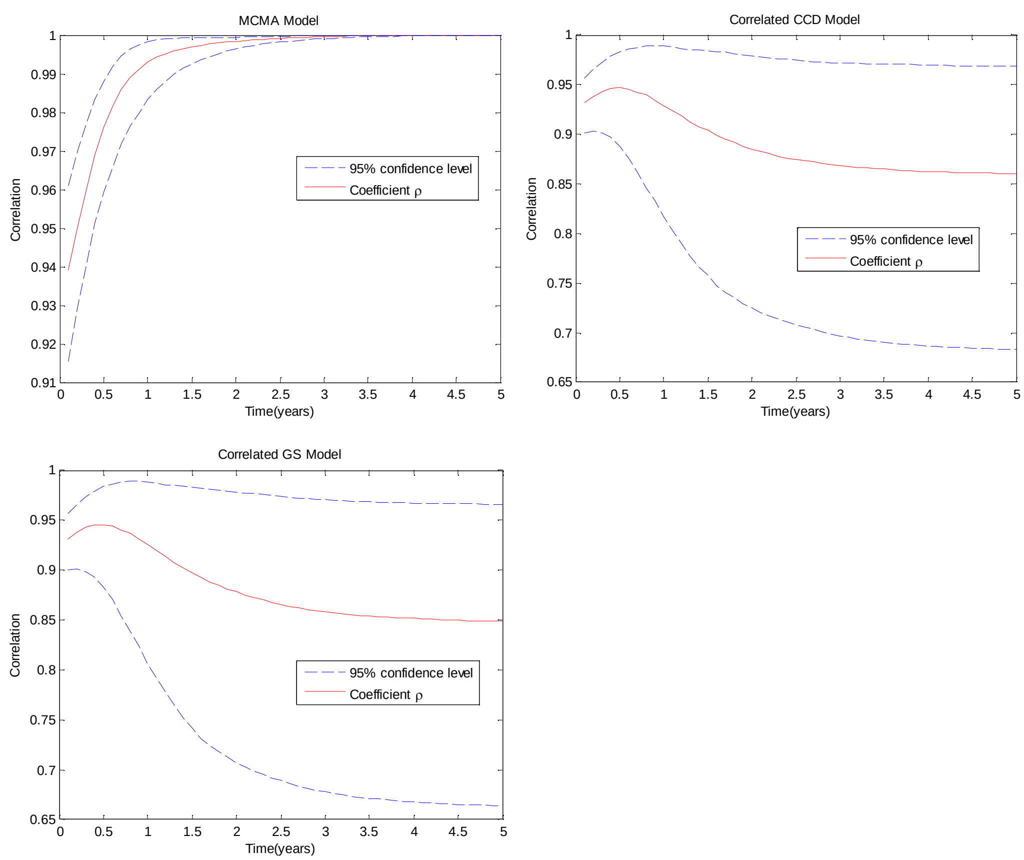

Figure 5: Correlation term structures (instantaneous futures correlation vs. time to maturity) of the WTI and Brent crude oil pair for the MCMA, correlated CCD, and correlated GS models. The 95% confidence levels are obtained by bootstrapping the model parameters.

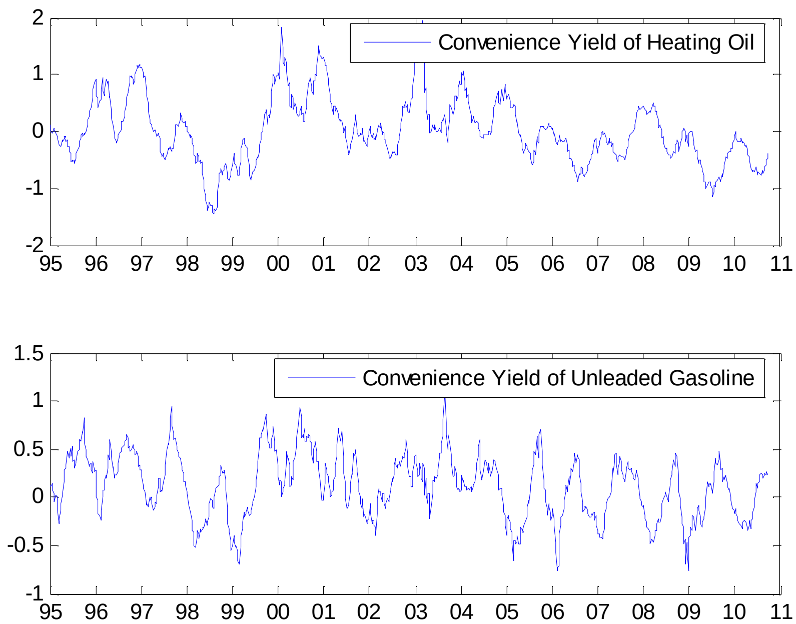

Figure 6: The implied convenience yield for the heating oil and unleaded gasoline pair. The implied convenience yield is from the MCMA model with the parameters obtained from Table 2.

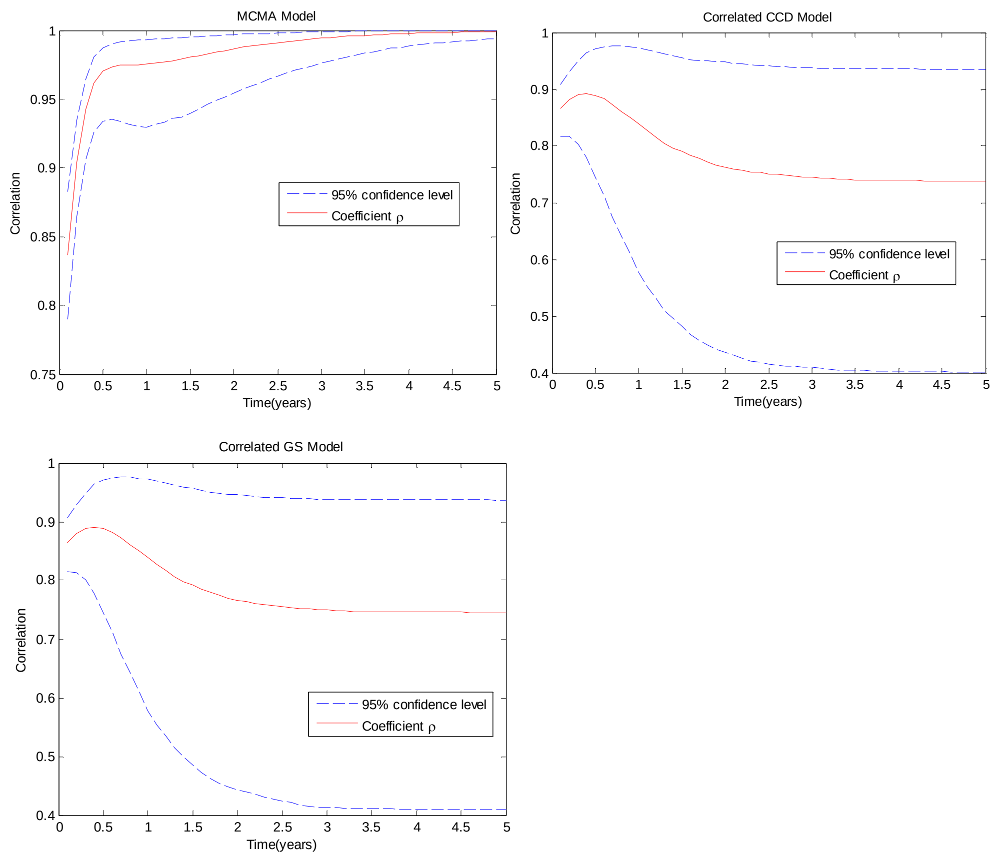

Figure 7: Correlation term structures (instantaneous futures correlation vs. time to maturity) of the heating oil and unleaded gasoline pair for the MCMA, correlated CCD, and correlated GS models. The 95% confidence levels are obtained by bootstrapping the model parameters.