INSTITUTO DE ECONOMÍA

DOCUMENTO de TRABAJO

477

2016

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www.economia.puc.cl • ISSN (edición impresa) 0716-7334 • ISSN (edición electrónica) 0717-7593

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July 30, 2016

Abstract

I compare two different systems of provision of discrete public goods: a centralized system, ruled by a benevolent dictator who has limited commitment power; and an anarchic system, based on voluntary contributions, where there is no ruler. If the public good is binary, then the public good provision problem is merely an informational one. In this environment, I show that any allocation which is implementable in a centralized system and is ex-post individually rational, is also implementable in Anarchy. However, as the number of alternatives available increases, the classical free riding problem described in Samuelson (1954) emerges, and eventually the centralized system becomes the preferred one.

JEL classification: D82, H41

Keywords: communication, free-riding, commitment power, voluntary provision of public goods

*I would like to thank Mustafa Dogan, Tzuo-Hann Law, George Mailath, Steven Matthews, Daniel Neuhann, Qiusha Peng, Andrew Postlewaite, Rakesh Vohra and Yanhao Wei for their helpful comments. All mistakes are mine.

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1 Introduction

The question of who should provide public goods is an old one in the economic literature. In a seminal work, Samuelson (1954) argues that, in a world with complete information, the market will typically be unable to provide an efficient allocation of public goods due to what I refer to as the "classical" free riding problem: each agent disregards the positive impact that his private contribution to the provision of public goods has on other agents. Hence, the government should be able to solve all inefficiencies simply by imposing socially optimal contributions on the agents.

However, as some authors have pointed out (for example, Hayek (1945)), the government does not know the agents' preferences for (public) goods, and, as a consequence, is bound to make decisions without being properly informed. By contrast, in a free market, agents make decisions based on prices, and prices contain information. Therefore, the market outcome should be more efficient as it is a function of the agents' private information while a centralized alternative is not.

The development of the literature of mechanism design applied to the provision of public goods has analyzed the general problem of constructing mechanisms that elicit reports from the agents in order to retrieve their private information. The famous revelation principle (see, e.g. Myerson (1979)) states that one can restrict attention to revelation mechanisms, where agents simply report their private type to a mediator, which then maps those reports into allocations. By thinking of the mediator as the government one can see that the revelation principle is the answer to Hayek's argument. Any mechanism outcome (including a market mechanism) can be replicated by the government using a revelation mechanism.

There are, however, two implicit assumptions one makes when equating the government to this mediator. First, one accepts that the government is benevolent and has the best interests of the people in mind. This is a rather controversial assumption and has been discussed quite thoroughly in the public choice theory literature.[1] The second assumption is that the government is able to commit to any public goods allocation it chooses. In the typical mechanism design approach to the problem of public good provision, the mediator maps the agents' truthful reports to units of the public good to provide and to transfers each individual must make. The assumption that the mediator has commitment power precludes any changes to this promised mapping once the agents' make their reports. In particular, even if, after some set of reports,

[1]See Buchanan and Tullock (1962) and the ensuing literature.

the mediator becomes convinced that there is a pareto improving alternative, he is assumed not to be able to take it. However, if the government is indeed this mediator, one has to wonder what is there to stop him. If being able to commit means to be able to write enforceable contracts, who would enforce the contracts written by the government? The purpose of this paper is to compare two alternative systems of provision of public goods: a centralized system governed by a *benevolent* dictator (BD) with limited commitment power and an *anarchic* system based on the voluntary contributions of the agents.

In either system, agents are able to communicate (send cheap talk messages) before decisions are made. In the BD system, it is the BD who decides what each agent's contribution to the public good should be after observing the agents' reports. I limit the ability of the BD to commit in two ways: I assume that she cannot commit not to make i) pareto improving changes and ii) inequality reducing changes. Loosely speaking, what this means is that, after any set of reports sent by the agents, the BD never selects an allocation for which there is an alternative which either makes all agents better off or reduces inequality.[2] In Anarchy, it is the agents' who chose their own contribution.

The main result of the paper is that, if the public good is binary, an anarchic system is able to implement any allocation such that i) it is implementable by the BD and ii) the ex-post utility of every agent is not negative. What makes this result possible is the fact that Samuelson's classical free-riding problem does not exist when the public good is binary. If the agents' valuations are known, there are many Nash equilibria of the voluntary contribution game where the efficient outcome is achieved.[3] The only problem is an informational one: agents do not know how everyone else values the public good.

In the final part of the paper, I also consider discrete non-binary public goods. In particular, I let the number of units of the public good $g \in \left\{\frac{1}{k}, \frac{2}{k}, ..., \frac{k}{k}\right\}$ and I interpret $k \in \mathbb{N}$ as determining the relative importance of the classical free riding problem relative to the underlying informational problem. If $k = 1$ (the binary case), there is no classical free riding problem, in which case, Anarchy seems to be the superior system. However, as k increases, it becomes increasingly difficult for the public good to be provided in Anarchy, precisely because agents disregard the positive externalities

[2] The case of an inequality concerned BD has been explored previously in the literature in Ledyard and Palfrey (1999) and Hellwig (2005) but in a context where the BD has commitment power.

[3] Think for example of an example with only two agents where each values the public good in 5, while the total cost of the good is 8. There are many Nash Equilibrium outcomes where the public good ends up being provided.

that their private contributions generate. As a result, if k is sufficiently large, the preferred system becomes the BD system. So, there seems to be a trade-off associated with the centralized provision of public goods. A BD is better equipped to deal with the classical free riding problem described in Samuelson (1954), but is less capable of accommodating the private information held by the agents as described in Hayek (1945). The relative strength of each of these two forces, measured by k, determines which of the two systems is preferred.

I believe this paper makes contributions to three different areas of the economic literature. First, to the literature on public goods. The classic literature on public good provision with incomplete information, which includes Groves (1973), d'Aspremont and Gerard-Varet (1979), Laffont and Maskin (1979) among others, typically assumes the mediator/BD has complete contracting ability. This paper relates more closely to the literature that reduces the commitment power of the BD. Schmidt (1996) provides an argument for the privatization of public firms. The idea is that, if the government is directly responsible for the firm and is unable to commit, it will receive private information that will make it less able to provide incentives for the agents employed by the firm to exert effort. Hence, the author argues, privatization (and subsequent regulation) can be seen as a useful commitment device by the government. The main difference from Schmidt (1996) to this paper is that the former focuses on the moral hazard problem rather than on the adverse selection problem the government faces.

Second, this paper may be interpreted in light of the literature on the decentralization of the government. It is possible to interpret the agents in my model as local representatives of different regions and ask the question: should the decision about a public good that affects all regions be made by a centralized government? Or should it be left to the local representatives to reach an agreement? The classical analysis of this problem is due to Oates (1972), where the author argues that decentralization will be preferred as long as the provision of the public good in a given region does not generate large enough positive spillovers on the other regions. Besley and Coate (2003) and Lockwood (2002) relax the assumption made in Oates (1972) that a centralized government selects a uniform policy for all its regions but still assume complete information. There are also several papers that analyze the same question under incomplete information but do not allow for communication among the regions (for example Kessler (2014) or Cho (2013)), which limits the benefits of decentralization. Klibanof and Poitevin (2013) is an exception in that the authors do allow for some bargaining to occur between the regions. However, when modelling the decentralized system, it is assumed that the regions are able to write contracts among themselves.

However, if it is possible for the regions to write contracts among themselves, then it seems reasonable to also allow the government to write contracts with the regions, which, by the revelation principle would be (at least weakly) preferred. For this reason, in my analysis, the regions (agents) are not allowed to write contracts.

Finally, my analysis of the anarchic system builds on the notion that allowing agents to communicate enhances considerably the set of allocations that can be implemented through an equilibrium. Matthews and Postlewaite (1989) show that, in a bilateral trade setting, the introduction of a cheap talk stage, prior to having the traders participate in a double action, allows the implementation of a much larger set of allocations. Agastya et al (2007) explore the same idea in the context of a binary public good provision problem and show that allowing agents to exchange "simple" messages ("yes" or "no" messages) in a voluntary contribution game enlarges the set of implementable allocations.

In section 2, I present the model and define more rigorously the two systems studied: BD system and Anarchy. In section 3, I present and discuss the main result. In section 4, I argue that there are appealing allocations that can be implemented in Anarchy but cannot be implemented with a BD. Finally, in section 5, I discuss the case where the public good is not binary. Section 6 concludes.

2 Model

2.1 Fundamentals

I consider a community with $N > 1$ agents. Each agent n is endowed with a private type $v_n \in [\underline{v}, \overline{v}]$ where $\overline{v} > \underline{v} \geq 0$, independent across n. Each v_n is drawn from a continuous CDF F_n, while each PDF is denoted by f_n. I assume that the public good $g \in \{0, 1\}$. The utility function of each agent n is given by $u_n = v_n g - t_n$ - it depends on whether the public good is provided and on the transfer $t_n \in \mathbb{R}$ that agent n makes. The cost of providing the public good is given by $c > 0$. I assume that $\overline{v} < c$ so that no agent is willing to provide the good by himself.

2.2 Benevolent Dictator System

In a BD system, there are two stages. In the first stage, each agent n sends a public message $m_n \in M_n$, where M_n is assumed to be an arbitrarily large set. In the second stage, given each vector $m \in M = (M_1, ..., M_N)$, there is a probability $p(m) \in \{0, 1\}$ that the public good is provided, and a transfer $t_n(m) \in \mathbb{R}_+$ is requested from each agent n, where $t = (t_1, ..., t_N)$, such that[4]

$$p(m) = 1 \Rightarrow \sum_{n=1}^{N} t_n(m) \geq c$$

A strategy for each agent n is a distribution $\sigma_n(v_n) \in \triangle M_n$ and a strategy profile $\sigma = (\sigma_1, ..., \sigma_N)$. Strategy profile σ is a Bayes-Nash equilibrium of the game induced by (p, t) if and only if, for all n, for all $v_n \in [\underline{v}, \overline{v}]$ and for all $m_n \in M_n$ such that $\sigma_n(v_n)(m_n) > 0$,

$$E^{\sigma}_{m_{-n}}(v_n p(m_n, m_{-n}) - t_n(m_n, m_{-n})) \geq E^{\sigma}_{m_{-n}}(v_n p(m'_n, m_{-n}) - t_n(m'_n, m_{-n}))$$

for all $m'_n \in M_n$, where $m_{-n} = (m_1, ..., m_{n-1}, m_{n+1}, ..., m_N)$ and $E^{\sigma}_{m_{-n}}$ represents the expectation formed by agent n relative to m_{-n}, given strategy profile σ.

It is generally accepted that if the BD has commitment power, she is able to select any (σ, p, t) provided that σ is a Bayes-Nash Equilibrium of the game induced by (p, t). I model the BD's limited commitment power by further restricting the set of (σ, p, t) that she can select. The idea is that there are some profiles (σ, p, t) that are not "reasonable", because they imply that the BD, after some message m, is foregoing alternatives that would be preferred if he was benevolent and inequality averse.

For all $m \in M$, let $\pi^{\sigma}(m) = (\pi^{\sigma}_1(m_1), ..., \pi^{\sigma}_N(m_N))$ denote the the beliefs of the BD after observing message m, given strategy profile σ. More rigorously,

$$\pi^{\sigma}_n(m_n)(v_n) = \frac{\sigma_n(v_n)(m_n) f_n(v_n)}{\int_{\underline{v}}^{\overline{v}} \sigma_n(v_n)(m_n) dF_n(v_n)}$$

[4]In Appendix B, I show that the result of section 3 holds even if random mechanisms are allowed, i.e., even if a distribution over (p, t) follows each message m.

for all m_n such that

$$\int_{\underline{v}}^{\overline{v}} \sigma_n(v_n)(m_n)\, dF_n(v_n) > 0$$

For a profile (σ, p, t) to be selectable by a BD with limited commitment power, it must have the following two properties.

Property A: Posterior Pareto Efficiency

Definition 1 *Profile (σ, p, t) has property A if and only if, for all $m \in M$,*

$$\sum_{n=1}^{N} \inf\left[supp\left(\pi_n^{\sigma}(m_n)\right)\right] > c \Rightarrow p(m) = 1$$

and

$$\sum_{n=1}^{N} t_n(m) = \begin{cases} c \text{ if } p(m) = 1 \\ 0 \text{ otherwise} \end{cases}$$

Property A guarantees that, after any message m, there is no alternative to $(p(m), t(m))$ which makes all agents better off with certainty.

Consider the first part of the definition. Notice that $\inf\left[supp\left(\pi_n^{\sigma}(m_n)\right)\right]$ represents the lowest valuation agent n might have, given the BD's beliefs following message m. Therefore, the BD knows that, if she was to demand a transfer of $\inf\left[supp\left(\pi_n^{\sigma}(m_n)\right)\right]$ or lower from agent n, his utility would certainly be positive, as long as the public good is provided. Therefore, if the inequality is true, it is possible for the BD to fund the provision of the public good, while guaranteeing that no agent makes a contribution that is larger than their valuation.

The second part of the definition means that there are no wasted transfers - if the public good is provided, then the sum of the contributions must not exceed the total cost, while if the good is not provided, no transfers are requested.

Property B: Consistent inequality aversion

Definition 2 *Profile (σ, p, t) has property B if and only, for all n and for all $m_n, m_n' \in M_n$ such that*

$$\pi_n^{\sigma}(m_n) \neq \pi_n^{\sigma}(m_n')$$

i)

$$\pi_n^\sigma(m_n)\ \ FOSD\ \pi_n^\sigma(m_n') \Rightarrow t_n(m_n, m_{-n}) \geq t_n(m_n', m_{-n})$$

for all $m_{-n} \in M_{-n}$ *and*

ii)

$$\exists m_{-n}' \in M_{-n} : t_n\left(m_n, m_{-n}'\right) > t_n\left(m_n', m_{-n}'\right) \Rightarrow t_n(m_n, m_{-n}) \geq t_n(m_n', m_{-n})$$

for all $m_{-n} \in M_{-n}$.

There are two parts to property B. Take two messages that induce a different belief by the BD. Part i) states that if the BD knows that agent n's valuation is larger when he sends message m_n than when he sends message m_n', then, all else the same, it must be that he demands a larger transfer from agent n if he receives message m_n than if he receives message m_n'. The motivation for this property is the assumption that the BD is inequality averse and, therefore, prefers that agents with larger valuations pay for most of the cost of the public good.

Part ii) is a consistency assumption. It states that if, there is some m_{-n}' such that message m_n leads to a larger transfer than m_n', then it is always the case that message m_n leads to a larger transfer than message m_n'. The idea is that, because the BD's beliefs about agent n only depend on the message agent n sends, which message leads to a larger transfer should be independent of the reports of the other agents. In a way, part ii) imposes that the order of the messages each agent sends is independent of what other agents report, while part i) (partially) characterizes that order.

Let Γ^{BD} be the set of profiles (σ, p, t) that are selectable by a BD.

Definition 3 *Profile* $(\sigma, p, t) \in \Gamma^{BD}$ *if and only if* (σ, p, t) *has properties* A *and* B, *and* σ *is a Bayes-Nash equilibrium of the game induced by* (p, t).

2.3 Anarchy

The only difference from Anarchy to the BD system is that the BD does not exist (or can commit not to intervene). Therefore, all is as in the BD system except that each agent freely chooses what transfer to make. In particular, there are two stages.

In the first stage, each agent sends a public message $m_n \in M_n$. In the second stage, after observing vector m, each agent chooses their own transfer. If the sum of transfers (weakly) exceeds c, the public good is provided.[5]

A strategy for each agent n is not only a reporting strategy $\sigma_n(v_n) \in \triangle M_n$ as in the BD system, but also a transfer choice $\gamma_n(v_n, m) \in \mathbb{R}_+$, given the agent's valuation v_n and the observed report m. Let $\sigma = (\sigma_1, ..., \sigma_N)$ and $\gamma = (\gamma_1, ..., \gamma_N)$. Strategy profile (σ, γ) is a Perfect Bayesian equilibrium of the Anarchy game described if and only if

i) for all n, for all $v_n \in [\underline{v}, \overline{v}]$ and for all $m \in M$,

$$\begin{aligned} & E^{\sigma,m}_{v_{-n}}\left(v_n \mathbf{1}\left\{\sum_{\widehat{n}=1}^{N} \gamma_{\widehat{n}}(v_{\widehat{n}}, m) \geq c\right\} - \gamma_n(v_n, m)\right) \\ \geq\ & E^{\sigma,m}_{v_{-n}}\left(v_n \mathbf{1}\left\{\sum_{\widehat{n}\neq n}^{N} \gamma_{\widehat{n}}(v_{\widehat{n}}, m) + x_n \geq c\right\} - x_n\right) \end{aligned}$$

for all $x_n \in \mathbb{R}_+$ where $E^{\sigma,m}_{v_{-n}}$ represents the expectation about v_{-n} formed by agent n, given σ and report m;

ii) for all n, for all $v_n \in [\underline{v}, \overline{v}]$ and for all $m_n \in M_n$ such that $\sigma_n(v_n)(m_n) > 0$,

$$\begin{aligned} & E^{\sigma}_{m_{-n},v_{-n}}\left(v_n \mathbf{1}\left\{\sum_{\widehat{n}=1}^{N} \gamma_{\widehat{n}}(v_{\widehat{n}}, m_n, m_{-n}) \geq c\right\} - \gamma_n(v_n, m_n, m_{-n})\right) \\ \geq\ & E^{\sigma}_{m_{-n},v_{-n}}\left(v_n \mathbf{1}\left\{\sum_{\widehat{n}=1}^{N} \gamma_{\widehat{n}}(v_{\widehat{n}}, m'_n, m_{-n}) \geq c\right\} - \gamma_n(v_n, m'_n, m_{-n})\right) \end{aligned}$$

for all $m'_n \in M_n$, where $E^{\sigma}_{m_{-n},v_{-n}}$ represents the expectation about m_{-n} and v_{-n} formed by agent n, given σ.

Let the set of all Perfect Bayesian equilibria (σ, γ) of the anarchic game be denoted by Γ^A.

[5] While, in order to complete the description of the anarchic system, I assume that once an agent has selected a positive transfer, he immediately loses it, this is not necessary for the main result of section 3. An alternative assumption could be that if the sum of transfers is not enough to provide the public good, then every contributing agent receives their transfer back.

3 Main result

The main result of the paper is that, if the public good is binary, every allocation which is implementable by a BD and is ex-post individually rational is also implementable in Anarchy. In order to formally state the result, it is necessary to first define a few concepts.

3.1 Definitions

An allocation is a function $(\rho, \tau) : [\underline{v}, \overline{v}]^N \to \{0, 1\} \times \mathbb{R}_+^N$, where $\rho(v) \in \{0, 1\}$ represents the probability the public good is provided if the agents' valuation vector is v, while $\tau_n(v)$ represents the transfer agent n makes, given vector v.

An allocation is implementable by a BD system if there is $(\sigma, p, t) \in \Gamma^{BD}$ such that, for all $v \in [\underline{v}, \overline{v}]^N$, for all m such that $\sigma(v)(m) > 0$,

$$\rho(v) = p(m) \text{ and } \tau(v) = t(m)$$

where $\sigma(v)(m) = \prod_{n=1}^{N} \sigma_n(v_n)(m_n)$. Let all allocations (ρ, τ) that are implementable by a BD system be denoted by Ψ^{BD}.

An allocation is implementable in Anarchy if there is $(\sigma, \gamma) \in \Gamma^A$ such that, for all $v \in [\underline{v}, \overline{v}]^N$, for all m such that $\sigma(v)(m) > 0$,

$$\rho(v) = \mathbf{1}\left\{\sum_{n=1}^{N} \gamma_n(v_n, m) \geq c\right\} \text{ and } \tau_n(v) = \gamma_n(v_n, m)$$

for all n. Let all allocations (ρ, τ) that are implementable in Anarchy be denoted by Ψ^A.

An allocation (ρ, τ) is ex-post individually rational if and only if, for all n, for all $v \in [\underline{v}, \overline{v}]^N$ and for all $v_n \in [\underline{v}, \overline{v}]$,

$$\rho(v) v_n - \tau_n(v) \geq 0$$

Let the set of all ex-post individually rational allocations (ρ, τ) be denoted by Ψ^{IR}.

3.2 Theorem

The formal statement of the main result is as follows.

Theorem 4 $\Psi^{BD} \cap \Psi^{IR} \subseteq \Psi^{A}$.

In order to prove the result, one must take some arbitrary $(\rho, \tau) \in \Psi^{BD} \cap \Psi^{IR}$ and show that $(\rho, \tau) \in \Psi^{A}$. If $(\rho, \tau) \in \Psi^{BD}$, then there must be some profile $(\sigma, p, t) \in \Gamma^{BD}$ which implements (ρ, τ). The goal is to find a perfect Bayesian equilibrium $(\widehat{\sigma}, \widehat{\gamma})$ of the anarchic game which implements the same allocation.

I divide the proof in two parts. In the first part, I characterize $(\widehat{\sigma}, \widehat{\gamma})$ and show that if $(\widehat{\sigma}, \widehat{\gamma}) \in \Gamma^{A}$, then $(\widehat{\sigma}, \widehat{\gamma})$ implements (ρ, τ) in Anarchy. In the second part, I show that $(\widehat{\sigma}, \widehat{\gamma}) \in \Gamma^{A}$. For the second part, I only provide a sketch of the proof in the main text and leave the formal details for Appendix A.

3.3 Proof (part i)

Let $(\widehat{\sigma}, \widehat{\gamma})$ be such that $\widehat{\sigma} = \sigma$ and

$$\widehat{\gamma}_n (v_n, m) = \begin{cases} t_n (m) \text{ if } p(m) = 1 \text{ and } t_n (m) \leq v_n \\ \text{0 otherwise} \end{cases}$$

for all n, $v_n \in [\underline{v}, \overline{v}]$ and $m \in M$. I show that $(\widehat{\sigma}, \widehat{\gamma}) \in \Gamma^{A}$ and it implements (ρ, τ).

The idea is that, in Anarchy, each agent n is supposed to report as they did in the BD system ($\widehat{\sigma} = \sigma$) and then make the transfer the BD would have made following each message vector m, provided that such transfer does not leave him with a negative utility.

The first thing to note is that, in Anarchy, if an agent does not deviate in the first stage, he will also not want to deviate in the second stage. Consider some arbitrary agent n and suppose he chooses his message m_n according to σ_n. Suppose also that he believes that every other agent plays according to $(\widehat{\sigma}, \widehat{\gamma})$. For any message vector m, agent n's best option at the second stage is to choose a transfer of $t_n (m)$, because one of two things must happen.

If $p(m) = 1$, a transfer of $t_n(m)$ is the minimum transfer which ensures the public good is provided (because (σ, p, t) has property A, which guarantees that there are no wasted transfers), so that it is strictly preferred to a larger transfer. And the best of the smaller transfers is a transfer of 0, in which case the agent's utility would also be 0, which is not strictly preferred, as transferring $t_n(m)$ guarantees a minimum utility of 0 to agent n (because $(\rho, \tau) \in \Psi^{IR}$).

If $p(m) = 0$, then no other agent will make any positive transfer and so the best choice for agent n is to choose a transfer of 0. However, because (σ, p, t) is such that there are no wasted transfers, $t_n(m) = 0$.

Therefore, there are two observations one can already make. First, $\widehat{\gamma}$ constitutes the set of best responses for each agent n, given their type and any first stage history. This means that, by showing that no agent wishes to deviate in the first stage, one shows that $(\widehat{\sigma}, \widehat{\gamma}) \in \Gamma^A$. And second, if $(\widehat{\sigma}, \widehat{\gamma}) \in \Gamma^A$, then (ρ, τ) is implementable in Anarchy by $(\widehat{\sigma}, \widehat{\gamma})$, simply because if agents play according to $(\widehat{\sigma}, \widehat{\gamma})$, they report as they did in the BD system and then end up choosing the transfer the BD would have chosen. What is left then is to show that no agent wishes to misreport at the first stage.

3.4 Proof (part ii)

In order to complete the proof, it is necessary first to characterize (σ, p, t) further. Take any two messages m_n', m_n'' sent by an arbitrary agent n such that

$$E^{\sigma}_{m_{-n}}\left(p\left(m_n', m_{-n}\right)\right) > E^{\sigma}_{m_{-n}}\left(p\left(m_n'', m_{-n}\right)\right)$$

and suppose that, if agent n's valuation is $\widehat{v}_n \in [\underline{v}, \overline{v}]$, then he prefers message m_n' to message m_n''. This means that

$$\begin{aligned} & \widehat{v}_n E^{\sigma}_{m_{-n}}\left(p\left(m_n', m_{-n}\right)\right) - E^{\sigma}_{m_{-n}}\left(t_n\left(m_n', m_{-n}\right)\right) \\ \geq\ & \widehat{v}_n E^{\sigma}_{m_{-n}}\left(p\left(m_n'', m_{-n}\right)\right) - E^{\sigma}_{m_{-n}}\left(t_n\left(m_n'', m_{-n}\right)\right) \end{aligned}$$

It follows that, if agent n's type is larger than $\widehat{v}_n$, the agent strictly prefers message m_n' to message m_n''.

This observation implies that each σ_n has essentially two possible forms. The first possibility is that σ_n is monotone, i.e. if agent n sends message m_n whenever $v_n \in \{v_n', v_n''\}$, it must be that agent n sends message m_n whenever $v_n \in [v_n', v_n'']$. Figure 1 displays an example of what such σ_n might look like for an arbitrary agent n.

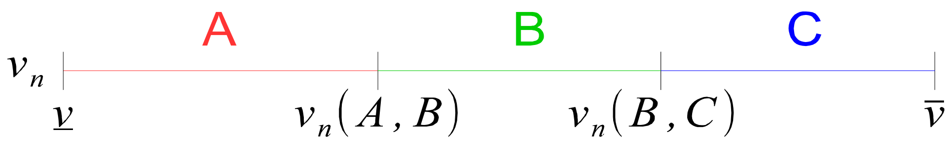

Figure 1: Example of a monotone reporting strategy for agent n

In the example, agent n sends a total of three messages: A, B, and C. Message A is sent if the agent's type is lower than $v_n(A, B)$, message B is sent if the agent's type is between $v_n(A, B)$ and $v_n(B, C)$ and message C is sent if the agent's type is larger than $v_n(B, C)$. If the agent's type is equal to $v_n(A, B)$, the agent is indifferent between A and B, while if it is equal to $v_n(B, C)$, he is indifferent between B and C. Furthermore,

$$E^{\sigma}_{m_{-n}}\left(p\left(A, m_{-n}\right)\right) \leq E^{\sigma}_{m_{-n}}\left(p\left(B, m_{-n}\right)\right) \leq E^{\sigma}_{m_{-n}}\left(p\left(C, m_{-n}\right)\right)$$

If the agent's valuation is larger, he prefers to select the message that leads to a larger probability that the public good is provided.

However, it is also possible that σ_n is not monotone as exemplified by figure 2.

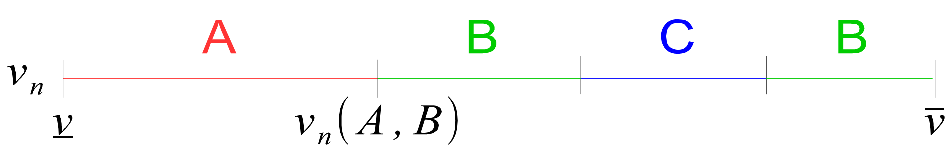

Figure 2: Example of a non-monotone reporting strategy for agent n

Notice that the set of types for which the agent sends message C is "sandwiched" in between the two sets of types for which the agent sends message B. This is only possible if

$$E^{\sigma}_{m_{-n}}\left(p\left(A, m_{-n}\right)\right) \leq E^{\sigma}_{m_{-n}}\left(p\left(B, m_{-n}\right)\right) = E^{\sigma}_{m_{-n}}\left(p\left(C, m_{-n}\right)\right)$$

In the rest of the section, I show that agent n does not want to deviate in his report if σ_n is as in figure 1 or as in figure 2. In appendix A, I provide a more complete and formal proof which also applies to the cases where σ_n involves infinite messages.

Analysis of figure 1:

Take some agent n and suppose σ_n is as in figure 1. I show that agent n does not want to deviate in the first stage in three steps.

Step 1: Agent n never "reports down".

Agent n "reports down" if he deviates to a message which was supposed to be sent only if his type was smaller. For example, if the agent's valuation

$$v_n \in [v_n(A,B), v_n(B,C)]$$

reporting down would mean to report A. I show that this deviation is not strictly beneficial to the agent. The other cases are analogous.

Notice that

$$t_n(A, m_{-n}) \leq \underline{v} < v_n$$

for all m_{-n}, where the first inequality follows because $(\rho, \tau) \in \Psi^{IR}$. If sending message A led, for some m_{-n}, to a transfer larger than $\underline{v}$, the agent's ex-post utility would sometimes be negative if his's type was exactly $\underline{v}$. Therefore, the agent knows that, in Anarchy, if he deviates to sending message A in the first stage, he will choose his own transfer just like the BD would have chosen. This means that a deviation to message A in Anarchy is equivalent to a deviation to message A in the BD system. But then, in the BD system, agent n prefers message B to message A whenever $v_n \in [v_n(A,B), v_n(B,C)]$, because $(\sigma, p, t) \in \Gamma^{BD}$. So, it follows that, also in Anarchy, the agent will not deviate to A.

Step 2: For any type reporting A, reporting B is better than reporting C.

First, notice that $p(B, m_{-n}) \leq p(C, m_{-n})$ for all m_{-n}, because (σ, p, t) has property A and $(\rho, \tau) \in \Psi^{IR}$. Basically, if the BD believes she can fund the provision of the public good while making all agents better off with certainty after observing vector (B, m_{-n}), that must also be true after observing (C, m_{-n}).

It is also the case that, for all m_{-n} such that $p(B, m_{-n}) = p(C, m_{-n}) = 1$,

$$\max\{v_n - t_n(B, m_{-n}), 0\} \geq \max\{v_n - t_n(C, m_{-n}), 0\}$$

for all $v_n \in [\underline{v}, \overline{v}]$, because (σ, p, t) has property B. After message C, the BD knows that the agent's type is larger than $v_n(B,C)$, while after message B, she knows it is smaller than $v_n(B,C)$. So, because the BD is inequality averse, she demands a larger transfer after message C than after message B.

It follows that some type $v_n \in [\underline{v}, \overline{v}]$ would only strictly prefer C to B if there was some m_{-n} such that

$$p(B, m_{-n}) = 0 < p(C, m_{-n}) = 1$$

and

$$t_n(C, m_{-n}) < v_n$$

Notice that, because (σ, p, t) has property A,

$$p(B, m_{-n}) = 0 \Rightarrow \sum_{\widehat{n} \neq n} t_{\widehat{n}}(B, m_{-n}) + v_n(A, B) \leq c$$

which simply means that, for the BD to choose not to provide the public good, it must be impossible to fund it and make sure that every agent has a positive ex-post utility. However, it is also the case that

$$p(C, m_{-n}) = 1 \Rightarrow \sum_{\widehat{n} \neq n} t_{\widehat{n}}(B, m_{-n}) + v_n(B, C) \geq c$$

given that $(\rho, \tau) \in \Psi^{IR}$. These two observations combined imply that, for any m_{-n} such that

$$p(B, m_{-n}) = 0 < p(C, m_{-n}) = 1$$

it must be that

$$t_n(C, m_{-n}) \geq v_n(A, B)$$

which implies that, if the agent's type $v_n \leq v_n(A, B)$, he would prefer to send C over B.

Step 3: For any type reporting A, reporting B is not strictly preferred.

If agent n's type is $v_n = \underline{v}$, the agent prefers A to B by step 2. Because $t_n(B, m_{-n}) \leq v_n(A, B)$, it is also the case that if $v_n = v_n(A, B)$, the agent does not strictly prefer B to A.

Let

$$\widehat{p}_n(m, v_n) = \begin{cases} p(m) \text{ if } t_n(m) \leq v_n \\ 0 \text{ otherwise} \end{cases}$$

and

$$\widehat{t}_n(m, v_n) = \begin{cases} \widehat{t}_n(m) \text{ if } t_n(m) \le v_n \\ 0 \text{ otherwise} \end{cases}$$

so that $\widehat{p}_n(m, v_n)$ and $\widehat{t}_n(m, v_n)$ represent, respectively, the probability that the public good is provided and the transfer of agent n in Anarchy, given message vector m and agent n's type v_n. It basically represents the outcome for agent n when his type is v_n, after m, given that he is optimizing his behavior.

Suppose there is some $v_n \in (\underline{v}, v_n(A, B))$ for which the agent prefers to deviate to B. It follows that

$$\begin{aligned} & v_n E^{\sigma}_{m_{-n}}(\widehat{p}_n(B, m_{-n}, v_n)) - E^{\sigma}_{m_{-n}}\left(\widehat{t}_n(B, m_{-n}, v_n)\right) \\ > \; & v_n E^{\sigma}_{m_{-n}}(\widehat{p}_n(C, m_{-n}, v_n)) - E^{\sigma}_{m_{-n}}\left(\widehat{t}_n(C, m_{-n}, v_n)\right) \end{aligned}$$

Imagine that the probability that the good is provided in Anarchy if agent n reports B is larger than if he reports A, i.e.

$$E^{\sigma}_{m_{-n}}(\widehat{p}_n(B, m_{-n}, v_n)) \ge E^{\sigma}_{m_{-n}}(p(A, m_{-n}))$$

In this case, it follows that

$$\begin{aligned} & v_n(A, B) E^{\sigma}_{m_{-n}}(\widehat{p}_n(B, m_{-n}, v_n)) - E^{\sigma}_{m_{-n}}\left(\widehat{t}_n(B, m_{-n}, v_n)\right) \\ > \; & v_n(A, B) E^{\sigma}_{m_{-n}}(p(A, m_{-n})) - E^{\sigma}_{m_{-n}}(t_n(A, m_{-n})) \end{aligned}$$

which, in turn, implies that

$$\begin{aligned} & v_n(A, B) E^{\sigma}_{m_{-n}}(\widehat{p}_n(B, m_{-n}, v_n(A, B))) - E^{\sigma}_{m_{-n}}\left(\widehat{t}_n(B, m_{-n}, v_n(A, B))\right) \\ > \; & v_n(A, B) E^{\sigma}_{m_{-n}}(p(A, m_{-n})) - E^{\sigma}_{m_{-n}}(t_n(A, m_{-n})) \end{aligned}$$

simply because agent n does better if he optimizes his behavior given his type. However, this means that, if the agent's type is $v_n(A, B)$ he strictly prefers message B to A, which is a contradiction. Therefore, it must be that the probability that the public good is provided is larger after message A:

$$E^{\sigma}_{m_{-n}}(\widehat{p}_n(B, m_{-n}, v_n)) < E^{\sigma}_{m_{-n}}(p(A, m_{-n}))$$

However, in this case, it must be that

$$\begin{aligned} & \underline{v}E^{\sigma}_{m_{-n}}\left(\widehat{p}_n\left(B, m_{-n}, v_n\right)\right) - E^{\sigma}_{m_{-n}}\left(\widehat{t}_n\left(B, m_{-n}, v_n\right)\right) \\ > \; & \underline{v}E^{\sigma}_{m_{-n}}\left(p\left(A, m_{-n}\right)\right) - E^{\sigma}_{m_{-n}}\left(t_n\left(A, m_{-n}\right)\right) \end{aligned}$$

which, for the same reasons, implies that

$$\begin{aligned} & \underline{v}E^{\sigma}_{m_{-n}}\left(\widehat{p}_n\left(B, m_{-n}, \underline{v}\right)\right) - E^{\sigma}_{m_{-n}}\left(\widehat{t}_n\left(B, m_{-n}, \underline{v}\right)\right) \\ > \; & \underline{v}E^{\sigma}_{m_{-n}}\left(p\left(A, m_{-n}\right)\right) - E^{\sigma}_{m_{-n}}\left(t_n\left(A, m_{-n}\right)\right) \end{aligned}$$

and so, if agent n's type is $\underline{v}$, he strictly prefers message B to A, which again is a contradiction. Therefore, it follows that there cannot be any $v_n \leq v_n(A, B)$ for which the agent prefers B to A.

By the same argument, if the agent's type $v_n \in [v_n(A, B), v_n(B, C)]$, he prefers to send message B to C, which completes the proof.

Analysis of figure 2:

Consider figure 2 and compare messages B and C. Recall that, because (σ, p, t) has property A and $(\rho, \tau) \in \Psi^{IR}$, the BD provides the public good as long as she can fund it while leaving all agents with a positive ex-post utility. Therefore, it must be that, for all m_{-n},

$$p(C, m_{-n}) \geq p(B, m_{-n})$$

Recall also that, because (σ, p, t) has property B (part ii), the BD acts in a consistent manner and so,

$$t_n(C, m_{-n}) \geq t_n(B, m_{-n}) \text{ for all } m_{-n}$$

or

$$t_n(C, m_{-n}) \leq t_n(B, m_{-n}) \text{ for all } m_{-n}$$

which implies that the two messages must result in the same outcome almost everywhere. As a result, if agent n's type is any $v_n \in [\underline{v}, \overline{v}]$, he is indifferent between reporting B and C. By following the steps from the analysis of figure 1, it is straightforward to verify that, if σ_n is as portrayed by figure 2, agent n will not deviate in Anarchy.

4 Is Anarchy better than the BD system?

The difference between the BD system and Anarchy is the freedom that exists in Anarchy for agents to choose their own transfers. On the one hand, this is a bad thing, as it gives the agents more opportunities to deviate from the behavior the BD would like to impose. However, the whole exercise of the previous section was exactly to show that the freedom that agents have to choose their own transfers in Anarchy does not prevent any of the allocations which are implementable by a BD to be implementable also in Anarchy, provided they are individually rational. Basically, when the public good is binary, the agents are never in a situation where they would want to choose a different transfer than the one the BD would have chosen. As I discuss in the next section, when the public good is not binary, things are a little different.

However, the fact that agents have the freedom to choose their own transfers also has a positive side to it in that what happens in the second stage becomes less restricted. In the BD system, if it is known that it is possible to provide the public good while ensuring that all agents have a positive ex-post utility, then the good must be provided by the BD. However, in Anarchy, it does not, as there is always a Bayes-Nash equilibrium of the second stage voluntary contribution subgame where no agent provides a positive transfer. Likewise, in a BD system, the transfers that agents make reflect the inequality aversion preference of the BD, while in Anarchy that needs not be the case.

Therefore, in general, the set of allocations which are implementable in Anarchy is larger than the set of allocations which are implementable by a BD and are ex-post individually rational. It is also often the case that some of the allocations which are implementable in Anarchy but not by a BD are particularly appealing as the following example illustrates.

Example 5 *Consider the case explored in Agastya et al. (2007) where there are only two agents, $v_n \sim U(\underline{v}, \overline{v})$ for $n = 1, 2$ and it is assumed that $\overline{v} + \underline{v} \leq c$. Let Ω denote the set of all allocations (ρ, τ) which are incentive compatible, interim individually rational, and ex-ante budget balanced. Notice that Ω can be interpreted as a set of "reasonable" allocations. In particular, it is the case that $\Psi^{BD} \subseteq \Omega$.*

For all $\lambda \in [0, 1]$, let $\omega(\lambda) \in \Omega$ be such that

$$\omega(\lambda) = \arg \max_{(\rho,\tau)\in\Omega} \lambda E(u_1|\rho, \tau) + (1 - \lambda) E(u_2|\rho, \tau)$$

where $E(u_n|\rho, \tau)$ represents the expected utility of agent n, given allocation (ρ, τ), for

$n = 1, 2$, so that the set of efficient allocations within Ω is given by $\bigcup_{\lambda \in [0,1]} \omega(\lambda)$. It can be shown that, for all $\lambda \in [0,1]$, if $(\rho, \tau) \in \omega(\lambda)$, then,

$$\rho(v) = \rho^{\lambda}(v) \equiv \begin{cases} 1 \text{ if } v_2 \geq m^{\lambda} v_1 + b^{\lambda} \\ 0 \text{ otherwise} \end{cases}$$

for all $v \in [\underline{v}, \overline{v}]^2$, where $m^{\lambda} < 0$ and $b^{\lambda} \in \mathbb{R}$ (see Agastya et al. (2007)). The authors use this insight to show that, for all $\lambda \in [0,1]$, there is an allocation $(\rho^{\lambda}, \tau^{\lambda}) \in \omega(\lambda)$, which is implementable in Anarchy.

However, no efficient allocation is implementable in a BD system, i.e. if $(\rho, \tau) \in \omega(\lambda)$ for some $\lambda \in [0,1]$, then (ρ, τ) is not implementable by a BD. Let

$$\rho^{*}(v) = \begin{cases} 1 \text{ if } v_2 \geq -v_1 + c \\ 0 \text{ otherwise} \end{cases}$$

for all $v \in [\underline{v}, \overline{v}]^2$ and notice that, for all $v \in [\underline{v}, \overline{v}]^2$, $\rho^{*}(v)$ represents the probability that the public good is provided if the BD knows v.

Simple algebra based on Agastya et al. (2007) shows that, for all $\lambda \in [0,1]$, $(m^{\lambda}, b^{\lambda}) \neq (-1, c)$, which implies $\rho^{\lambda} \neq \rho^{*}$. Furthermore, it is also the case that $\rho^{\lambda}(v) \geq \rho^{*}(v)$ for all $v \in [\underline{v}, \overline{v}]^2$ and for any $\lambda \in [0,1]$. Figure 3 provides a comparison between ρ^{λ} for some $\lambda \in [0,1]$ and ρ^{*}.

For any allocation $(\rho^{\lambda}, \tau) \in \omega(\lambda)$ to be implementable by a BD, it would be necessary that each agent reported truthfully, at least for some types. In particular, in figure 3, it must be that if $v_n > \widehat{v}_n$, agent n must be reporting truthfully for $n = 1, 2$.[6] *But, if this was the case, because the BD is not assumed to be able to commit and is benevolent, he must select to provide the public good if and only if $\rho^{*}(v) = 1$.*

5 Discrete public good

In this section, while I continue to assume that the public good is discrete, I no longer focus only on the binary case. In particular, I assume that $g \in \left\{\frac{1}{k}, \frac{2}{k}, ..., \frac{k}{k}\right\}$ for some $k \geq$

[6] Suppose not and imagine that, for some type $v_1' > \widehat{v}_1$, agent 1 sends a message which is also sent if $v_1 = v_1'' \in [\underline{v}, \overline{v}]$ such that $v_1' \neq v_1''$. This would imply that the probability that the good is provided is the same if $v_1 = v_1'$ and $v_1 = v_1''$ which would make the implementation of the ρ^{λ} depicted in figure 3 impossible.

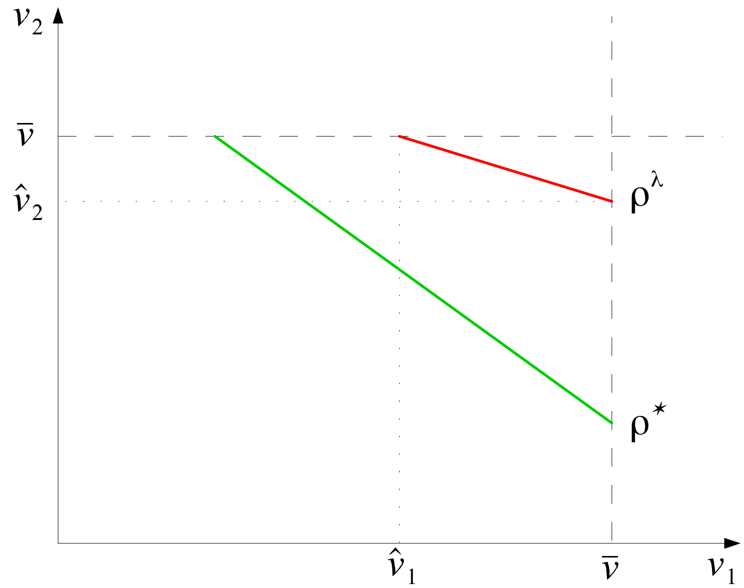

Figure 3: Comparison between ρ^λ and ρ^*. Above the red (green) line, $\rho^\lambda = 1$ ($\rho^* = 1$), while below the red (green) line, $\rho^\lambda = 0$ ($\rho^* = 0$).

1 and that the cost of providing g units of the public good is given by cg. Accordingly, an allocation $(\widehat{\rho}, \widehat{\tau}) : [\underline{v}, \overline{v}]^N \to \left\{\frac{1}{k}, \frac{2}{k}, ..., \frac{k}{k}\right\} \times \mathbb{R}_+^N$ maps the agents' types to units of the public good to be provided and a transfer vector.

Consider the BD system. Notice that properties A and B still continue to make sense if the dictator is benevolent and inequality averse, even if the public good is not binary. In particular, in what concerns property A, if the BD knows that the sum of the agents' valuations is larger than c, she must provide $g = 1$ units of the public good, because anything less would be pareto dominated. Therefore, the set of allocations $(\widehat{\rho}, \widehat{\tau})$ which are implementable by a BD system and are ex-post individually rational are essentially the same regardless of k.[7]

However, in Anarchy, things are different, because, if $k > 1$, the classic free-riding problem arises. In particular, the maximum transfer each agent makes in Anarchy is $\frac{1}{k}\overline{v}$. To see this, imagine that, after some message $m \in M$, there is some agent n who

[7] The number of allocations which are implementable by a BD and ex-post individually rational is only increasing with k if there is some allocation (ρ, τ) when $k = 1$ such that there is a $v \in [\underline{v}, \overline{v}]^N$ for which all agents report truthfully and $\sum_{n=1} v_n = c$. If that happens, additional allocations can be implemented by having agents report the same as with (ρ, τ) and the BD making the same decisions as with (ρ, τ) except when she knows that the agents' types are given by v. In that case, she can select any $g \in \left\{\frac{1}{k}, ..., \frac{k-1}{k}\right\}$. These new allocations are essentially the same as (ρ, τ) simply because they induce the same behavior by the agents and the BD almost everywhere.

makes a transfer of $\frac{1}{k}\overline{v} + \varepsilon$ for some $\varepsilon > 0$. This would lead to the provision of $\frac{j}{k}$ units of the public good, for some $j > 0$. If agent n was to reduce his transfer in $\left(\frac{1}{k}\overline{v} + \delta\right)$ for

$$\delta \in \left(0, \min\left\{\frac{1}{k}\left(c - \overline{v}\right), \varepsilon\right\}\right)$$

the units of the public that would be provided would be $\frac{j-1}{k}$. Therefore, the benefit for the agent of this deviation would be

$$\frac{1}{k}\left(\overline{v} - v_n\right) + \delta > 0$$

Given that each agent never transfers more than $\frac{1}{k}\overline{v}$, it follows that the maximum amount of units of the public good which can be provided in Anarchy is $\frac{N}{k}\overline{v}$, which becomes smaller as k increases, i.e. Anarchy becomes increasingly unappealing.[8]

In a way, parameter k represents how important the classic free-riding problem is relative to the underlying informational problem. If k is small, then classic free-riding is of little importance (if $k = 1$ it has no importance) and so, the best system seems to be Anarchy. But as k grows, Anarchy becomes inefficient because the free-riding problem becomes the dominant one, and so the BD system performs better.

6 Conclusion

The main result of this paper is that, if the public good is binary, all allocations which are implementable in a BD system and are ex-post individually rational are implementable in Anarchy. Naturally, the definition of what is a BD system has a subjective component to it and it could be that the reader is reluctant to accept some of the properties assumed (property A: posterior pareto efficiency and property B: inequality aversion). In particular, the reader might wonder whether the result would still follow if the inequality aversion assumption was abandoned. In appendix C, I show by way of example that both assumptions are necessary, i.e. there are ex-post individually rational allocations which have, for example, property A but not property B, which are not implementable in Anarchy. Nevertheless, this does not imply that every allocation which is implementable in Anarchy has properties A and B, as

[8] In a similar context, Barbieri (2012) also highlights some of the problems that the voluntary provision of public goods with pre-play communication may have with non-binary goods.

section 4 illustrates. On the other hand, it is the case that every allocation which is implementable in Anarchy is ex-post individually rational.

Therefore, the only advantage of the BD system as I have defined it is that it allows for the implementation of allocations which are not ex-post individually rational. Such allocations have a debatable appeal for two different reasons. The first is a practical reason. If an agent anticipates that he will be made worse off from the provision of the public good, he might, for example, be tempted to move to a region where the public good is not provided. The second is an ethical reason, as such allocations imply that the BD would have to coerce the agents into making transfers which make them worse off.

7 Appendix

In appendix A, I complete the proof of the theorem. In appendix B, I show that the theorem also holds if the BD is able to randomize on his second stage choice of transfers. Finally, in appendix C, I provide counterexamples to the theorem whenever either property A or B fail, as well as when the agents' types are not continuous.

7.1 Appendix A

In this section, I formally prove that $(\widehat{\sigma}, \widehat{\gamma}) \in \Gamma^A$. In part i), in the text, I have already show that $\widehat{\gamma}$ represents the best response for each agent, given $\widehat{\sigma}$ and any first stage history. What is left is to show that no agent wishes to deviate in the first stage from $\widehat{\sigma}$.

Let $(\rho, \tau) \in \Psi^{BD} \cup \Psi^{IR}$ be the underlying allocation one and $(\sigma, p, t) \in \Gamma^{BD}$ the profile which implements (ρ, τ) in the BD system. Take any n, any $v_n \in [\underline{v}, \overline{v}]$ and any $m_n \in M_n$ such that $\sigma_n(v_n)(m_n) > 0$

Lemma 1: If agent n's type is v_n, he does not strictly prefer to report any $m_n' \in M_n$ such that

$$E^{\sigma}_{m_{-n}}\left(p\left(m_n', m_{-n}\right)\right) < E^{\sigma}_{m_{-n}}\left(p\left(m_n, m_{-n}\right)\right)$$

Proof of Lemma 1.

Notice that, because $(\sigma, p, t) \in \Gamma^{BD}$, it follows that

$$v_n E^{\sigma}_{m_{-n}}(p(m_n, m_{-n})) - E^{\sigma}_{m_{-n}}(t_n(m_n, m_{-n})) \geq v_n E^{\sigma}_{m_{-n}}(p(m'_n, m_{-n})) - E^{\sigma}_{m_{-n}}(t_n(m'_n, m_{-n}))$$

which implies that

$$v'_n E^{\sigma}_{m_{-n}}(p(m_n, m_{-n})) - E^{\sigma}_{m_{-n}}(t_n(m_n, m_{-n})) > v'_n E^{\sigma}_{m_{-n}}(p(m'_n, m_{-n})) - E^{\sigma}_{m_{-n}}(t_n(m'_n, m_{-n}))$$

for all $v'_n > v_n$. Therefore, it must be that

$$t_n(m'_n, m_{-n}) \leq \sup(supp(\pi^{\sigma}_n(m'_n))) \leq v_n$$

for all m_{-n}, where the first inequality comes from the fact that $(\rho, \tau) \in \Psi^{IR}$. Hence, the fact that $(\sigma, p, t) \in \Gamma^{BD}$ guarantees that type v_n does not deviate to m'_n.

Lemma 2: If agent n's type is v_n, he does not strictly prefer to report any $m'_n \in M_n$ such that

$$E^{\sigma}_{m_{-n}}(p(m'_n, m_{-n})) = E^{\sigma}_{m_{-n}}(p(m_n, m_{-n}))$$

Proof of Lemma 2.

Notice that, because (σ, p, t) has property A and from the fact that $(\rho, v) \in \Psi^{IR}$, it follows that

$$\int_{m_{-n} \in M_{-n}} \mathbf{1}\{p(m'_n, m_{-n}) \neq p(m_n, m_{-n})\} \, dG^{\sigma}_n(m_{-n}) = 0$$

where G^{σ}_n represents the density of m_{-n}, given σ. Because (σ, p, t) has property B (part ii),

$$\int_{m_{-n} \in M_{-n}} \mathbf{1}\{t_n(m'_n, m_{-n}) \neq t_n(m_n, m_{-n})\} \, dG^{\sigma}_n(m_{-n}) = 0$$

Therefore, the deviation payoff for type v_n of reporting m'_n is equal to reporting m_n almost everywhere, which implies it is not strictly preferable.

Lemma 3: If there is $m'_n \in M_n$ and $m''_n \in M_n$ such that

$$E^{\sigma}_{m_{-n}}(p(m'_n, m_{-n})) > E^{\sigma}_{m_{-n}}(p(m''_n, m_{-n})) > E^{\sigma}_{m_{-n}}(p(m_n, m_{-n}))$$

then, if agent n's type is v_n, he prefers (maybe weakly) to report m''_n than m'_n.

Proof of Lemma 3.

Notice that $\pi^\sigma_n(m'_n)$ $FOSD$ $\pi^\sigma_n(m''_n)$. To see this, let $\widehat{v}_n$ be such that

$$\widehat{v}_n E^\sigma_{m_{-n}}(p(m''_n, m_{-n})) - E^\sigma_{m_{-n}}(t_n(m''_n, m_{-n})) = \widehat{v}_n E^\sigma_{m_{-n}}(p(m'_n, m_{-n})) - E^\sigma_{m_{-n}}(t_n(m'_n, m_{-n}))$$

Then, it is clear that, after message m''_n, it is known that agent n's type is not greater than $\widehat{v}_n$, while after message m'_n it is known that agent n's type is not smaller than $\widehat{v}_n$.

Because (σ, p, t) has property B (part i), it follows that if for all m_{-n} such that

$$p(m'_n, m_{-n}) = p(m''_n, m_{-n}) = 1$$

then

$$\max\{v_n - t_n(m'_n, m_{-n}), 0\} \leq \max\{v_n - t_n(m''_n, m_{-n}), 0\}$$

Therefore, the difference for agent n between the interim expected payoff of reporting m'_n and m''_n is no greater than

$$\int\limits_{m_{-n}\in M_{-n}} \mathbf{1}\{p(m'_n, m_{-n}) > p(m''_n, m_{-n})\} \max\{v_n - t_n(m'_n, m_{-n}), 0\}\, dG^\sigma(m_{-n})$$

Take any m_{-n} such that

$$p(m'_n, m_{-n}) = 1 > p(m''_n, m_{-n}) = 0$$

which implies that

$$\sum_{\widehat{n}\neq n} \inf(supp(\pi^\sigma_n(m_{\widehat{n}}))) + \inf(supp(\pi^\sigma_n(m''_n))) \leq c$$

by property A. Because $\pi^\sigma_n(m''_n)$ $FOSD$ $\pi^\sigma_n(m_n)$, it follows that

$$\inf(supp(\pi^\sigma_n(m''_n))) \geq v_n$$

Therefore,

$$\sum_{\widehat{n}\neq n} \inf(supp(\pi^\sigma_{\widehat{n}}(m_{\widehat{n}}))) + v_n \leq c$$

Hence, because

$$t_{\widehat{n}}\left(m_n', m_{-n}\right) \leq \inf\left(supp\left(\pi_{\widehat{n}}^{\sigma}\left(m_{\widehat{n}}\right)\right)\right)$$

for all $\widehat{n} \neq n$ because $(\rho, t) \in \Psi^{IR}$, it must be that

$$t_n\left(m_n', m_{-n}\right) \geq v_n$$

which implies that

$$\int\limits_{m_{-n} \in M_{-n}} \mathbf{1}\left\{p\left(m_n', m_{-n}\right) > p\left(m_n'', m_{-n}\right)\right\} \max\left\{v_n - t_n\left(m_n', m_{-n}\right), 0\right\} dG^{\sigma}\left(m_{-n}\right) = 0$$

Lemma 4: If agent n's type is v_n, he does not strictly prefer to report any $m_n' \in M_n$ such that

i)

$$E_{m_{-n}}^{\sigma}\left(p\left(m_n', m_{-n}\right)\right) > E_{m_{-n}}^{\sigma}\left(p\left(m_n, m_{-n}\right)\right)$$

and

ii) there is no $m_n'' \in M_n$ such that

$$E_{m_{-n}}^{\sigma}\left(p\left(m_n', m_{-n}\right)\right) > E_{m_{-n}}^{\sigma}\left(p\left(m_n'', m_{-n}\right)\right) > E_{m_{-n}}^{\sigma}\left(p\left(m_n, m_{-n}\right)\right)$$

Proof of Lemma 4.

Case 1: $v_n = \inf\left(supp\left(\pi_n^{\sigma}\left(m_n\right)\right)\right)$. I claim that the agent does not deviate to m_n' by the same argument as in Lemma 4. Because $\pi_n^{\sigma}\left(m_n'\right)$ $FOSD$ $\pi_n^{\sigma}\left(m_n\right)$ and (σ, p, t) has property B, if $p\left(m_n', m_{-n}\right) = p\left(m_n, m_{-n}\right) = 1$,

$$\max\left\{v_n - t_n\left(m_n', m_{-n}\right), 0\right\} \leq \max\left\{v_n - t_n\left(m_n, m_{-n}\right), 0\right\}$$

Furthermore, for all m_{-n} such that $p\left(m_n', m_{-n}\right) = 1 > p\left(m_n, m_{-n}\right) = 0$, it must be that $t_n\left(m_n', m_{-n}\right) \geq v_n$ by property A and the fact that $(\rho, t) \in \Psi^{IR}$.

Case 2: $v_n \neq \inf\left(supp\left(\pi_n^{\sigma}\left(m_n\right)\right)\right) \equiv x_n$ and suppose the agent strictly prefers to deviate to m_n'. Let $\widehat{p}_n\left(m, v_n\right)$ and $\widehat{t}_n\left(m, v_n\right)$ be defined as in the text and notice that

$$v_n E_{m_{-n}}^{\sigma}\left(\widehat{p}_n\left(m_n', m_{-n}, v_n\right)\right) - E_{m_{-n}}^{\sigma}\left(\widehat{t}_n\left(m_n', m_{-n}, v_n\right)\right) > v_n E_{m_{-n}}^{\sigma}\left(p\left(m_n, m_{-n}\right)\right) - E_{m_{-n}}^{\sigma}\left(t_n\left(m_n, m_{-n}\right)\right)$$

If $E^{\sigma}_{m_{-n}}\left(\widehat{p}_n\left(m'_n, m_{-n}, v_n\right)\right) \leq E^{\sigma}_{m_{-n}}\left(p\left(m_n, m_{-n}\right)\right)$, then

$$\begin{aligned} & x_n E^{\sigma}_{m_{-n}}\left(\widehat{p}_n\left(m'_n, m_{-n}, v_n\right)\right) - E^{\sigma}_{m_{-n}}\left(\widehat{t}_n\left(m'_n, m_{-n}, v_n\right)\right) \\ > \; & x_n E^{\sigma}_{m_{-n}}\left(p\left(m_n, m_{-n}\right)\right) - E^{\sigma}_{m_{-n}}\left(t_n\left(m_n, m_{-n}\right)\right) \end{aligned}$$

Given that

$$\begin{aligned} & x_n E^{\sigma}_{m_{-n}}\left(\widehat{p}_n\left(m'_n, m_{-n}, v_n\right)\right) - E^{\sigma}_{m_{-n}}\left(\widehat{t}_n\left(m'_n, m_{-n}, v_n\right)\right) \\ \leq \; & x_n E^{\sigma}_{m_{-n}}\left(\widehat{p}_n\left(m'_n, m_{-n}, x_n\right)\right) - \widehat{t}_n E^{\sigma}_{m_{-n}}\left(\widehat{t}_n\left(m'_n, m_{-n}, x_n\right)\right) \end{aligned}$$

it follows that, if the agent's type is $x_n = \inf\left(supp\left(\pi^{\sigma}_n\left(m_n\right)\right)\right)$, he also prefers to report m'_n than to report m_n in Anarchy, which is a contradiction to case 1.

Suppose instead that $E^{\sigma}_{m_{-n}}\left(\widehat{p}_n\left(m'_n, m_{-n}, v_n\right)\right) > E^{\sigma}_{m_{-n}}\left(p\left(m_n, m_{-n}\right)\right)$. Let $\widehat{v}_n \geq v_n$ be such that

$$\widehat{v}_n E^{\sigma}_{m_{-n}}\left(p\left(m_n, m_{-n}\right)\right) - E^{\sigma}_{m_{-n}}\left(t_n\left(m_n, m_{-n}\right)\right) = \widehat{v}_n E^{\sigma}_{m_{-n}}\left(p\left(m'_n, m_{-n}\right)\right) - E^{\sigma}_{m_{-n}}\left(t_n\left(m'_n, m_{-n}\right)\right)$$

If follows that

$$\begin{aligned} & \widehat{v}_n E^{\sigma}_{m_{-n}}\left(\widehat{p}_n\left(m'_n, m_{-n}, v_n\right)\right) - E^{\sigma}_{m_{-n}}\left(\widehat{t}_n\left(m'_n, m_{-n}, v_n\right)\right) \\ > \; & \widehat{v}_n E^{\sigma}_{m_{-n}}\left(p\left(m_n, m_{-n}\right)\right) - E^{\sigma}_{m_{-n}}\left(t_n\left(m_n, m_{-n}\right)\right) \end{aligned}$$

which implies that

$$\begin{aligned} & \widehat{v}_n E^{\sigma}_{m_{-n}}\left(\widehat{p}_n\left(m'_n, m_{-n}, \widehat{v}_n\right)\right) - E^{\sigma}_{m_{-n}}\left(\widehat{t}_n\left(m'_n, m_{-n}, \widehat{v}_n\right)\right) \\ > \; & \widehat{v}_n E^{\sigma}_{m_{-n}}\left(p\left(m_n, m_{-n}\right)\right) - E^{\sigma}_{m_{-n}}\left(t_n\left(m_n, m_{-n}\right)\right) \end{aligned}$$

so that, if the agent's type is $\widehat{v}_n$, he strictly prefers to report m'_n to m_n in Anarchy.

Notice that there must be some $\widetilde{m}_n \in M_n$ such that

$$E^{\sigma}_{m_{-n}}\left(p\left(m'_n, m_{-n}\right)\right) = E^{\sigma}_{m_{-n}}\left(p\left(\widetilde{m}_n, m_{-n}\right)\right)$$

and

$$\inf\left(supp\left(\pi^{\sigma}_n\left(\widetilde{m}_n\right)\right)\right) = \widehat{v}_n$$

To see this, suppose not, i.e. for all $\widetilde{m}_n \in M_n$ such that

$$E^{\sigma}_{m_{-n}}\left(p\left(m'_n, m_{-n}\right)\right) = E^{\sigma}_{m_{-n}}\left(p\left(\widetilde{m}_n, m_{-n}\right)\right)$$

it follows that

$$\inf\left(supp\left(\pi_n^{\sigma}\left(\widetilde{m}_n\right)\right)\right)-\widehat{v}_n>\varepsilon$$

for some $\varepsilon>0$. Let m_n'' be such that $\sigma_n\left(\widehat{v}_n+\varepsilon\right)\left(m_n''\right)>0$. It must be that

$$E_{m_{-n}}^{\sigma}\left(p\left(m_n',m_{-n}\right)\right)>E_{m_{-n}}^{\sigma}\left(p\left(m_n'',m_{-n}\right)\right)>E_{m_{-n}}^{\sigma}\left(p\left(m_n,m_{-n}\right)\right)$$

which is ruled out by ii).

By Lemma 2, if the agent's type is $\widehat{v}_n$, he does not strictly prefer to send m_n' to $\widetilde{m}_n$ in Anarchy. And because

$$t_n\left(\widetilde{m}_n,m_{-n}\right)\leq\widehat{v}_n$$

he does not strictly prefer to send $\widetilde{m}_n$ to m_n in Anarchy. Therefore, if the agent's type is $\widehat{v}_n$, he does not strictly prefer to send m_n' to m_n in Anarchy, which is a contradiction.

7.2 Appendix B

In appendix B, I allow the BD to randomize between her second stage options. After each message $m\in M$, rather than selecting a probability $p\left(m\right)\in\{0,1\}$ and a transfer vector $t\left(m\right)\in\mathbb{R}_+^N$, the BD selects a distribution $\mu\left(m\right)\in\triangle\left(\{0,1\}\times\mathbb{R}_+^N\right)$ such that

$$\int\limits_{(p,t)\in\{0,1\}\times\mathbb{R}_+^N}\mu\left(m\right)\left(p,t\right)d\left(p,t\right)=1$$

For μ to be selectable by a BD, it must be that the equivalent of properties A and B hold, which I denote by properties A' and B' respectively.

Definition 6 *Profile (σ,μ) has property A' if and only if, for all $m\in M$,*

$$\sum_{n=1}^{N}\inf\left[supp\left(\pi_n^{\sigma}\left(m_n\right)\right)\right]>c\Rightarrow p=1$$

and

$$\sum_{n=1}^{N}t_n=\begin{cases}c\ \textit{if}\ p=1\\0\ \textit{otherwise}\end{cases}$$

for all $(p,t)\in\{0,1\}\times\mathbb{R}_+^N$ such that $\mu\left(m\right)\left(p,t\right)>0$.

Definition 7 *Profile* (σ, μ) *has property* B' *if and only if, for all* n*, for all* $m_n, m'_n \in M_n$ *such that*

$$\pi_n^\sigma(m_n) \neq \pi_n^\sigma(m'_n)$$

i)

$$\pi_n^\sigma(m_n) \ \ FOSD \ \pi_n^\sigma(m'_n) \Rightarrow t_n \geq t'_n$$

for any $m_{-n} \in M_{-n}$*, for any* $t_n, t'_n \in \mathbb{R}_+$ *for which there is* (p, t_{-n}) *and* (p', t'_{-n}) *such that*

$$\mu(m_n, m_{-n})(p, t_n, t_{-n}) > 0 \text{ and } \mu(m'_n, m_{-n})(p', t'_n, t'_{-n}) > 0$$

ii) if

$$\exists m'_{-n} \in M_{-n} : t_n > t'_n$$

for some $t_n, t'_n \in \mathbb{R}_+$ *for which there is* (p, t_{-n}) *and* (p', t'_{-n}) *such that*

$$\mu(m_n, m_{-n})(p, t_n, t_{-n}) > 0 \text{ and } \mu(m'_n, m_{-n})(p', t'_n, t'_{-n}) > 0$$

then

$$\widehat{t}_n \geq \widehat{t}'_n$$

for all $\widehat{t}_n, \widehat{t}'_n \in \mathbb{R}_+$ *for which there is* $(\widehat{p}, \widehat{t}_{-n})$ *and* $(\widehat{p}', \widehat{t}'_{-n})$ *such that*

$$\mu(m_n, m_{-n})(\widehat{p}, \widehat{t}_n, \widehat{t}_{-n}) > 0 \text{ and } \mu(m'_n, m_{-n})(\widehat{p}', \widehat{t}'_n, \widehat{t}'_{-n}) > 0$$

These two definitions are extensions of properties A and B and can be interpreted as applying them to each (p, t) which is chosen by the BD with positive probability.

If the BD is able to select random mechanisms, she will be able to implement random allocations. In this context, an allocation is a distribution $\kappa(v) \in \triangle(\{0,1\} \times \mathbb{R}_+^N)$ for each $v \in [\underline{v}, \overline{v}]^N$, such that

$$\int_{(p,t)\in\{0,1\}\times\mathbb{R}_+^N} \kappa(v)(p,t)\, d(p,t) = 1$$

Allocation κ is ex-post individually rational if and only if, for all $v \in [\underline{v}, \overline{v}]^N$ and for any $(p, t) \in \{0,1\} \times \mathbb{R}_+^N$ such that $\kappa(v)(p,t) > 0$,

$$pv_n - t_n \geq 0$$

for all n.

I show that any allocation κ that is implementable by a BD and is ex-post individually rational is also implementable in Anarchy, provided that agents are allowed to randomize between their transfers.[9]

The problem that the BD's randomization creates is that it forces agents to coordinate. Imagine, for example, that, after some message m, the BD randomizes between two transfer vectors t' and t'' such that

$$\sum_{n=1}^{N} t'_n = \sum_{n=1}^{N} t''_n = c$$

and $v_n \geq \max\{t'_n, t''_n\}$ for all n. It is not enough for each agent n to simply randomize between t'_n and t''_n because, in that case, it is possible that, with some probability, the sum of transfers does not equal c. Rather, it is necessary that the agents coordinate on their randomization so that if some agent n' chooses transfer $t'_{n'}$ all the other agents choose t'_n and not t''_n.

There are two steps to the argument. In the first step, I show how the result follows if there is an exogenous public signal, which allows agents to coordinate on their transfer selection. In the second part, I show how it is possible to "create" a public signal by allowing agents to communicate more than just their type in a way similar to Matthews and Postlewaite (1989).

Step 1:

Let $\theta \sim U[0,1]$ be a public signal, which is realized after messages have been sent but before agents choose their transfer. The idea is to use the inverse transform method to show that agents can coordinate their play in Anarchy through the public signal.

First, notice that, if $p = 0$, then $t_n = 0$ for all n whenever $\mu(m)(p,t) > 0$ for any $m \in M$, which means that the BD is essentially randomizing on the transfer vector. Let $\widehat{\mu}(m) \in \triangle(\mathbb{R}^N_+)$ be such that $\widehat{\mu}(m)(t) = \widehat{\mu}(m)(p,t)$ for all $p \in \{0,1\}$.

Let $\prec_l$ represent the lexicographical order so that

$$t' \prec_l t''$$

[9] An allocation is implementable by a BD and in Anarchy in an analogous way to the main text.

if

$$t_1' < t_1'' \vee (t_1' = t_1'' \wedge t_2' < t_2'') \vee (t_1' = t_1'' \wedge t_2' = t_2'' \wedge t_3' < t_3'') \ldots$$

and let $H(m) : \mathbb{R}_+^N \to [0,1]$ be such that

$$H(m)(t') = \int\limits_{t \in \left\{ \left\{ \widehat{t} \in \mathbb{R}_+^N : \widehat{t} \prec t' \right\} \cup \{t'\} \right\}} \widehat{\mu}(m)(t)\, d(t)$$

for any $m \in M$. Finally, let $H^{-1}(m) : [0,1] \to \mathbb{R}_+^N$ be such that

$$H^{-1}(m)(z) = \min \left\{ t \in \mathbb{R}_+^N : H(m)(t) = z \right\}$$

It is possible to implement any allocation κ which is implementable by a BD through (σ, μ) and is ex-post individually rational through the anarchic equilibrium profile (σ, γ), where γ is such that

$$\gamma_n(m, \theta) = \begin{cases} H^{-1}(m)(\theta) \text{ if } H^{-1}(m)(\theta) \leq v_n \\ 0 \text{ otherwise} \end{cases}$$

Notice that $\gamma_n(m, \theta)$ is a best response for agent n, given type v_n, for all $m \in M$ and $\theta \in [0,1]$. Furthermore, if (σ, γ) is indeed an Anarchic equilibrium, it implements κ, because, after each m, each transfer vector is chosen with the same probability in the BD system and in Anarchy. And, finally, no agent misreports in the first stage because the steps of the argument of Appendix A still hold.[10]

Step 2:

It is possible to use the agents' ability to send messages in order to create what essentially becomes a public signal, by using a variation of the method used in Matthews and Postlewaite (1989).

In the anarchic equilibrium profile which implements allocation κ in Anarchy, each agent n sends a two dimensional message. The first element comes from the same set M_n as in the BD system, while the second element is a number $z_n \in [0,1]$. In order to implement κ, agents report the first element as they did in the BD system and randomize uniformly on the second element.

[10] If, for example, a BD randomizes between two positive transfers and chooses transfer t' with probability $\lambda \in [0,1]$ and transfer t'' with probability $1-\lambda$, in what concerns the agents' decisions in the first stage, it is essentially as if a single transfer $t = \lambda t' + (1-\lambda) t''$ is chosen instead, so that the arguments of Appendix A apply.

Given all reports $z = [z_1, ..., z_N]$ there is a function $\widehat{\theta} : [0,1]^N \to [0,1]$, which essentially becomes the public signal of step 1 that agents use to coordinate. I now proceed to define function $\widehat{\theta}$.

Let $\widehat{f} : [0,1]^2 \to [0,1]$ be defined as in Matthews and Postlewaite (1989): $f(z_1, z_2)$ is equal to the number whose binary expansion has a "1" in the nth position if and only if the binary expansion of z_1 has the same digit in nth position as does the binary expansion of z_2. The idea is that the distribution of $\widehat{f}$ is uniform on $[0,1]$ as long as either z_1 or z_2 are also uniform on $[0,1]$ and are independent.

Define $\widehat{g}^2 = f$ and $\widehat{g}^n : [0,1]^n \to [0,1]$ such that

$$\widehat{g}^n(z_1, ..., z_n) = \widehat{f}\left(\widehat{g}^{n-1}(z_1, ..., z_{n-1}), z_n\right)$$

for all $n \in \{3, N\}$. Finally, define $\widehat{\theta} \equiv \widehat{g}^N$. Notice that, if $z_n \sim U(0,1)$ for at least $N-1$ elements and all z_n are independent, then $\widehat{\theta}(z) \sim U(0,1)$.

By construction of $\widehat{\theta}$, no agent can individually influence the distribution of $\widehat{\theta}$, so randomizing over the set $[0,1]$ is a best response for each agent. Therefore, the anarchic equilibrium profile of step 1 is still an anarchic equilibrium in this context (simply by replacing θ for $\widehat{\theta}$ and allowing each agent n to report a number z_n) and implements allocation κ.

7.3 Appendix C

The purpose of this appendix is to demonstrate that the conditions described in the text under which the theorem holds are as "tight" as possible. I divide appendix C into three parts. In each of them, I relax one assumption that was made in the main text, and provide a counterexample to the theorem.

7.3.1 If the agents' types are discrete

Consider an example with three agents. Assume that $v_1 \in \{0.1, 0.3, 0.52\}$, $v_2 \in \{0.409, 0.51\}$, $v_3 \in \{0.3, 0.5\}$, with $c = 1.23$. Consider a BD system where, in equilibrium, agents 2 and 3 report truthfully (send message H_n if their type is the largest and L_n if their type is the smallest, for $n = 1, 2$), while agent 1 reports L_1 if $v_1 \in \{0.1, 0.3\}$ and H_1 if $v_1 = 0.52$.

Let $p(m)$ be given by the following table

$v_3 = H_3$	L_2	H_2
H_1	1	1
L_1	0	0

and

$v_3 = L_3$	L_2	H_2
H_1	0	1
L_1	0	0

so that, for example, $p(H_1, L_2, H_3) = 1$, while

$$t_1(H_1, L_2, H_3) = 0.52,\ t_2(H_1, L_2, H_3) = 0.4,\ t_3(H_1, L_2, H_3) = 0.31$$

$$t_1(H_1, H_2, H_3) = 0.29,\ t_2(H_1, H_2, H_3) = 0.5,\ t_3(H_1, H_2, H_3) = 0.44$$

$$t_1(H_1, H_2, L_3) = 0.52,\ t_2(H_1, H_2, L_3) = 0.41,\ t_3(H_1, H_2, L_3) = 0.3$$

Notice that both properties A and B. Furthermore, the agents' reporting profile is an equilibrium in a BD system if $\Pr\{v_1 = 0.52\} = \Pr\{v_2 = 0.51\} = \Pr\{v_3 = 0.5\} = \frac{1}{2}$ and the allocation which is implemented is ex-post individually rational.

However, in Anarchy, agent 1 has an incentive to deviate on his report if he is of type $v_1 = 0.3$. If he does not deviate, he receives an expected payoff of 0, while a deviation to H_1 in Anarchy gives him an expected payoff of $\frac{0.01}{4} > 0$. This is because, after reporting H_1, the agent only chooses to provide a positive transfer in the event that the other agents play (H_2, H_3), because only then is the transfer (0.29) smaller than the agent's valuation.

The reader might wonder how things would change if, instead of discrete types, we were to have continuous types where all types which are not the ones in the example have a very small density. At first glance, it might appear as though there should be no difference. However, if we were to have a continuum of types, when agent 1's type is $\widehat{v}_1 = 0.44333$, he would be indifferent between the two messages. So if the agent has a larger type he should report H_1; otherwise he should report L_1. However, this would violate the ex-post individual rationality constraint because there would be types (from $\widehat{v}_1$ to 0.52) for which agent 1 would have a strictly negative ex-post payoff on certain circumstances.

7.3.2 If property A does not hold

Consider an example with three agents. Assume that $v_1 \in [0.7, 2]$, $v_2 \in [0.9, 2]$, $v_3 \in [0.2, 2]$ and $c = 3$. Consider a BD system where, in equilibrium, agent 1 reports L_1 if $v_1 \leq 1.1286$, M_1 if $v_1 \in (1.1286, 1.3176]$ and H_1 if $v_1 > 1.3176$; agent 2 sends message

L_2 if $v_2 \leq 1.0667$, M_2 if $v_2 \in (1.0667, 1.5105]$ and H_2 if $v_2 \geq 1.5105$; and agent 3 sends message L_3 if $v_3 \leq 1.314$ and H_3 if $v_3 > 1.314$.

Function p is given by

$m_3 = H_3$	H_2	M_2	L_2
H_1	1	1	1
M_1	1	1	0
L_1	1	0	0

and

$m_3 = L_3$	H_2	M_2	L_2
H_1	1	0	0
M_1	0	0	0
L_1	0	0	0

Function t_1 is given by

$m_3 = H_3$	H_2	M_2	L_2
H_1	0.9	1.3	0.8
M_1	0.9	0.9	0
L_1	0.7	0	0

and

$m_3 = L_3$	H_2	M_2	L_2
H_1	1.3	0	0
M_1	0	0	0
L_1	0	0	0

Function t_2 is given by

$m_3 = H_3$	H_2	M_2	L_2
H_1	1.1	1	0.9
M_1	0.9	0.9	0
L_1	1.3	0	0

and

$m_3 = L_3$	H_2	M_2	L_2
H_1	1.5	0	0
M_1	0	0	0
L_1	0	0	0

Function t_3 is given by

$m_3 = H_3$	H_2	M_2	L_2
H_1	1	0.7	1.3
M_1	1.2	1.2	0
L_1	1	0	0

and

$m_3 = L_3$	H_2	M_2	L_2
H_1	0.2	0	0
M_1	0	0	0
L_1	0	0	0

Notice that property B is satisfied but property A is not (for example, if $m = (L_1, M_2, H_3)$, the BD would be able to fund the public good while making everyone better off and yet $p((L_1, M_2, H_3)) = 0$). Furthermore, the agents' reporting profile is an equilibrium in a BD system if

$$\Pr\{v_1 \leq 1.1286\} = 0.2, \ \Pr\{v_1 \in (1.1286, 1.3176]\} = 0.3, \ \Pr\{v_1 > 1.3176\} = 0.5$$

$$\Pr\{v_2 \leq 1.0667\} = 0.25, \ \Pr\{v_2 \in (1.1286, 1.5105]\} = 0.35, \ \Pr\{v_2 > 1.5105\} = 0.4$$

and

$$\Pr\{v_2 \leq 1.314\} = 0.6,\ \Pr\{v_2 > 1.314\} = 0.4$$

and the allocation which is implemented is ex-post individually rational.

However, in Anarchy, agent 1 has an incentive to deviate on his report if he is of type $v_1 = 1.1286$ (his type is such that, in a BD system, he would be indifferent between reporting L_1 and M_1). By reporting L_1 (by not deviating), the agent receives an expected payoff of

$$0.4 * 0.4 * (1.1286 - 0.7) = 0.0686$$

while, by deviating to H_1, the agent receives an expected payoff of

$$0.4 * 0.4 * (1.1286 - 0.9) + 0.25 * 0.4 * (1.1286 - 0.8) = 0.0694$$

because, if the realized messages are either (H_1, M_2, H_3) or (H_1, H_2, L_3), the agent chooses a transfer that is different than the one the BD would have chosen for him.

7.3.3 If property B does not hold

Consider an example with 2 agents. Assume that $v_n \in [0.3, 2]$ for $n = 1, 2$ and $c = 1$. Consider a BD system where agent n reports L_n if $v_n < x_n$, M_n if $x_n \leq v_n < y_n$ and H_n if $v_n \geq y_n$. Let p be given by the following table:

	H_2	M_2	L_2
H_1	1	1	1
M_1	1	1	0
L_1	1	0	0

while t is given by

	t_1	t_2
H_1H_2	0.75	0.25
H_1M_2	0.7	0.3
H_1L_2	0.7	0.3
M_1H_2	0.4	0.6
M_1M_2	0.5	0.5
L_1H_2	0.3	0.7

Finally, let $x_1 = 0.6$, $y_1 = 1.8$, $x_2 = 0.5$, $y_2 = 1.025$.

Notice that property A is satisfied but property B is not, because $t_2(H_1, H_2) < t_2(H_1, M_2)$. Notice also that the allocation induced in the BD system is ex-post individually rational. For the reporting strategy of the agents to form an equilibrium of the BD system it is enough that $\Pr\{v_1 \geq y_1\} = 0.1$, $\Pr\{v_1 \in [x_1, y_1)\} = 0.7$, $\Pr\{v_2 \geq y_2\} = 0.4$ and $\Pr\{v_1 \in [x_2, y_2)\} = 0.4$.

However, in Anarchy, if agent 2's type is $v_2 = x_2 = 0.5$, a deviation to H_2 would return an payoff of

$$0.1 * 0.25 > 0.1 * 0.2$$

where the RHS is the agent's payoff if he reports M.

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