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Procurement Auction with Strategic Sellers and a Binding Buying Constraint

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PROCUREMENT AUCTION WITH STRATEGIC SELLERS AND A BINDING BUYING CONSTRAINT

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Abstract

The present paper solves a sequential procurement auction in which the auctioneer needs to buy a fixed amount of inventory over a finite time horizon. Bidders strategically submit supply functions which are aggregated by the auctioneer until they reach the required amount. The auctioneer has an outside option which is assumed to be exogenous and highly costly. Results show that the equilibrium price of the second stage is fixed at the price cap and that there are multiple equilibria reports in pure strategies. We study a symmetric case which considers interior solutions only, and a maximal case, which derives a corner solution where the more productive firm monopolizes the industry in the first period. The optimal price posted by the auctioneer in the first stage is higher in the symmetric case than the maximal case as it has more space to manipulate firms' reports. In spite of this, both cases are preferable to a single-period mechanism since the auctioneer manages to save expenses through the purchase of inventory units at a price strictly lower than the price cap.

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1 Introduction

The present paper considers a sequential procurement auction model that seeks to analyze the effect of time and quantity constraints on the auctioneer in the strategic interaction with the bidders, as well as among bidders themselves. There are several situations in which auctions may have time limits, budget limits, and other sorts of agreements, which act as practical time and quantity constraints. This study in particular is focused on the effect of these last two.

Chile is one of many countries that actively publish procurement auctions through organizations such as The Ministry of Public Works (MOP). A good example of a time constraint is the case of an auction for the application of the National Socioeconomic Characterization Survey (Casen) that in 2013 was declared deserted by the Ministry of Social Development due to the lack of suppliers (San Cristóbal, 2013). The urgency of finding a guarantor in charge of carrying out the process was such that the Government enacted a direct agreement for it to proceed, paying 10% more than the initial amount contemplated in the auction (El Dínamo, 2013). The reason for the time limit necessity was because they needed to execute the survey in less than a month since otherwise the data collection would be inconsistent with previous realizations of it. This case shows how the time limit constraint faced by the government led it to enact direct agreements and a higher payment for the service it required.

However, there are other cases in which the auctioneer has more control over the conditions of an auction instead of facing a binding constraint as was seen previously for the case of Chile. One example is the case of electricity auctions in Brazil, where there are two separate environments for electricity trading: a regulated and a free contracting environment. The existence of these two environments provides the government with the option of managing its expenses by buying less electricity whenever prices are considered high (Rego and Parente (2013)).

Moreover, there are also empirical studies that share the idea of a dynamic auction that will be used in the present paper. Ji and Li (2008) analyze multi-round auctions with secret and fixed reserve prices based on a highway construction contract. Ji and Li (2008) study how the government holds an auction over time waiting for offers to get lower. This idea of strategically holding the auction over time is going to be important for the purposes of the present paper, as it studies if bidders hold up the auction offers through time in order to pressure the government and make it raise its price offers due to its time and capacity constraints.

We aim to analyze a sequential procurement auction with the distinct property that the auctioneer is forced to buy a fixed amount of inventory (also referred to as “target inventory”) during a fixed period of time. Bidders produce the inventory demanded through a private cost function that depends on a parameter which reflects how efficiently they produce. The game consists of two stages. In the first stage, the auctioneer first posts a uniform price, to which bidders respond through independent supply function submissions. In the second stage, bidders submit first and then the auctioneer chooses the unit price such that it manages to buy the remaining quantity needed to reach the target inventory. Intuitively, both time and inventory constraints feed back to the previous stage of the auction, encouraging sellers to refrain from offering high efficiency parameters at the beginning, in order to force the auctioneer to buy at a higher price in the second stage. Conversely, the auctioneer is interested in designing a way to bind its expenses and prevent the above from happening. In this respect, one aspect of this problem that is worth noting is the disadvantageous situation in which the auctioneer is involved.

2 Literature

Our analysis is mainly based on the literature of procurement auctions. However, there is a considerable amount of literature which also contributes useful insights for the analysis that will be carried out next. Among these, Su (2007) develops a model composed of a monopoly which offers a finite amount of inventory over a finite time period, and adjusts prices dynamically in pursuance of maximizing profits. Unsold units of inventory have zero value once the time for operation is fulfilled. This description has a lot in common with the topic presented in this paper. In both cases there is a pressure of selling (buying) a fixed amount. In particular, both share the main purpose of designing a price schedule that would benefit the monopolist the most, which in this case corresponds to the auctioneer. Nevertheless, in Su (2007) customers arrive continuously during the selling season and are heterogeneous regarding two dimensions: valuation for the product and level of patience. Here, the number of bidders in the model is constant between the two rounds of the auction, and they differentiate each other only in one dimension: cost efficiency. Similarly to Su (2007), Chen and Zhang (2009) develop a model about dynamic pricing facing strategic customers, but diverge from the purposes of this paper due to the use of price discrimination, since our model works with uniform prices.

The problem introduced in this paper resembles the “use it or lose it” incentive on public agencies. Karnani (2017) studies the strategic behavior of firms in procurement auctions during the end of the year period (December), finding that the environment becomes less competitive and that price offers are closer to the price cap. Interestingly, these results are parallel to the ones presented here, albeit derived from different sources. In Karnani (2017), differences in capacity constraints explain the rise of price offers, as the possibility that large firms would become single bidders increases as their competitors exhaust their capacity (marginal cost increase from zero to infinite). However, our model presents smoother asymmetries regarding differences on the slopes of firms’ supply functions, which are determined by their productive parameters. Here, the fact that price offers reach the price cap is an equilibrium result induced by the auctioneer’s inelastic demand.

Regarding procurement auction literature, Jofre-Bonet and Pesendorfer (2003) model a repeated auction game using bid data of highway contracts. They study whether capacity constraints, understood as the amount of uncompleted contracts each bidder has from auctions won in the past, can affect their ability to win in present and future auctions. However, this case departs from the objective of the present article because the restriction is on bidders. Instead, in this paper the restriction is on the amount the auctioneer needs to buy, and it is assumed that bidders do not have capacity restrictions that would give them more or less advantage against others, except for their cost differences, which will be reflected on their supply submissions. Furthermore, Jofre-Bonet and Pesendorfer (2003) present a dynamic model where a new auction offer is released in each period, whereas the present work analyzes one single auction in two periods of time. Another discrepancy worth mentioning is the methodology used. Jofre-Bonet and Pesendorfer (2003), as well as Ji and Li (2008) base their bidding models on the framework of Markov dynamic decision processes. Yet, the present paper will follow the Supply Function Equilibrium (SFE) concept, which was started by Klemperer and Meyer (1989). Holmberg (2008) applies this last technique in the context of electricity auctions, in the interest of finding conditions under which a unique and symmetric SFE exists. He considers a market with symmetric producers, a price cap, perfect inelastic demand, and a capacity constraint that binds producers with positive probability. In addition to sharing the methodology with the game presented here, Holmberg (2008) also shares the condition that bidders behave symmetrically, but most importantly for the purposes of this article, a perfect inelastic demand. In spite of this, Holmberg's main focus is the capacity constraint faced by bidders. Thus, the present paper makes further contributions by extending the game to a two-stage auction, shifting the attention from the capacity constraint of bidders to the purchase constraint faced by the auctioneer. Another paper that studies bidding behavior in electricity auctions is Hortaçsu and Puller (2008), but like Holmberg (2008), it evaluates a one-stage auction, and most importantly, the auctioneer is unrestricted on how much energy has to be transacted. The same can be seen in Wolfram (1997), but the latter also considers theoretical insights concerning multi-unit auctions, which is a common feature with the problem presented here.

As was mentioned in the previous section, the purpose of the present article is to model a competitive, multi-unit, two-staged auction with a binding buying constraint. The key feature is the fact that bidders are aware of the urgency in which the auctioneer is implicated, which changes the nature of the problem in the way bidders respond. Essentially, this study contributes beyond usual auction models where transactions are defined by prices and quantities. In this case quantity is fixed, and the price is forced to be such that the target quantity is achieved. Thus, our purpose is to evaluate how bidders strategically interact when this constraint is binding, and how the auctioneer could buy its target inventory in the most convenient terms while preventing the price from becoming too high. In this sense, this work takes parts of existing literature and builds upon them by adding this buying restriction, changing the focus from constraints applied on bidders to a constraint applied on the auctioneer. In particular, it uses SPE as a modeling technique, as it has been used in works like Vives (2011) and in particular Laplace (2016). The present paper goes in the spirit of Su (2007) by restricting the seller, but in a context of multi-round auctions as treated, for instance, in Lu and Tong (2008). To the best of our knowledge, no research has been done regarding this particular situation in which the auctioneer has a deadline and inventory to comply with.

3 The model

Consider one auctioneer that must buy $\bar{Q}$ units of inventory over two periods of time, $t \in \{1, 2\}$. Both the target inventory and time horizon are public information. There are N bidders (firms), denoted by $i = 1, ..., N$, that only produce the good demanded by the auctioneer with an increasing and convex cost function given by $C_{it}(Q_{it}) = \frac{Q_{it}^2}{2\delta_i}$, with $\frac{\partial C_{it}}{\partial \delta_i} < 0$. The parameter $\delta_i \in [0, +\infty]$ captures bidders' productive efficiency, which is unknown to the auctioneer but known between firms and constant over time[1]. Appendix B provides an abbreviate introduction of the model when these parameters are private information. For simplicity, we assume that both the bidders and the auctioneer have a discount factor equal to one. Every period, a procurement auction is carried out where each firm decides on how to report their productive efficiency through a supply function submission. Instead, the auctioneer chooses a price for each period in the interest of minimizing the expenditure needed to acquire the target inventory. In the first period, the auctioneer begins by announcing a uniform price, to which bidders respond by submitting their respective submissions. In the second period, the price is determined endogenously: now firms play first, and the auctioneer adjusts the price until the sum of submitted supplies in that period is equal to the remaining units of inventory required to fulfill the target inventory. This adjustment on price is anticipated by bidders, since the auctioneer is committed *ex-ante* to do so. Formally, in the first period the auctioneer announces P_1 to which firms respond by submitting independent supply functions (i.e. marginal cost functions) denoted by:

$$\widetilde{C'_{i1}}(Q_{i1}) = \frac{Q_{i1}}{\beta_{i1}}$$

Note that the reports are not necessarily honest, which means that $\beta_{it} \in [0, +\infty]$ may not be δ_i. Each firm will sell the amount that result from equating the posted price with their supply function:

$$P_1 = \frac{Q_{i1}}{\beta_{i1}} \Rightarrow Q_{i1} = P_1 \beta_{i1}$$

As the amount sold in the second period comes from the same procedure, we can generalize the last equation:

$$Q_{it} = P_t \beta_{it} \tag{1}$$

[1]Firms compete under complete information but the auctioneer is uninformed regarding firms' productivities. Alternatively, the auctioneer could be informed about firms' productivities but would not be able to contract upon their cost structure.

The total amount bought in the first period corresponds to $Q_1 = \sum_{i=1}^{N} Q_{i1}$. In the second period, the auctioneer is committed to raise the price until the total amount purchased in both periods sum $\bar{Q}$. Considering (1), this commitment can be appreciated in the following condition:

$$Q^R \equiv Q_2 = \bar{Q} - Q_1 \quad \Leftrightarrow \quad Q^R = \bar{Q} - P_1(\beta_{11} + \beta_{21}) \tag{2}$$

We define Q^R as the residual demand at $t = 2$ to simplify notation. Since $Q_2 = \sum_{i=1}^{N} \beta_{i2} P_2$, we can express P_2 as:

$$P_2 = \frac{Q^R}{\sum_{i=1}^{N} \beta_{i2}} \tag{3}$$

Notice that the auctioneer cannot surprise bidders on the second period since the price it could choose is limited by the constraint of buying $\bar{Q}$ before that period ends. Thus, although setting higher prices on the first period may seem attractive for the auctioneer, as it may increase sales and reduce Q^R, it reduces the auctioneer's marginal income by lowering the unit price margin when the prices of each period are more similar with each other. In addition, we assume that bidders compete symmetrically in equilibrium and that the auctioneer has an outside option of buying inventory units at cost K. This prevents the price from exploding and making the inventory unaffordable[2]. Throughout this paper, we assume that bidders are efficient enough to be capable of serving the entire market even when the amount of inventory required by the auctioneer is high. Since firms have an increasing cost function in the production of inventory units, this assumption is not negligible. However, it does not make the price cap unimportant. Essentially, K is sufficiently high so that the outside option is never taken on the equilibrium path, and also plays the role of a threat that prevents the price from becoming undefined. For this to be true we need a parametric condition.

Assumption 3.1 *Independent of how large the target inventory $\bar{Q}$ is, the marginal cost of the least productive firm of producing an extra unit of inventory would always be under the price cap K:*

$$K > \frac{\bar{Q}}{2\delta_i}, \quad \text{if } \delta_i \leqslant \delta_{-i} \quad \forall -i \in N$$

Note this assumption asserts that firms will always be more productive than the outside option, which is the most interesting case because otherwise the auctioneer would end up paying K[3]. We will start analyzing the cases of whether the price cap is restrictive or not restrictive (i.e. the threat of the outside option is non credible) separately, due to the interesting intuitions that each provide.

The paper proceeds as follows: Section 4 solves the sequential auction by backward induction. For this,

[2] Alternatively referred to as the social cost of not serving part of the residual demand.

[3] If firms were inefficient, the auctioneer would appeal to its alternative option. For the case of two firms, if only one them was efficient, it would charge K for the remaining units that the unproductive firm was unable to offer.

two types of equilibrium are selected as solutions of the second period: a symmetric and a maximal case, where the latter admits an analytic and a numerical solution. Section 5 discusses important insights derived from results obtained in the previous section, while section 6 compares the model presented in this work with a single period mechanism. This comparison serves the purpose of evaluating how efficient our auction model is compared to the single-period one in terms of the costs it has for the auctioneer. Section 7 concludes.

4 Equilibrium

4.1 Second period with unrestricted price cap ($K = \infty$)

In this case, the price cap K is equal to infinity, which means that the price in the second period is unbounded. We will start by analyzing the firms' profit maximization problem. Since P_2 is not set, and will be chosen to meet $\bar{Q}$, firms can manipulate its determination through their reports. Under strategic behavior, each firm solves:

$$\max_{\beta_{i2}} \pi_{i2} = P_2 Q_{i2} - \frac{Q_{i2}^2}{2\delta_i} \qquad s.t. \quad P_2 = \frac{Q^R}{\sum_{i=1}^N \beta_{i2}}$$

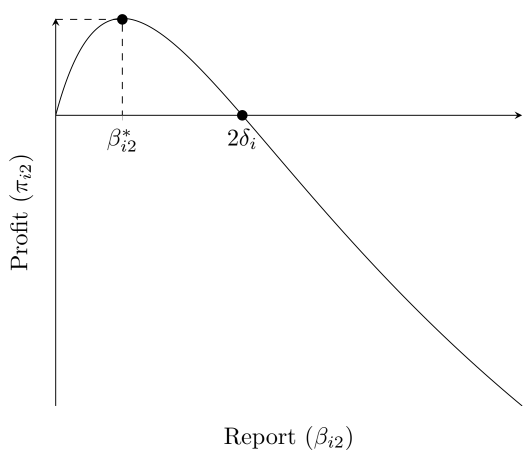
This can be rewritten as:

$$\max_{\beta_{i2}} \left[\frac{Q^R}{\beta_{i2} + \sum_{j \neq i} \beta_{j2}} \right]^2 \left(\beta_{i2} - \frac{\beta_{i2}^2}{2\delta_i} \right)$$

Lemma 4.1.1 *The solution for the firm's profit maximization problem will be restricted to positive profit values delimited by* $\beta_{i2} \in [0, 2\delta_i]$.

Proof. See Appendix A

Figure 1: Firm Profit



Lemma 4.1.2 *First order conditions lead to a symmetric best response function given by:*

$$\beta_{i2} = \frac{\delta_i \sum_{j \neq i} \beta_{j2}}{\delta_i + \sum_{j \neq i} \beta_{j2}}$$

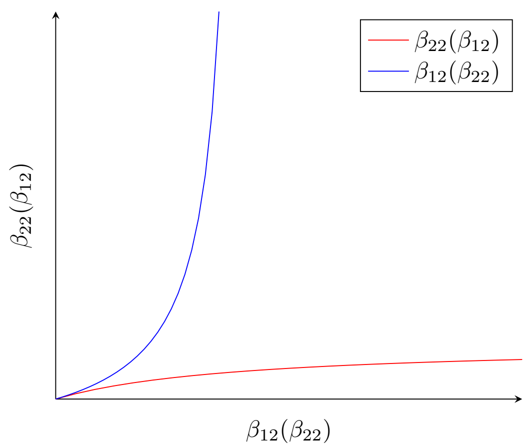
Proof. See Appendix A

Let $B = \sum_{j \neq i} \beta_{j2}$ to simplify notation. It can be seen that reports are strategic complements since β_{i2} is increasing in β_{j2} for any $j \neq i$[4]. The reason for the latter is as follows: From equation (3), P_2 is a decreasing and convex function of bidders' reports[5]. Thus, in the margin, as B is higher, variations of β_{i2} have little impact on P_2, whereas for lower values of B, β_{i2} has more impact on P_2, which means more space to manipulate P_2. Therefore, for higher values of B, there are more incentives to report higher values of β_{i2} and to compete as aggressively as the rest of the bidders by offering larger amounts of inventory to the auctioneer[6], since lower values of β_{i2} would barely increase the price. On the contrary, because P_2 is sensible to changes in individual reports, for lower values of B the incentive of bidding lower values of β_{i2} to make P_2 rise overpowers the incentive of increasing β_{i2} to offer larger amounts of inventory to the auctioneer.

Proposition 4.1.1 *For the case of two players, the firms' optimal report is $\beta_{i2} = 0$, which is unfeasible because it leads to an undefined P_2. Therefore, there is no equilibrium under strategic behavior.*

Proof. See Appendix A

Figure 2: Best responses



Proposition 4.1.1 predicts a strong result in the sense that even though firms cannot report zero productive

[4] Since $\frac{\partial \beta_{i2}}{\partial B} = \frac{\delta_i^2}{(\delta_i + B)^2} > 0$.

[5] Note $\frac{\partial P_2}{\partial B} = \frac{-Q^R}{(\beta_{i2} + B)^2} < 0$, $\frac{\partial^2 P_2}{\partial B^2} = \frac{2Q^R}{(\beta_{i2} + B)^3} > 0$.

[6] Recall $Q_{i2} = \beta_{i2} P_2$.

efficiency, they will submit slight values of it because the temptation of entangling the auctioneer is too high. Interestingly, without collusive behavior, all firms report minimum efficiency parameters in order to raise the unit price they are paid for their offers. Figure 2 shows best responses functions for $N = 2$, where it can be seen that they only coincide at the origin.

4.2 Second period with restrictive price cap ($K < \infty$)

Now we analyze the case where the price cap K is relevant, as it binds the price of the second period and prevents it from exploding. Essentially, the auctioneer adds the supply submissions made by the bidders at $t = 2$ until they reach the auctioneer's desired amount, Q^R. At this point, if the cleared unit price is below K, the auctioneer pays the bidders said price times the amount each bidder offered at that price. Yet, if the cleared price is above K, then the auctioneer would only buy the amount offered up to price K. Hence, now it is possible that the auctioneer may buy less than Q^R. In this respect, firms maximize the following:

$$\max_{\beta_{i2}} \left[\min \left\{ \frac{Q^R}{\beta_{i2} + B}, K \right\} \right]^2 \left(\beta_{i2} - \frac{\beta_{i2}^2}{2\delta_i} \right) \tag{4}$$

Lemma 4.2.1 *The relevant support of firm reports considered in their profit maximization problem is* $\beta_{i2} \in [0, \delta_i]$.

Proof. See Appendix A

We will now begin to analyze the extreme cases. The first one occurs when B is sufficiently small, such that even if the firm reports the highest reasonable value, $\beta_{i2} = \delta_i$, the price would still be above K. In other words, the firm faces a constant price K independent of its report. This can be expressed by the following condition:

$$\frac{Q^R}{\delta_i + B} \geqslant K \quad \Leftrightarrow \quad B \leqslant \frac{Q^R - K\delta_i}{K} \tag{5}$$

Therefore, when (5) holds, (4) becomes the maximization of the second term only, which is a parabola maximized at $\beta_{i2} = \delta_i$. Thus, firms report truthfully. The second extreme case occurs when B is sufficiently high such that even if the firm reports the lowest feasible value, $\beta_{i2} = 0$, the price would still be below K. This can be expressed by:

$$\frac{Q^R}{B} \leqslant K \quad \Leftrightarrow \quad B \geqslant \frac{Q^R}{K} \tag{6}$$

Since the firm faces an endogenous price, the price cap K becomes irrelevant and therefore (4) is the same as the problem of the previous section. As a result, conditional on (6), the optimal report is $\beta_{i2}(B) = \frac{\delta_i B}{\delta_i + B}$.

Lemma 4.2.2 *When the aggregate reports from other firms are $B \leqslant \frac{Q^R - K\delta_i}{K}$, the optimal report of firm i at period $t = 2$ is $\beta_{i2}(B) = \delta_i$, whereas when the aggregate reports from other firms are $B \geqslant \frac{Q^R}{K}$, the optimal report of firm i at period $t = 2$ is $\beta_{i2}(B) = \frac{\delta_i B}{\delta_i + B}$.*

Now that we know each firm's optimal response to both extreme cases, we aim to understand how they respond when $B \in \left[\frac{Q^R - K\delta_i}{K}, \frac{Q^R}{K}\right]$. To do this, we must calculate a threshold on reports to distinguish which of the two prices minimized in (4) the firm is facing. In this context, the endogenous price would be lower than the price cap if:

$$\beta_{i2} \geqslant \frac{Q^R - BK}{K}$$

This can be used to rewrite the firm's maximization problem as:

$$\max_{\beta_{i2} \in [0, \delta_i]} \left\{ \max_{\beta_{i2} \geqslant \frac{Q^R - BK}{K}} \left(\frac{Q^R}{\beta_{i2} + B} \right)^2 \left(\beta_{i2} - \frac{\beta_{i2}^2}{2\delta_i} \right), \max_{\beta_{i2} \leqslant \frac{Q^R - BK}{K}} K^2 \left(\beta_{i2} - \frac{\beta_{i2}^2}{2\delta_i} \right) \right\}$$

Lemma 4.2.3 *The optimal report for firm i at period $t = 2$, $\beta_{i2}(B)$, when the aggregate reports by other firms is $B \in \left[\frac{Q^R - K\delta_i}{K}, \frac{Q^R}{K}\right]$, is given by:*

$$\beta_{i2}^*(B) = \max_{\left\{\delta_i \geqslant \beta_{i2} \geqslant \frac{Q^R - BK}{K}\right\}} \left\{ \left(\frac{\delta_i B}{\delta_i + B} \right), \left(\frac{Q^R - BK}{K} \right) \right\}$$

Proof. See Appendix A

Lemma 4.2.3 establishes that optimal reports correspond to the maximum of two functions that depend on the joint reports of the other firms, where one of the functions is increasing and the other decreasing.

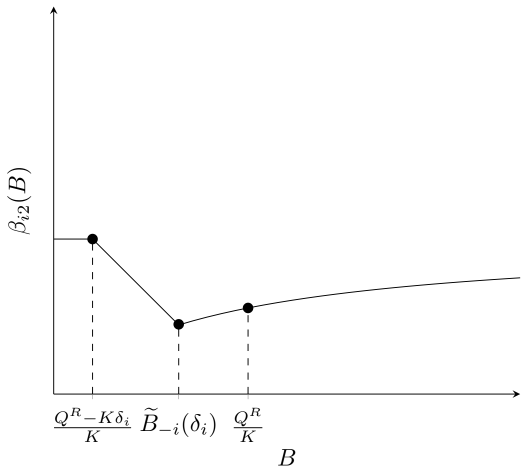
Proposition 4.2.1 *The best response function of firm i in period $t = 2$ in response to the aggregate reports by other firms, $\beta_{i2}(B)$, can be separated in three sections:*

$$\beta_{i2}(B) = \begin{cases} \delta_i, & \text{if } B \leqslant \frac{Q^R - K\delta_i}{K} \\ \frac{Q^R - BK}{K}, & \text{if } \frac{Q^R - K\delta_i}{K} \leqslant B \leqslant \widetilde{B}_{-i}(\delta_i) \\ \frac{\delta_i B}{\delta_i + B}, & \text{if } \widetilde{B}_{-i}(\delta_i) \leqslant B \end{cases}$$

Proof. See Appendix A

Figure 3 illustrates three segments for the best response function $\beta_{i2}(B)$. The first one corresponds to low values of B, specifically to $[0, \frac{Q^R - K\delta_i}{K}]$. As was mentioned before, in this zone, a firm's optimal choice is

Figure 3: Best responses



to report truthfully at $\beta_{i2}(B) = \delta_i$, since it faces a fixed price K regardless of its submission. Interestingly, the case with a restrictive price cap diverges from the case with an un-restrictive price cap since in the former, as the firm is unable to manipulate the price, reducing its report is meaningless because it would not make the price go higher, whereas in the latter, lower values of B imply lower reports, making P_2 become undefined in the extreme. Moreover, from (1) we know that lower reports signify lower amounts of inventory offered to the auctioneer, reinforcing the argument that reporting a productive efficiency less than δ_i is never preferred because it leads to less units sold at the price cap K. Thus, when all other firms are reporting low productivity parameters (i.e. high costs of production), firm i responds by offering as much as possible, taking advantage of the high price K.

Intermediate values of B are divided into two main zones: the second and the third sections of the best response function. While in both sections the firm is able to manipulate the price through its report, there is a contrast between them because reports change from being strategic substitutes to strategic complements. The point when this change occurs is defined as $\widetilde{B}_{-i}(\delta_i) \in B$ and it is denoted by[7]:

$$\widetilde{B}_{-i}(\delta_i) = \frac{(Q^R - 2K\delta_i) + \sqrt{4(\delta_i K)^2 + (Q^R)^2}}{2K} \tag{7}$$

Let us start by analyzing the second section of the best response function. Here, reports are strategic substitutes since the optimal report is decreasing in β_{2j} for any $j \neq i$. In particular, as the rest of the firms report higher values of productivity parameters, firm i is willing to report lower values of productive efficiency in the interest of maintaining a high price, since it is aware of the ability to manipulate prices in this segment. Nevertheless, such interaction ends when the aggregate reports of the rest of the firms is too high (larger or equal to $\widetilde{B}_{-i}(\delta_i)$) for the firm i to find it optimal to maintain a high price even when it is possible. This is the

[7]See proof of Proposition 4.2.1 in Appendix A.

third segment of the best response function. Despite the ability of affecting the price, the incentive to offer a larger amount of inventory dominates the incentive of raising P_2. Thus, reports are strategic complements since higher values of B lead to higher reports from firm i.

Furthermore, it is worth mentioning that for values of B larger than or equal to $\frac{Q^R}{K}$, the firm is incapable of affecting the price. This occurs because the rest of the reports are so high that even if firm i reports its lowest productivity, the price would still be under K. In contrast to the previous section, where $K = \infty$ and reports are always increasing on the aggregate reports from other firms, under $K < \infty$, the optimal response can either be constant, decreasing or increasing throughout B's domain.

Proposition 4.2.2 *In the second period, for the case of two players, there are multiple equilibria in pure strategies given by:*

$$\forall i \in N, \quad \beta_{i2}(\beta_{j2}) \in \Big[\max\Big\{\frac{Q^R - \delta_j K}{K}, \frac{Q^R - \widetilde{B}_j(\delta_i)K}{K}\Big\}, \widetilde{B}_i(\delta_j)\Big]$$

$$\beta_{j2}(\beta_{i2}) = \frac{Q^R - \beta_{i2}K}{K}$$

Proof. From Proposition 4.2.1, we know that for certain values of aggregate reports from other firms, the best response function of an arbitrary firm i is denoted by $\frac{Q^R - BK}{K}$. Since in this case there are only two firms, the aggregate report faced by a firm is just the report of the other firm. Let $i = 1$ and $j = 2$. We start with the conjecture that firm 2 is playing according to its decreasing section of its best response function:

$$\beta_{22}(\beta_{12}) = \frac{Q^R - \beta_{12}K}{K} \tag{8}$$

Clearing β_{12}, we derive the analogous expression:

$$\beta_{12}(\beta_{22}) = \frac{Q^R - \beta_{22}K}{K}$$

We note that if we clear β_{12} from (8), we derive the analogous expression but with respect to firm 1. Thus, as long as β_{12} and β_{22} are located in the domain's section that leads the opposing firm to answer according to its decreasing section, the strategies played by each firm are mutual best responses. In other words, if firm 2 is playing its best response to firm 1, it automatically means that firm 1 is also playing its best response to firm 2 therefore both firm's best responses are mutually contestable. For (8) to hold, there are two conditions that must be satisfied:

(i) $\frac{Q^R - \delta_2 K}{K} \leqslant \beta_{12} \leqslant \widetilde{B}_1(\delta_2)$

(ii) $\frac{Q^R-\delta_1 K}{K} \leqslant \frac{Q^R-\beta_{12}K}{K} \leqslant \widetilde{B}_2(\delta_1)$

Condition (i) is a direct inference from Proposition 4.2.1 and defines the relevant domain of firm 1's report by denoting between which values it must be located in order that firm 2 responds in accordance with its decreasing section. Condition (ii) shows that for (8) to be an equilibrium, firm 2's best response must be located in the section where the optimal response of firm 1 is $\beta_{12}(\beta_{22}) = \frac{Q^R-\beta_{22}K}{K}$, which happens when $\beta_{22}(\beta_{12})$ is contained in $[\frac{Q^R-\delta_1 K}{K}, \widetilde{B}_2(\delta_1)]$. Additionally, the left hand side inequality of condition (ii) leads to $\beta_{12} \leqslant \delta_1$, which is true as seen in Lemma 4.2.1. The right hand side inequality of condition (ii) leads to $\beta_{12} \geq \frac{Q^R-\widetilde{B}_2(\delta_1)K}{K}$.

To make sure that β_1 is located in the section of interest, condition (i) and condition (ii) can be written as:

$$\max\left[\frac{Q^R-\delta_2 K}{K}, \frac{Q^R-\widetilde{B}_2(\delta_1)K}{K}\right] \leqslant \beta_{12} \leqslant \widetilde{B}_1(\delta_2)$$

It can be shown that there are feasible values of β_{12} that satisfy these conditions, since a space between their lower and upper bound exists. That is, if we compare both $\frac{Q^R-\delta_2 K}{K}$ and $\frac{Q^R-\widetilde{B}_2(\delta_1)K}{K}$ with $\widetilde{B}_1(\delta_2)$:

$$\frac{Q^R-\delta_2 K}{K} \lesseqgtr \widetilde{B}_1(\delta_2) \quad \Leftrightarrow \quad \frac{Q^R-\delta_2 K}{K} \lesseqgtr \frac{(Q^R-2K\delta_2)+\sqrt{4(\delta_2 K)^2+(Q^R)^2}}{2K} \quad \Leftrightarrow \quad 0 \lesseqgtr 4(K\delta_2)^2$$

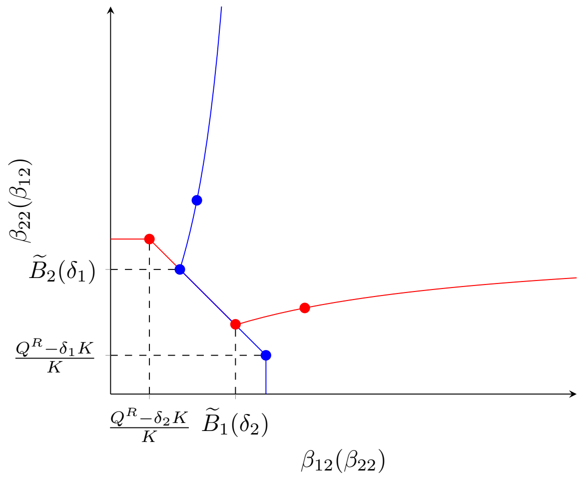
Therefore $\frac{Q^R-\delta_2 K}{K} < \widetilde{B}_1(\delta_2)$,

$$\frac{Q^R-\widetilde{B}_2(\delta_1)K}{K} \lesseqgtr \widetilde{B}_1(\delta_2) \quad \Leftrightarrow \quad \frac{Q^R}{K} \lesseqgtr (\widetilde{B}_2(\delta_1)+\widetilde{B}_1(\delta_2))$$
$$\Leftrightarrow \quad \frac{Q^R}{K} \lesseqgtr \frac{Q^R-2K\delta_2+Q^R-2K\delta_1+\sqrt{4(\delta_2 K)^2+(Q^R)^2}+\sqrt{4(\delta_1 K)^2+(Q^R)^2}}{2K} \quad \Leftrightarrow \quad 0 \lesseqgtr (\delta_1+\delta_2)^2$$

Therefore $\frac{Q^R-\widetilde{B}_2(\delta_1)K}{K} < \widetilde{B}_1(\delta_2)$. □

Figure 4 depicts the best response functions for firms 1 and 2. It can be seen that both intersect in the section in which reports are strategic substitutes, which gives delimited supports for both β_{12} and β_{22} such that each firm's best response corresponds to the decreasing section of its respective best response function. Through all this intersection, there are multiple asymmetric equilibria, as well as a symmetric one given by $\beta_{12} = \beta_{22} = \frac{Q^R}{2K}$.

Figure 4: Best responses ($N = 2$)



Proposition 4.2.3 *For N firms, there are multiple equilibria in pure strategies given by:*

$$\begin{aligned}
&\text{If} \quad \forall i,\\
&(i) \quad \sum_{j\neq i}\beta_j \leqslant \widetilde{B}_{-i}(\delta_i),\\
&(ii) \quad \sum_{j\neq i}\beta_j \geqslant \frac{Q^R - K\delta_i}{K},\\
&(iii) \quad \sum_{j\neq i}\beta_j \geqslant (N-1)\frac{Q^R}{K} - \sum_{j\neq i}\widetilde{B}_i(\delta_{-i}),\\
&\text{Therefore,} \quad \beta_i\left(\sum_{j\neq i}\beta_j\right) = \frac{Q^R - K\sum_{j\neq i}\beta_j}{K}
\end{aligned}$$

Proof. See Appendix A

Proposition 4.2.3 generalizes Proposition 4.2.2. Condition (i) and (ii) assert that for a given firm i, aggregate reports of the rest of the firms must be between $\frac{Q^R-K\delta_i}{K}$ and $\widetilde{B}_{-i}(\delta_i)$ for firm i to optimally respond with $\frac{Q^R-BK}{K}$. This must be true for every $i \in N$. Therefore, all possible combinations of aggregate reports must be contained in the interval of interest. Condition (iii) enable us to assume that all firms are indeed optimally reporting in the decreasing section of their respective best response function.

Until now, we have learned that in the second period the auctioneer ends up paying K per unit of inventory. Though the price cap is intended to play the role of a threat to prevent the price from being too high, it ultimately becomes the cleared price from firms' submissions[8]. This result is unusual because, in

[8] The price of the second period is cleared as $P_2 = \frac{Q^R}{\sum \beta_{i2}}$. Let $N = 2$. We know that in the equilibrium path, $\beta_{22} =$

a Bertrand world of two firms competing on prices, the cleared price is the lowest marginal cost between both firms. In fact, the second period of our model can be observed as a competition on prices since P_2 is determined endogenously through firm's reports, while the amount is fixed at Q^R. Because firms compete on who sells larger amounts of inventory units, we should expect them to report their truthful efficiency parameters so that they will be as productive as possible in our model. This would make their supply functions flatter and therefore the cleared price would be strictly lower than K. Surprisingly, however, the result is the opposite: firms manage to optimally report in such a way that the auctioneer pays the same price it would have paid for its outside option, even though taking it is not optimal[9]. It is interesting how the end result is an equilibrium behavior, despite the impression that firms have colluded.

In the next section we will study the first stage of the auction with two players and a restrictive price cap. Because the second stage of the auction is characterized by a zone of multiple equilibria in pure strategies, we select two equilibria of this zone as solutions of the second period so the auction design can be solved accordingly. In particular, we consider a symmetric and a maximal case.

4.3 First period with restrictive price cap ($K < \infty$): Symmetric case

Consider the symmetric case where firms report the same productive efficiency at $t = 2$. From Proposition 4.2.2, we know that the best response functions of both firms correspond to their decreasing section, $\beta_{i2} = \frac{Q^R - K\beta_{j2}}{K}$ for $i \neq j \in N$. Therefore, if $\beta_{21}(\beta_{12}) = \beta_{12}(\beta_{21}) = \beta$:

$$\beta = \frac{Q^R}{2K}$$

Note that these reports are strictly positive and that, even if reports of each firm in the first period differ, firms still report equally in the second period[10]. Moreover, this case is valid because it is contained in the multiple equilibria zone defined by the interval of optimal reports.

Lemma 4.3.1 *In the symmetric case, given that the optimal report of firm 2 in the second period is $\beta_{22} = \frac{Q^R}{2K}$, the optimal report of firm 1 in the second period, $\beta_{12} = \frac{Q^R}{2K}$, must belong to* $\left[\max\left\{\frac{Q^R - \delta_2 K}{K}, \frac{Q^R - \widetilde{B}_2(\delta_1)K}{K}\right\}, \widetilde{B}_1(\delta_2)\right]$

Proof. See Appendix A

In the first period, firms use backward induction to anticipate their profits from the second period in order to solve their optimal reports in the first period. Since the equilibrium price of the second period corresponds to $P_2 = K$, each firm will maximize their total profits, which corresponds to the sum of profits over the two

$\frac{Q^R - K\beta_{12}}{K} \Leftrightarrow (\beta_{22} + \beta_{12}) = \frac{Q^R}{K}$. Therefore, in this case $P_2 = \frac{Q^R}{\frac{Q^R}{K}} \equiv K$. This also holds for N firms from Proposition 4.2.3.

[9] Since K is too high in comparison to the firm's marginal cost of producing inventory units (Assumption 3.1).

[10] By comparing $\frac{Q^R}{2K} \lesseqgtr 0 \Leftrightarrow Q^R > 0$, otherwise the second period would not be carried out.

periods:

$$\pi_i(\beta_{i1}) = P_1^2\left(\beta_{i1} - \frac{(\beta_{i1})^2}{2\delta_i}\right) + K^2\left[\frac{Q^R}{2K} - \left(\frac{Q^R}{2K}\right)^2\frac{1}{2\delta_i}\right] \tag{9}$$

Note that the residual demand depends on the reports given by both firms on the first period. Considering (2), we can rewrite (9) as:

$$\max_{\beta_{i1}} \pi_i(\beta_{i1}) = P_1^2\left(\beta_{i1} - \frac{(\beta_{i1})^2}{2\delta_i}\right) + K^2\left[\frac{\bar{Q} - P_1\beta_{i1} - P_1\beta_{j1}}{2K} - \frac{(\bar{Q} - P_1\beta_{i1} - P_1\beta_{j1})^2}{8\delta_i K^2}\right] \tag{10}$$

For simplicity, in this section we restrict to interior solutions only. There are important intuitions regarding this maximization problem. First, a relevant comparison to (10) is the case where there is no second period. If this were the case, firms would only maximize the first term of the expression above which results in a truthful reporting[11]. Hence, we want to understand how the presence of the second period distorts the optimal report from $\beta_{i1} = \delta_i$. Consider the second term of the maximization presented in (10). Note that the total profits of firm i are increasing on Q^R and therefore decreasing on β_{i1}[12]. This means that profits from the second period pressure reports downwards, which lead firms to optimally submit productivity parameters lower than δ_i. That is, firms claim to be more inefficient than they really are.

Lemma 4.3.2 *First order conditions lead to a symmetric best response function given by:*

$$\beta_{i1}(\beta_{j1}) = \frac{2\delta_i(2P_1 - K) + \bar{Q} - P_1\beta_{j1}}{5P_1}$$

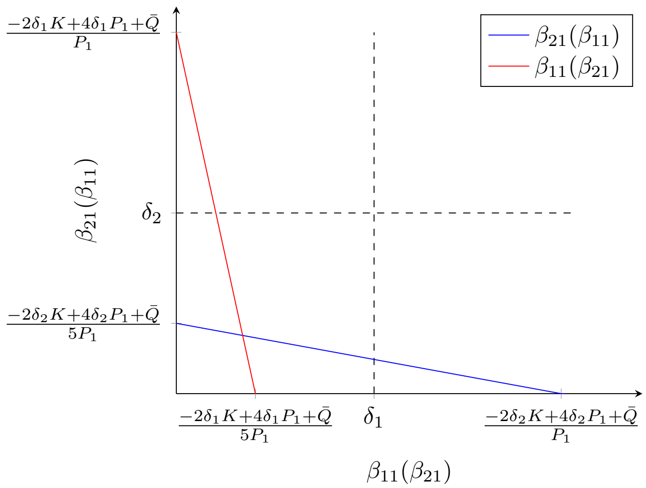
Proof. See Appendix A.

Figure 5 illustrates the best response functions for each firm. It can be seen that reports are strategic substitutes since the optimal report of firm i is decreasing in β_{j1}. Intuitively, when firm j reports higher productivity parameters, firm i reports lower productivity parameters because it prefers to maintain a higher residual demand for the next period even if that means selling less units in the present period while firm j sells more units of inventory. This occurs because the effect of having a fixed price K in $t = 2$ is stronger than the marginal income of selling today at P_1. Moreover, dashed lines on each firm's efficiency parameters

[11] Let $\hat{\pi}(\beta_i) = P_1^2\left(\beta_i - \frac{(\beta_i)^2}{2\delta_i}\right)$. It is easy to see that $\frac{\partial\hat{\pi}}{\partial\beta_i} = P_1^2\left(1 - \frac{\beta_i}{\delta_i}\right) \geqslant 0$, $\frac{\partial^2\hat{\pi}}{\partial\beta_i^2} = -\frac{(P_1)^2}{\delta_i} < 0$. Equating $\frac{\partial\hat{\pi}}{\partial\beta_i} = 0$, we find the maximum at $\beta_i = \delta_i$.

[12] If we take the first derivative of firm i's total profits π_i with respect to the residual demand Q^R, we obtain $\frac{\partial\pi_i}{\partial Q^R} = \frac{1}{2K} - \frac{Q^R}{4K^2\delta_i}$. For this to be strictly positive, $2K\delta_i > Q^R$ must hold. From Assumption 3.1, $2K\delta_i > \bar{Q}$, and since $\bar{Q} \geqslant Q^R$, total profits are increasing on the residual demand as $2K\delta_i > \bar{Q} \geqslant Q^R \Leftrightarrow 2K\delta_i > Q^R$. Finally, from $\bar{Q} - P_1\beta_{i1} - P_1\beta_{j1}$, we know that $\frac{\partial Q^R}{\partial\beta_{i1}} < 0$.

Figure 5: Best Responses



are included to show how both declare to be less efficient than they really are[13].

Proposition 4.3.1 *Firms' optimal reports at $t = 1$, when there are symmetric equilibria at $t = 2$, are the following:*

$$\beta_{11} = \frac{(\delta_2 - 5\delta_1)(K - 2P_1) + 2\bar{Q}}{12P_1}$$

$$\beta_{21} = \frac{(\delta_1 - 5\delta_2)(K - 2P_1) + 2\bar{Q}}{12P_1}$$

Proof. Direct from Lemma 4.3.2. Take arbitrary $i = 1$ and $j = 2$, then solve the system of equations.

Proposition 4.3.1 presents optimal reports for each firm in the first period, which are symmetric and strictly positive if the following condition holds: $2\bar{Q} + (K - 2P_1)(\delta_j - 5\delta_i) > 0,\ \forall\ i, j = 1, 2$[14]. Interestingly, we can see that, conditional on equal reports at $t = 2$, firms optimally report analogous efficiencies at $t = 1$. Indeed, the difference between them depends on how their own parameters differ with each other[15]. Furthermore, comparative statics are not direct except that reports are increasing on the target inventory, which is reasonable because as more units of inventory are demanded by the auctioneer, more units are offered by firms [16]. In addition, we should expect a positive relation between both firms' reports and posted prices in the first period, otherwise, lower P_1 would lead to higher β_{i1}, implying that firms would prefer selling

[13] Consider $\beta_{21}(\beta_{11}) = \frac{-2\delta_2K+4\delta_2P_1+\bar{Q}-P_1\beta_{11}}{5P_1}$. If $\beta_{11} = 0$, then $\beta_{21} = \frac{-2\delta_2K+4\delta_2P_1+\bar{Q}}{5P_1}$. If $\beta_{21} = 0$, then $\beta_{11} = \frac{-2\delta_2K+4\delta_2P_1+\bar{Q}}{P_1}$. Comparing $\frac{-2\delta_2K+4\delta_2P_1+\bar{Q}}{5P_1} \lesseqgtr \delta_2 \Leftrightarrow \bar{Q} < (2K\delta_2 + P_1\delta_2)$ which holds from Assumption 3.1.

[14] By comparing $\beta_{11} \lesseqgtr 0$. Analogous for firm 2.

[15] Since $\frac{(\delta_2-5\delta_1)(K-2P_1)+2\bar{Q}}{12P_1} \lesseqgtr \frac{(\delta_1-5\delta_2)(K-2P_1)+2\bar{Q}}{12P_1} \Leftrightarrow \delta_1 \lesseqgtr \delta_2$.

[16] Since $\frac{\partial\beta_{i1}}{\partial\bar{Q}} = \frac{1}{6P_1} > 0$.

more units at lower prices while reducing the residual demand for the next period. For this to happen the following parametric condition must hold: $K(5\delta_i - \delta_j) > 2\bar{Q}$[17]. In this context, as P_1 rises, the marginal income perceived by firms from selling in the first period gets closer to the one perceived from selling in the second period (i.e. P_1 gets closer to K). Hence, in the margin, this price difference between periods compensates a marginal cost difference that favors $t = 1$, making β_{i1} increase. Moreover, if we calculate the partial effect of δ_i on β_{i1}, it can be seen that the sign depends on how close P_1 is to K[18]. When K and P_1 are sufficiently close, efficiency parameters have a positive relation with firms' reports. Conversely, when both prices deviate sufficiently, the relation turns negative. The idea behind this is that when K is far enough from P_1, firms would prioritize maintaining high Q^R by decreasing β_{i1}. Interestingly, the partial effect of δ_j on β_{i1} is exactly the opposite of δ_i on β_{i1}[19]. Finally, based on the intuition given, we should expect β_{i1} to relate negatively with K, which happens if $\delta_j < 5\delta_i$[20].

Now we proceed to the auctioneer's problem. The auctioneer wants to design a way to minimize its expenses made on the purchase of the target inventory $\bar{Q}$. In the first period, it posts a uniform price first, to which the firms respond by announcing a supply function by giving a report of their productivity parameter δ_i. Until now we have seen how firms perform in the first period when they anticipate a symmetric equilibrium in the second period. This is anticipated by the auctioneer, which is also aware that the price of the second period would be endogenously fixed at K. Let the total expenditure be defined as the sum of the expense made in each time period:

$$\mathcal{E}(P_1) = P_1 Q_1 + K Q_2$$

As the amount bought in each period corresponds to the sum of both firms $Q_t = Q_{1t} + Q_{2t}$ and given that they both report $\beta = \frac{Q^R}{2K}$ in the second period, we can rewrite total expenditure as:

$$\mathcal{E}(P_1) = P_1^2(\beta_{11} + \beta_{21}) + KQ^R \quad \Leftrightarrow \quad \mathcal{E}(P_1) = P_1^2(\beta_{11} + \beta_{21}) + K(\bar{Q} - P_1(\beta_{11} + \beta_{21}))$$

The auctioneer solves the following minimization problem:

$$\min_{P_1} \mathcal{E}(P_1) = P_1(P_1 - K)(\beta_{11} + \beta_{21}) + K\bar{Q} \tag{11}$$

For this problem to make sense, the endogenous price P_1 must be strictly less than K, since offering a higher price only leads to higher expenses. If the auctioneer posts $P_1 = 0$, the first term of the minimization is zero, which means it does not purchase in the first period and waits for the second period to buy $\bar{Q}$. On

[17] Since $\frac{\partial \beta_{i1}}{\partial P_1} = \frac{5K\delta_i - 2\bar{Q} - \delta_j K}{12(P_1)^2} \Leftrightarrow 5K\delta_i - \delta_j K \lesseqgtr 2\bar{Q}$.
[18] Since $\frac{\partial \beta_{i1}}{\partial \delta_i} = \frac{-5K + 10P_1}{12P_1} \lesseqgtr 0 \Leftrightarrow 2P_1 \gtreqless K$.
[19] Since $\frac{\partial \beta_{i1}}{\partial \delta_j} = \frac{K - 2P_1}{12P_1} \lesseqgtr 0 \Leftrightarrow K \gtreqless 2P_1$.
[20] Since $\frac{\partial \beta_{i1}}{\partial K} = \frac{-5\delta_i + \delta_j}{12P_1} \lesseqgtr 0 \Leftrightarrow -5\delta_i + \delta_j > 0$.

the other hand, if the auctioneer posts $P_1 = K$, the auctioneer does not buy inventory units from the firms and purchases the whole target inventory from its outside option at a unit price K. In this context, the first term of the minimization represents the savings made by the auction design from the case where there is no transaction between the auctioneer and the firms. An equivalent way of presenting the auctioneer's decision problem is given by:

$$\max_{P_1} \mathcal{S}(P_1) = (K - P_1)[P_1(\beta_{11} + \beta_{21})]$$

Where $\mathcal{S}(P_1)$ is a savings function that depends on the first period price. This formulation is very useful because it shows the classic trade-off faced by a monopolist: setting a higher price incites firms to offer larger amounts of inventory so that the auctioneer can save units of inventory to be purchased in the next period. However, setting a higher price also decreases the auctioneer's margin, because as P_1 gets closer to K, it reduces the unit price saved for each unit of inventory. If the reports of each firm at $t = 1$ did not depend on P_1, the problem would be trivial, but since they do, the auctioneer must contemplate each firm's optimal report when solving its problem. Let P_1^s be the optimal price that minimize the auctioneer's expenditure. From Proposition 4.3.1, we can replace β_{11} and β_{21} in (11):

$$\min_{P_1} \mathcal{E}(P_1) = P_1(P_1 - K)\left(\frac{\bar{Q} - K(\delta_1 + \delta_2) + 2P_1(\delta_1 + \delta_2)}{3P_1}\right) + K\bar{Q} \tag{12}$$

Proposition 4.3.2 *Optimal price announcement for the auctioneer at* $t = 1$*, when both firms report* $\beta_{i2} = \frac{Q^R}{2K}$ *at* $t = 2$*, is given by:*

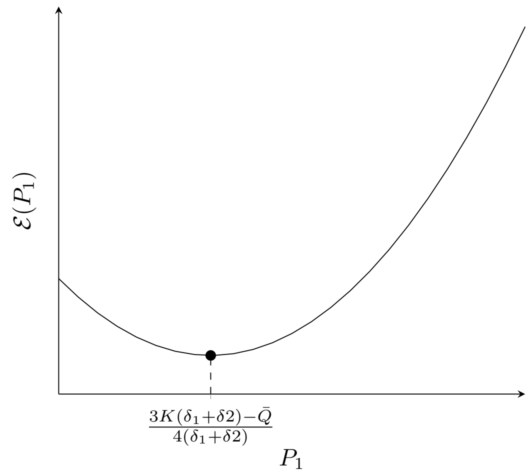
$$P_1^s = \frac{3K(\delta_1 + \delta_2) - \bar{Q}}{4(\delta_1 + \delta_2)} \quad \Leftrightarrow \quad P_1^s = \frac{3K}{4} - \frac{\bar{Q}}{4(\delta_1 + \delta_2)}$$

Proof. See Appendix A

Figure 6 illustrates the expenditure function minimized by the auctioneer and the optimal price. This price clears the trade-off faced by the auctioneer and it is strictly lower than K[21]. We can infer interesting insights from simple comparative statics. First, the target inventory relates negatively with the optimal price. This means that for higher $\bar{Q}$, it is preferred to save through setting a lower P_1 so as to increase the price margin $(K - P_1)$, in spite of buying less units of inventory today. Moreover, the auctioneer is interested on the aggregate productivity efficiency $(\delta_1 + \delta_2)$, rather than individual parameters. The aggregate productivity is positively related to P_1 which means that higher $(\delta_1 + \delta_2)$ leads to higher unit prices offers from the auctioneer to the bidders. This occurs because when P_1 is closer to K, firms with higher productivity parameters will report higher β_{i1}, therefore the auctioneer will be able to buy more units of inventory at $t = 1$. In this sense, P_1 relates positively with K, since the auctioneer knows that it needs

[21] $P_1^s \equiv \frac{3K}{4} - \frac{\bar{Q}}{4(\delta_1+\delta_2)} \lesseqgtr K \Leftrightarrow 0 < \frac{\bar{Q}}{4(\delta_1+\delta_2)} + \frac{K}{4}$.

Figure 6: Auctioneer Expenditure



to maintain prices from both periods close to each other in order to attract inventory offers in the first period.

We can evaluate the optimal price on (12) and derive the total expense made by the auctioneer:

$$\mathcal{E}^s \equiv \mathcal{E}(P_1^s) = \frac{-[\bar{Q} + K(\delta_1 + \delta_2)]^2}{24(\delta_1 + \delta_2)} + K\bar{Q} \tag{13}$$

There are some intuitions we can notice from analyzing the symmetric case. Given that firms know how many units were sold in the first period, there is no uncertainty regarding the cleared amount to be sold in the second period, which is fixed at Q^R. When both firms submit equal productivities at $t = 2$ the slopes of their supply functions are the same, therefore, the market is divided evenly between them for the auctioneer to reach its target inventory ($Q_{12} = Q_{22} = \frac{Q^R}{2}$). Moreover, firms report being less productive than they really are in both periods[22]. However, while in the second period their reports are independent of their productivities, recall that in the first period the difference between reports will actually depend on the difference in their productivities. This suggests that firms are more competitive in the first period than the second one as more productive firms report higher submissions, making their supply functions flatter, allowing them to offer larger amounts of inventory units than they would with a steeper supply curve. An interesting question that emerges from this analysis is why firms still offer positive amounts of inventory in the first period when the posted price is lower than the one cleared in the second period. Since we assume that firm's marginal costs are always under K, it seems more preferable for them to sell units only in the second period rather than the first one. Nevertheless, the reason they submit parameters at $t = 1$ anyways is that they fear that if they do not, the other firm could take advantage of this and sell the entire $\bar{Q}$ first, leaving the other one out of the market. This allows the auctioneer to actually post a price lower than K, enabling it to save expenses. This is explained in more detail in section 5.

[22]From Figure 5 we already know firms declare to be less productive than they really are. For the second period we compare $\frac{Q^R}{2K} \gtrless \delta_i \Rightarrow \bar{Q} < 2K\delta_i + P_1(\beta_{i1} + \beta_{j1})\ \forall i$ which holds from Assumption 3.1.

4.4 First period with restrictive price cap ($K < \infty$): Maximal case

Now consider an equilibrium selection for the second period of the auction such that the sum of the profits of firm 1 and firm 2 ($\pi_{12}(\beta_{12}) + \pi_{22}(\beta_{22})$) is maximized. This selection is referred as the "maximal case". The idea is to study how both firms behave when they anticipate large aggregate profits and compare this case with the case when both behave equally at $t = 2$. In this context, we return to Proposition 4.2.2 which derives the following firm's optimal reports for the second period:

$$\beta_{i2}(\beta_{j2}) \in \Big[\max\Big\{\frac{Q^R - \delta_j K}{K}, \frac{Q^R - \widetilde{B}_j(\delta_i)K}{K}\Big\}, \widetilde{B}_i(\delta_j)\Big]$$
$$\beta_{j2}(\beta_{i2}) = \frac{Q^R - \beta_{i2}K}{K}$$

Suppose arbitrarily that $i = 1$ and $j = 2$. Note that whenever β_{i2} takes one of its feasible extreme values, which could be either the lowest or highest bound of the interval described above, there are three possible equilibria:

(i) $\beta_{12} = \frac{Q^R - K\delta_2}{K}, \quad \beta_{22} = \delta_2$

(ii) $\beta_{12} = \frac{Q^R - K\widetilde{B}_2(\delta_1)}{K}, \quad \beta_{22} = \widetilde{B}_2(\delta_1)$

(iii) $\beta_{12} = \widetilde{B}_1(\delta_2), \quad \beta_{22} = \frac{Q^R - K\widetilde{B}_1(\delta_2)}{K}$

Cases (ii) and (iii) are analogous. Recall that the equilibrium price of the second period is $P_2 = K$. Thus, a general expression for total profits of both firms in the second period is given by:

$$\pi_{12}(\beta_{12}) + \pi_{22}(\beta_{22}) = K^2\Big(\beta_{12} - \frac{\beta_{12}^2}{2\delta_1}\Big) + K^2\Big(\beta_{22} - \frac{\beta_{22}^2}{2\delta_2}\Big) \tag{14}$$

Let us assume without loss of generality that $\delta_1 < \delta_2$, which means that firm 1 is less productive than firm 2 (i.e. higher marginal cost). If we were to maximize (14), we would like β_{22} to be as high as possible and β_{12} to be as low as possible. Indeed, because best response functions are decreasing on the equilibrium path (strategic substitutes), in the second period, whenever we impose firm 1's report to be as low as possible, we are indirectly imposing β_{22} to be as high as possible. In this respect, the following two sections solve cases (i) and (ii). The former has an analytic solution while the latter has a numerical solution.

4.4.1 Analytic solution

Let the selected equilibrium of the second period be the case where firm 1 reports one of its two lowest feasible equilibrium reports:

$$\beta_{12} = \frac{Q^R - \delta_2 K}{K} \tag{15}$$

Note that β_{12} would be strictly positive if the following condition holds: $\bar{Q} > \delta_2 K + P_1(\beta_{11} + \beta_{21})$[23]. Thus, whether or not the less productive firm participates in the second period is not obvious. Firm 2 responds to (15) accordingly:

$$\beta_{22} = \frac{Q^R - K\left(\frac{Q^R - \delta_2 K}{K}\right)}{K} \quad \Leftrightarrow \quad \beta_{22} = \delta_2$$

In this case, it is direct that β_{22} is strictly positive, as efficiency parameters are assumed to be positive as well. Additionally, notice that this equilibrium selection does not always hold. In particular, $\frac{Q^R - \delta_2 K}{K}$ should be strictly higher than $\frac{Q^R - \widetilde{B}_2(\delta_1)K}{K}$ for it to be firm 1's lowest feasible report. For this to be true, the following condition must be satisfied:

$$Q^R(\delta_1 + \delta_2) > 2K\delta_2\left(\delta_1 + \frac{\delta_2}{2}\right) \tag{16}$$

This is a restrictive non-parametric condition that requires firm 2 to be highly efficient, which is not verifiable *ex-ante* because the residual demand depends on the optimal reports made by firms in the first period.

Once we know how both firms behave in the second period, we can back up to the first period and solve each firm's maximization problem. In contrast to the symmetric case, here each firm faces a different problem, as their reports on the second period are no longer the same. Replacing the definition of Q^R in (15), each firm maximizes total profits as follows:

$$\max_{\beta_{11}} \pi_{11}(\beta_{11}) = P_1^2\left(\beta_{11} - \frac{\beta_{11}^2}{2\delta_1}\right) + K^2\left[\frac{-\delta_2 K - (\beta_{11} + \beta_{21})P_1 + \bar{Q}}{K} - \frac{(-\delta_2 K - (\beta_{11} + \beta_{21})P_1 + \bar{Q})^2}{2\delta_1 (K)^2}\right]$$
$$\max_{\beta_{21}} \pi_{21}(\beta_{21}) = P_1^2\left(\beta_{21} - \frac{\beta_{21}^2}{2\delta_2}\right) + \frac{\delta_2 K^2}{2}$$

Let us start considering an interior solution. Following the analysis made in the symmetric case, we can study how the second period distorts each firm's optimal report from an honest submission, $\beta_{i1} = \delta_i$, which would be the result if firms were only maximizing profits from the first period. Note that profits from the second period perceived by firm 1 are decreasing on β_{11}[24].The reason for this is that profits are increasing on $(Q^R - \delta_2 K)$, which produces a negative effect on the first period optimum, resulting in a lower report than the actual parameter δ_1. Unlike firm 1, firm 2 reports honestly by submitting $\beta_{21} = \delta_2$. This is attributable to the fact that profits perceived by firm 2 in the second period do not depend on reports from the first period.

[23]Since $\beta_{12} \lesseqgtr 0 \Leftrightarrow \bar{Q} > \delta_2 K + P_1(\beta_{11} + \beta_{21})$.

[24]Profits perceived by firm 1 must be increasing on $(\bar{Q} - \beta_{11}P_1 - \beta_{21}P_1 - \delta_2 K)$. Let $Q' = (\bar{Q} - \beta_{11}P_1 - \beta_{21}P_1 - \delta_2 K)$. Note that $\frac{\partial \pi_{11}}{\partial Q'} \equiv \left(K - \frac{Q'}{\delta_1}\right) > 0 \Leftrightarrow K(\delta_1 + \delta_2) > Q^R$, which holds from Assumption 3.1. Furthermore, this condition is consistent with $Q^R(\delta_1 + \delta_2) > K\delta_2(2\delta_1 + \delta_2)$ needed for this equilibrium selection, since $\frac{K\delta_2(2\delta_1+\delta_2)}{(\delta_1+\delta_2)} > K(\delta_1 + \delta_2) \Leftrightarrow 0 < \delta_1^2$.

Lemma 4.4.1 *First order conditions lead to best response functions given by:*

$$\beta_{11}(\beta_{21}) = \frac{P_1\delta_1 - K(\delta_1+\delta_2) + \bar{Q} - \beta_{21}P_1}{2P_1}$$
$$\beta_{21}(\beta_{11}) = \delta_2$$

Proof. See Appendix A

Lemma 4.4.1 presents each firm's best response function, which arises from the first order condition of each firm's respective problem. It can be seen that firm 2 reports honestly regardless of what firm 1 reports, whereas firm 1's report relates negatively with the reports given by firm 2. Unlike the symmetric case, this section also inspects the case where firm 1 announces the least feasible productivity parameter, that is, $\beta_{11} = 0$. In other words, it reports having infinite marginal costs of producing inventory goods.

Proposition 4.4.1 *Firms' optimal reports at $t = 1$, when there are maximal aggregate profits in the second period through $\beta_{12} = \frac{Q^R - \delta_2 K}{K}$ and $\beta_{22} = \delta_2$, are given by:*

$$\beta_{11} = \max\left\{0, \frac{P_1\delta_1 - K(\delta_1+\delta_2) + \bar{Q} - \delta_2 P_1}{2P_1}\right\}$$
$$\beta_{21} = \delta_2$$

Proof. Direct from Lemma 4.4.1.

From Proposition 4.4.1, firm 2's optimal report is not only strictly positive but also constant, and amounts to its highest feasible value, which is its real productive efficiency. Surprisingly, under the assumption that firm 1 is less productive than firm 2 ($\delta_1 < \delta_2$), it is easy to show that for any feasible parameters that satisfy this assumption would derive a negative report for firm 1 in the first period, which is not possible because the model considers positive parameters only.[25]. Thus, $\beta_{11} = 0$, which means that firm 1 refrains from reporting and that firm 2 would be the only one offering inventory units in the first stage of the auction.

Now we proceed to solve the auctioneer's problem. Naturally, the auctioneer internalizes how firms will behave in the first period. The general structure of the problem faced by the auctioneer in this case is the same as the symmetric case, except that here the productivity reports from each firm differ notably as they are asymmetric in the second period and there is a corner solution in the first period. Total expenditure

[25] From comparing $\beta_{11} \lesseqgtr 0 \Leftrightarrow P_1 + \bar{Q} < K(\delta_1+\delta_2) + P_1(\delta_2 - \delta_1)$. We know $\bar{Q} < K(\delta_1+\delta_2)$ from Assumption 3.1 and that $\delta_1 < \delta_2$ assumed at the beginning of this section.

made by the auctioneer in the purchase of $\bar{Q}$ is given by:

$$\mathcal{E}(P_1) = P_1^2(\beta_{11} + \beta_{21}) + K^2(\beta_{12} + \beta_{22}) \quad \Leftrightarrow \quad \mathcal{E}(P_1) = P_1^2(\beta_{11} + \beta_{21}) + K^2\Big(\frac{Q^R}{K}\Big)$$

Using the definition of residual demand, this is equivalent to:

$$\mathcal{E}(P_1) = P_1(P_1 - K)(\beta_{11} + \beta_{21}) + K\bar{Q} \tag{17}$$

Note that since optimal the reports in the first period do not depend on the posted price P_1, the expenditure function is much simpler than the one derived in the symmetric case. Let P_1^m be the optimal price for the maximal case. From Proposition 4.4.1, we can replace $\beta_{11} = 0$ and $\beta_{21} = \delta_2$ in (17) so that the auctioneer minimizes the following:

$$\min_{P_1} \mathcal{E}(P_1) = P_1(P_1 - K)(\delta_2) + K\bar{Q}$$

Proposition 4.4.2 *The optimal price announcement for the auctioneer at $t = 1$, when firms report $\beta_{12} = \frac{Q^R - \delta_2 K}{K}$ and $\beta_{22} = \delta_2$ at $t = 2$, is given by:*

$$P_1^m = \frac{K}{2}$$

Proof. See Appendix A

Proposition 4.4.2 derives the optimal price, which is strictly lower than K. Interestingly, it is notably different from the optimal price obtained in the symmetric case, as here the price chosen by the auctioneer is only sensible to changes on the price cap and is independent of firms' productivity parameters as well as the target inventory. Intuitively, P_1^m clears the auctioneer's trade-off between raising the price in order to purchase more inventory units and reducing the price in order to increase savings through buying at a lower price than K. We can evaluate P_1^m on the total expenditure function $\mathcal{E}(P_1)$:

$$\mathcal{E}^m \equiv \mathcal{E}(P_1^m) = \frac{-\delta_2 K^2}{4} + K\bar{Q} \tag{18}$$

We denote $\mathcal{E}^m$ as the total expenditure made by the auctioneer in this maximal case. Since this depends, on the reports made by each firm in the first period, $\mathcal{E}^m$ only depends on the productivity of firm 2 as firm 1 refrains from reporting at $t = 1$. This contrasts with the symmetric case where the total expenditure made by the auctioneer depends on the firms' aggregate efficiency. Furthermore, we can compare the optimal prices determined in the maximal and the symmetric cases as follows:

$$P_1^{\mathrm{m}} \equiv \frac{K}{2} \lesseqgtr \frac{3K}{4} - \frac{\bar{Q}}{4(\delta_1 + \delta_2)} \equiv P_1^{\mathrm{s}}$$
$$\Leftrightarrow \bar{Q} \lesseqgtr K(\delta_1 + \delta_2)$$

From Assumption 3.1, we can conclude that $\bar{Q} < K(\delta_1 + \delta_2)$, therefore $P_1^{\mathrm{m}} < P_1^{\mathrm{s}}$. The auctioneer optimally posts a lower price when firms' reports differ than when they report symmetrically. The main reason for this is that the auctioneer faces no competition in the first period of the maximal case, as firm 2 is the only one offering inventory units. Moreover, the fact that firm 2's report does not depend on P_1 in neither of the two time periods limits the capacity of the auctioneer to induce higher reports when increasing P_1. Conversely, in the symmetric case, reports from both firms are sensible to changes in P_1 in both periods. As a result, whenever the auctioneer changes this price, apart from the direct effect this has on the cleared amount[26], present in both cases, there is an indirect effect on firms' reports, which is present in the symmetric case and absent in the maximal one unless firm 1 participates in the second period. Hence, since the auctioneer has less space of manipulation in the maximal case, it has less incentives to raise the posted price P_1, explaining why the optimal price of the maximal case is cleared below the symmetric one.

The maximal case shows us that when firms act in the interest of maximizing aggregate profits at $t = 2$ and their productivities differ, there is a corner solution in the first period. Regarding the second period, the presence of competition is inconclusive because we cannot infer whether firm 1 reports a positive parameter or not[27]. Firm 2 reports honestly in both periods, whereas firm 1 may participate in the second period by reporting a lower productivity parameter than its true value[28]. Consequently, the firm with a higher productivity parameter is more competitive than the less productive one because it submits supply functions with lower slopes in both periods, enabling it to obtain larger fractions of the market by selling more. In this respect, in the symmetric case firms do not behave as aligned as they do in the maximal one. In the former, firms face equal trade-off at $t = 2$ when they compete for all possible combinations of productivities they may have. In the latter, production in the first period is concentrated in the firm that is relatively more productive because they know the price of the second period would be fixed at K, hence it is suboptimal for firm 1 to produce units at a higher cost and for a lower price than K instead of leaving higher residual demand for the next period. This does not imply firms behave in a collusive manner *ex-ante*, as this conclusion arises as an equilibrium behavior conditional on a particular solution selection at $t = 2$.

[26] Recall equation (1).

[27] We mentioned that for firm 1 to participate in the second period, the following must hold: $\bar{Q} > \delta_2 K + P_1(\beta_{11} + \beta_{21})$. If we evaluate this with the solutions derived, we obtain $\bar{Q} \lesseqgtr \frac{3}{2} K \delta_2$, from which we cannot derive a conclusion.

[28] Since $\beta_{12} \lesseqgtr \delta_1 \Leftrightarrow \bar{Q} < K(\delta_1 + \delta_2) + P_1(\beta_{11} + \beta_{21})$ which holds from Assumption 3.1.

4.4.2 Numerical solution

Let the selected equilibrium of the second period be the case where firm 1 reports one of its two lowest feasible equilibrium reports. Using equation (7), we can express these reports by:

$$\beta_{12} = \frac{Q^R - \widetilde{B}_2(\delta_1)K}{K} \quad \Leftrightarrow \quad \beta_{12} = \frac{Q^R - \left[\frac{(Q^R - 2K\delta_1) + \sqrt{4(\delta_1 K)^2 + (Q^R)^2}}{2K}\right]K}{K} \tag{19}$$

$$\beta_{22} = \widetilde{B}_2(\delta_1) \quad \Leftrightarrow \quad \beta_{22} = \frac{(Q^R - 2K\delta_1) + \sqrt{4(\delta_1 K)^2 + (Q^R)^2}}{2K} \tag{20}$$

In contrast with the analytic solution, for this equilibrium selection to be maximal, we need $\frac{Q^R - \widetilde{B}2(\delta 1)}{K}$ to be strictly higher than $\frac{Q^R - \delta_2 K}{K}$. For this to be true, the following condition must be satisfied:

$$Q^R(\delta_1 + \delta_2) < 2K\delta_2\left(\delta_1 + \frac{\delta_2}{2}\right) \tag{21}$$

Note that this is the opposite of equation (16) and can also be written as:

$$(\bar{Q} - P_1(\beta_{11} + \beta_{21}))(\delta_1 + \delta_2) < 2K\delta_2\left(\delta_1 + \frac{\delta_2}{2}\right) \tag{22}$$

Furthermore, notice that the selected reports for each firm at $t = 2$ depend on each firm's reports at $t = 1$ through Q^R[29]. It is relatively simple to verify that under this equilibrium selection, profit maximization problems faced by firms in the first period do not have analytical solutions, due to the variety of orders in (19) and (20) for the variables of interest (β_{11} and β_{21}). Thus, we must solve this case numerically. Before doing this, it is important that the initial values used for the simulation satisfy the next three conditions:

(i) $2K\delta_i > \bar{Q}, \forall i = 1, 2$

(ii) $\delta_1 < \delta_2$

(iii) $(\bar{Q} - P_1(\beta_{11} + \beta_{21}))(\delta_1 + \delta_2) < 2K\delta_2\left(\delta_1 + \frac{\delta_2}{2}\right)$

The first one corresponds to Assumption 3.1 enunciated in section 3. The second one is a necessary condition for the maximal cases outlined at the beginning of this section, specifically in section 4.4, and the third one is the condition just described above (22).

Now we proceed to describe the necessary steps for solving the auction design for this equilibrium selection. We will describe two documents. The first one is descriptive as it is made for defining variables, in order to derive an expression for firms' profits in the first period, and the auctioneer's expenses, which are the ones that will be maximized and minimized respectively in the second document. The second document contains the simulation itself, and follows the main intuition from the analytical cases studied previously as it starts by solving each firm's profit maximization problem, followed by the auctioneer's minimization problem. Both documents are included in Appendix C which corresponds to the codes used in Matlab.

[29] See equation (2).

Document 1: Defining variables

- Step 1 Define the model variables and clarify sub indexes when needed.
- Step 2 Express the model variables in their reduced form.
- Step 3 Express the model variables in their extensive form. Replace when needed (Step 4 - Step 10).
- Step 4 Replace "QR" (line 15) and "root" (line 16) in "`B1_kink`" (line 17).
- Step 5 Replace "QR" (line 15) and "`B1_kink`" (line 17) in "`beta_12`" (line 18).
- Step 6 Replace "QR" (line 15) and "`beta_12`" (line 18) in "`beta_22`" (line 19).
- Step 7 Replace "`beta_12`" (line 18) in "`profit_12`" (line 20).
- Step 8 Replace "`beta_22`" (line 19) in "`profit_22`" (line 21).
- Step 9 Replace "`profit_12`" (line 20) in "`profit_11`" (line 22).
- Step 10 Replace "`profit_22`" (line 21) in "`profit_21`" (line 23).

Variables "`profit_11`", "`profit_21`" and "`auctioneer_exp`" are the ones that will be maximized and minimized in Document 2, which is described below.

Document 2: Simulation

- Step 1 Define the initials values for the model parameters (lines 6-19).
- Step 2 Define grids for the variables of interest: "`beta_11`", "`beta_21`" and "`p_1`" (lines 23-27).
- Step 3 Firm 1's maximization problem (lines 44-64). A **for loop** fixes "`beta_21`", making it iterate through "`grid_beta_21`". For each iteration, it finds the "`beta_11`" such that "`profit_11`" is maximized by using the **fsolve** command.
- Step 4 Generate vector of 2 columns. First column are the initial "`beta_21`" and second column contains the optimal "`beta_11`" found in Step 5. Name it "`BR_firm1`" (line 58).
- Step 5 Create new variable of the column vector with the optimal "`beta_11`" and name it "beta11op" (line 62).

- STEP 6 Firm 2's maximization problem (lines 74-100). A **for loop** fixes "`beta_11`", making it iterate through "beta11op". For each iteration, it finds "`beta_21`" such that "`profit_21`" is maximized by using **fsolve** command.

- STEP 7 Generate vector of 2 columns. First column is the "beta11op" vector and second column contains the optimal "`beta_21`" found in Step 6. Name it "`BR_firm2`" (line 96).

- STEP 8 Create new column vector with optimal "`beta_11`" and name it "beta21op" (line 98).

- STEP 9 Redefine "dif" as the maximum distance between each firm's best response function iteration with its previous iteration to make each function converge until it is below "tolerance" (lines 107-109).

- STEP 10 Rename initial grids of firms' reports to their optimal reports defined in step 5 and step 8 (lines 112-113).

- STEP 11 Generate **while loop** that contains steps 3 to 10 with a stop rule "dif>tolerance".

- STEP 12 Define 4 variables for each column of "`BR_firm1`" and "`BR_firm2`". Name them "`b_21`", "`b_11_optimum`", "`b_11`" and "`b_21_optimum`" accordingly (lines 121-124).

- STEP 13 Define the difference between "`b_21`" and "`b_11_optimum`" as "`dif_beta_21`" and minimize it with **fsolve** command. Name minimum position "`position_3`". (lines 134-136).

- STEP 14 Evaluate "`b_21_optimum`" and "`b_11`" in "`position_3`" and call them "`beta_21_STAR`" and "`beta_11_STAR`". These correspond to the coordinates where the best response functions intersect (lines 138-139).

- STEP 14 Evaluate "`auctioneer_exp`" in "`beta_11_STAR`" and "`beta_21_STAR`" (line 145).

- STEP 15 Generate **for loop** that contains steps 3 to 14 and make it iterate "`p_1`" through each element of "`grid_p1`" .

- STEP 16 Minimize "`auctioneer_exp`" using **fsolve** command and call minimum position "`position_4`" (line 151).

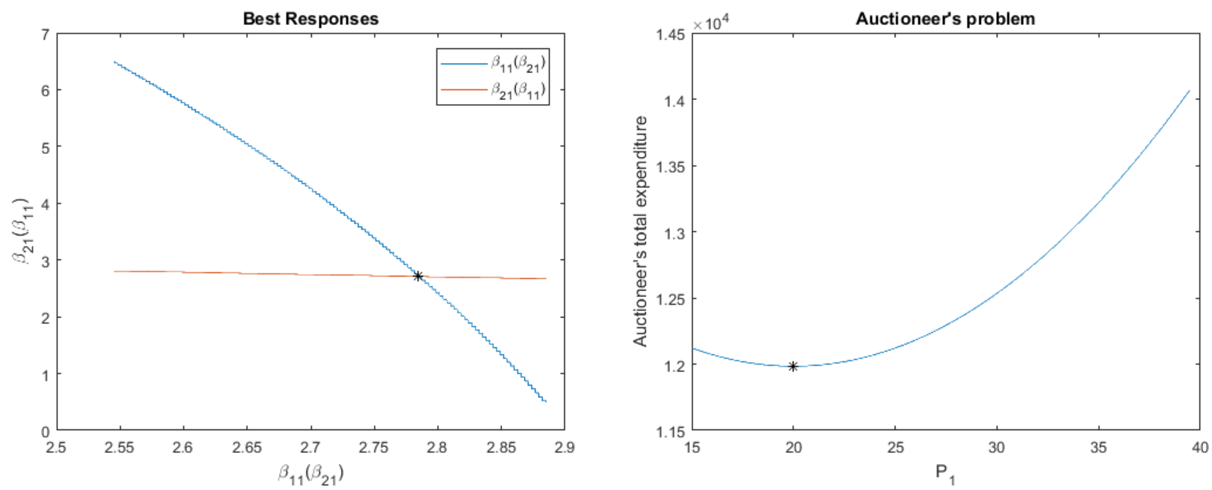
- STEP 17 Evaluate "`grid_p1`" in `position_4`" and name it "`p1_STAR`" (line 152).

- STEP 18 Graph best response functions of firm 1 and firm 2 in one plot and "`auctioneer_exp`" in another plot (lines 161-179).

- STEP 19 Use results to calculate optimal "`beta_12`" and "`beta_22`" (lines 181-185).

The results derived from the simulation are $\beta_{11}^{*} = 2.7847$, $\beta_{21}^{*} = 2.7102$ and $P_1^{*} = 20$. The first two are the equilibrium reports of firms 1 and 2, respectively, in the first period of the auction. Figure 7 (a) illustrates each firm's best response function and a mark is made to show where these two intersect. The third result is the optimal price that enables the auctioneer minimizes the expenditure made through the auction. Figure 7 (b) illustrates the auctioneer's expense function over the feasible price for the first period and the point where expenditures are minimized.

Figure 7



(a) (b)

From the results mentioned before, we can calculate equilibrium reports for the second stage thus obtaining $\beta_{12}^{*} = 1.8536$ and $\beta_{21}^{*} = 4.2623$. As $\delta_1 = 3.28$ and $\delta_2 = 6.5$, we can observe that both firms report being less productive than they really are in both periods, which differs from the analytical solution of the maximal case where the more productive firm reported its real efficiency parameter in both periods. Another notorious result is that firm 1 produces in both periods, whereas in the analytic case we found a corner solution at $t = 1$ where firm 2 kept the entire market. This is more beneficial from the auctioneer's perspective because there is more competition between firms. Recall that in the analytic case, it was required that firm 2 should be more productive than firm 1 (equation (16)). Based on the results obtained in this section, where δ_1 is nearly one half of δ_2, firm 2 would have to be at least 50% more productive than firm 1 to be under the conditions described in the analytic solution. Finally, it is worth mentioning that P_1^{*} is much smaller than the price cap $K = 40$, which suggests that the auctioneer prefers to buy fewer units at a lower price than posting a price closer to K to increase its purchases.

5 Discussion

The existence of two periods in the model has implications worth noting. The first one is that firms can distribute their production over time in order to smooth their costs. This is a natural consequence attributable to firms' increasing marginal cost functions on the amount produced in the model. Recall that there is no discount between the two periods. The second one is a strategic intertemporal interaction between firms. Even though firms know that in the second period the price would be fixed at K, both are willing to sell positive amounts of inventory at a lower price in the first period because they know that, if one them does not, the other firm may offer the entire target inventory first, leaving the rest with zero profit. Hence, the first stage of the auction induces firms to compete, making submissions less aggressive for the auctioneer who is able to purchase some units at a price below the price cap. This explains how competition changes between the first period and the second period. From section 4.1, we know that whenever the auctioneer's demand is known by firms and they compete freely on prices (i.e. without a restrictive price cap), the cleared price explodes up to K. This occurs because firms' submissions would not affect the quantity needed to be purchased at the end of the period, encouraging them to charge what benefits them the most. Conversely, when the auctioneer announces a fixed price first and lets firms compete on quantities, as happens in the first stage, the auctioneer obtains some utility through the acquisition of inventory units cheaper than K.

Another striking aspect regarding the symmetric case of our model is that total expenditures perceived by the auctioneer depend on the sum of firms' efficiency parameters instead of depending on each parameter individually[30]. As a result, the optimal price of the first stage is the same independently of whether one firm is far more productive than the other or whether both are equally productive. This is not obvious in an auction context. For instance, in second price auction models generalized by Vickrey (1961), firms would sell goods at the marginal cost of the second most productive firm. In this context, whether firms are similar or not in terms of efficiency is crucial because firms with closer costs will enforce more competitive pressure between them, determining how much the auctioneer would end up paying in the first period. In this sense, in a second price auction the auctioneer prefers bidders to be similar in terms of efficiency, while, under the symmetric solution of our model, the auctioneer is only concerned with the aggregate efficiency rather than the structure of the industry. This observation is not applicable to the analytical solution of the maximal case studied, since in that scenario the auctioneer faces a monopoly in the first stage of the auction, making its expenses depend only on one firm's efficiency.

[30] See equation (13) from sections 4.3.

6 Comparison with a one-period mechanism

In this section, we will study the convenience of the mechanism presented in this work by contrasting it with a single period mechanism. We aim to isolate the strategic intertemporal effect that arises in our main model from the one-period auction introduced in this section, since both would be identical except for this component. In particular, consider the case where the auctioneer only has one period to purchase its target inventory $\bar{Q}$ and performs two auctions of $\frac{\bar{Q}}{2}$ each in the same period. The reason why the auction is made in two rounds is to avoid discrepancies with the paper's main model, since making firms produce only once would not be fair due to the functional form of their costs. Note that in both rounds there is no uncertainty regarding the amount to be bought by the auctioneer, so that firm's submissions in one round do not affect the result of the other, turning them into two independent auctions. Consequently, firms solve the same problem as in section 4.2, except that they maximize over $\frac{\bar{Q}}{2}$ (exogenous), instead of Q^R (endogenous). This results in a cleared unit price of K, leading the auctioneer to spend $K\bar{Q}$ for all the units it demands[31]. Notice that in this new model the existence of an outside option matters, as the price would explode otherwise. Essentially, we intend to compare the total expenditure made by the auctioneer in a single period mechanism with our sequential auction model, which can be respectively generalized as:

(i) $K\bar{Q}$

(ii) $P_1(P_1\beta_{11} + P_1\beta_{21}) + K(\bar{Q} - P_1(\beta_{21} + \beta_{22}))$

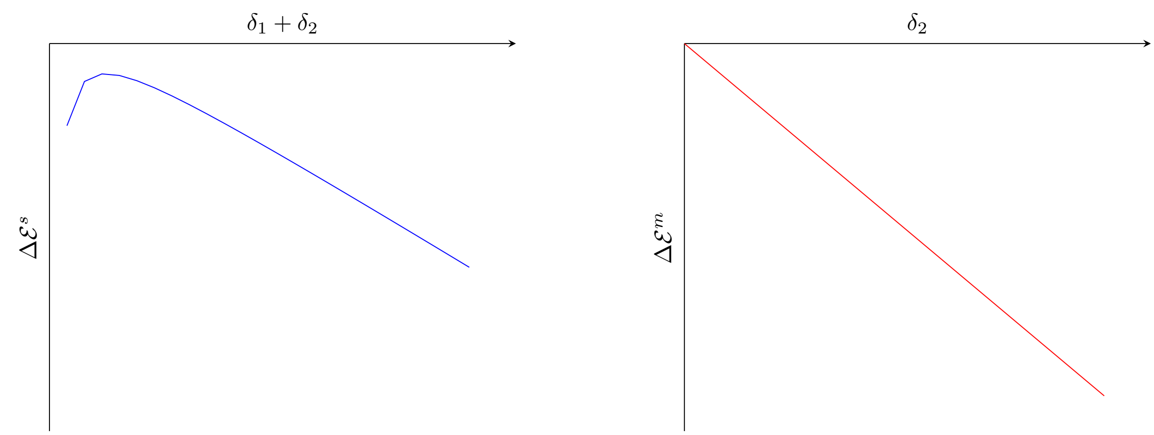
Observe that (ii) is cheaper than (i), as in (i) all units are sold at the price cap, which means that the auctioneer will save money by reducing its expenses as long as it buys units in the first period at prices strictly lower than K. Thus, savings attained when implementing a two-period auction model, instead of two auctions in one period, correspond to the difference between the two terms listed above. Consider the symmetric case and the analytic solution of the maximal case studied previously. We are interested in analyzing how the auctioneer's savings change as the firms' efficiency increases. In order to do this, we calculate the portion of savings made in each case relative to the total expenditure made on the single period model ($K\bar{Q}$). These are denoted as $\Delta\mathcal{E}^s$ and $\Delta\mathcal{E}^m$ for the symmetric and maximal case respectively, where the first one depends on the aggregate productivity of the industry ($\delta_1 + \delta_2$), while the second one only depends on the productivity of the more efficient firm (δ_2). Using equations (13) and (18), we calculate the portion of savings for the symmetric and the maximal case:

$$\Delta\mathcal{E}^s = \frac{\mathcal{E}^s - K\bar{Q}}{K\bar{Q}} \quad \Leftrightarrow \quad \Delta\mathcal{E}^s = \frac{-[\bar{Q} + K(\delta_1 + \delta_2)]^2}{24K\bar{Q}(\delta_1 + \delta_2)} \tag{23}$$

$$\Delta\mathcal{E}^m = \frac{\mathcal{E}^m - K\bar{Q}}{K\bar{Q}} \quad \Leftrightarrow \quad \Delta\mathcal{E}^m = \frac{-K\delta_2}{4\bar{Q}} \tag{24}$$

[31] Total expenditures made by the auctioneer is given by $K\frac{\bar{Q}}{2} + K\frac{\bar{Q}}{2} = K\bar{Q}$.

Note that both expressions are negative as they are savings for the auctioneer. Figure 8 illustrates the relationship between the savings from the symmetric case (23) and the maximal case (24) with the aggregate efficiency of firms and the efficiency of firm 2 respectively. Recall that in the maximal case we assume that $\delta_1 < \delta_2$. Therefore, Figure 8 (b) holds as long as firm 1 is less productive than firm 2 for all feasible values of δ_2 being considered. It can be observed that, as productivities go to infinity, savings tend to $-\infty$ in both cases.



(a) Symmetric case (b) Maximal case

Figure 8: Expenditure savings

One way of understanding the savings trends for both cases is to study how the posted price of the first period changes as the aggregate efficiency of the industry increase indefinitely. Indeed, if we take the limit of the optimal prices derived in Proposition 4.3.2 and Proposition 4.4.2 when $(\delta_1 + \delta_2)$ goes to infinity, we obtain the following:

$$\lim_{(\delta_1+\delta_2)\to+\infty} P_1^s \equiv \lim_{(\delta_1+\delta_2)\to+\infty} \left(\frac{3K}{4} - \frac{\bar{Q}}{4(\delta_1+\delta_2)}\right) = \frac{3K}{4}$$

$$\lim_{(\delta_1+\delta_2)\to+\infty} P_1^m \equiv \lim_{(\delta_1+\delta_2)\to+\infty} \left(\frac{K}{2}\right) = \frac{K}{2}$$

We can see that when the sum of productivities is sufficiently high, the optimal price for the auctioneer approaches $\frac{3K}{4}$ in the symmetric case and $\frac{K}{2}$ in the maximal case. Note that the optimal price in the symmetric case relates positively with the sum of firms' efficiencies, whereas in the maximal case it is independent from any change in firms' efficiencies. In spite of this, in both cases the optimal price never gets close enough for it to be equal to K as $(\delta_1 + \delta_2)$ rises. Therefore, since a difference between the price cap and the optimal price will always exist, the auctioneer's savings are undefined in the limit because they never stop increasing. Furthermore, we can observe from Figure 8 that the path of savings in the symmetric case has an inverted u-shape while the maximal case is linear. In the first one, changes on productivities affect both the optimal price P_1^s and firms' optimal reports β_{11} and β_{21}, whereas in the second one, changes on productivities leave P_1^m unchanged while increases in δ_2 are indeed the same increases suffered by firm 2's optimal report as it reports truthfully in both periods. Indeed, this explains why variations in savings fluctuate in the symmetric case but are constant in the maximal one.

These results illustrate how the implications for the auctioneer's savings vary upon which equilibrium occurs during the second period. We have seen that, when comparing two distant equilibrium selections in terms of the optimal strategies followed by firms, the auctioneer reduces expenditure in both scenarios as the industry become infinitely efficient in the production of the inventory it demands.

7 Concluding remarks

This paper presents a model with an auctioneer that is forced to buy a fixed amount of inventory over a finite time horizon. As these conditions are known by strategic bidders (firms), the auctioneer must design a mechanism that would allow it to minimize its expenses. We learned that when quantities are fixed and firms compete on prices, the cleared price rises until it reaches the price cap. Thus, from the auctioneer's perspective, it would be more convenient to post a fixed price and let firms compete on quantities, but this would not be credible since firms know the auctioneer must achieve a target inventory before the second period ends. Hence, the auctioneer combines these two alternatives by fixing a price in the first period while letting the price be endogenously determined in the second period. This mechanism is preferable to the case where the auctioneer buys all units in one period, since acquiring units at a price strictly lower than the price cap provides it the chance to save expenses.

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Appendix A: Lemmas and Proofs

Lemma 4.1.1

Proof. Firm's maximization:

$$\max_{\beta_{i2}} \pi(\beta_{i2}) = \left[\frac{Q^R}{\beta_{i2} + \sum_{j\neq i} \beta_{j2}}\right]^2 \beta_{i2}\left(1 - \frac{\beta_{i2}}{2\delta_i}\right)$$

For profits to be null, either $Q^R = 0$, $\beta_{i2} = 0$, or $\beta_{i2} - \frac{\beta_{i2}^2}{2\delta_i} = 0$ must be true. The first option is not possible because if there was not a positive amount of residual demand, the game would be over on the first period since the auctioneer would have already purchased its target inventory $\bar{Q}$. If the second option is true, the profit function would contain the origin. If the third condition is true, then $\beta_{i2} = 2\delta_i$. Therefore, the function crosses the x axis in two points, $(0, 0)$ and $(2\delta_i, 0)$.
Taking the first derivative of the profit function we get:

$$\pi'_{i2}(\beta_{i2}) = \frac{(Q^R)^2}{\delta_i(\beta_{i2} + \sum_{j\neq i} \beta_{j2})^3}\left[\delta_i \sum_{j\neq i} \beta_{j2} - \beta_{i2}\left(\sum_{j\neq i} \beta_{j2} + \delta_i\right)\right]$$

Equating it to zero, we can express the critical value as:

$$\beta_{i2}^* = \frac{\delta_i \sum_{j\neq i} \beta_{j2}}{\delta_i + \sum_{j\neq i} \beta_{j2}}$$

There is only one critical value since cases for which the first derivative does not exist are discarded due to non negative conditions. The second derivative is:

$$\pi''_{i2}(\beta_{i2}) = \frac{(Q^R)^2[\sum_{j\neq i} \beta_{j2}(2\beta_{i2} - 4\delta_i) + 2\beta_{i2}\delta_i - (\sum_{j\neq i} \beta_{j2})^2]}{\delta_i(\beta_{i2} + \sum_{j\neq i} \beta_{j2})^4}$$

Evaluating the critical value (β_{i2}^*) in the second derivative:

$$\pi''_{i2}(\beta_{i2}^*) = \frac{-(Q^R)^2(\sum_{j\neq i} \beta_{j2} + \delta_i)^4}{\delta_1(\sum_{j\neq i} \beta_{j2})^3[2\delta_i + \sum_{j\neq i} \beta_{j2}]^3} < 0$$

Thus the profit function is strictly concave at β_{i2}^*. □

Lemma 4.1.2

Proof. Direct from Lemma 4.1.1, taking first order condition on the firm's maximization problem. □

Proposition 4.1.1

Proof. For $N = 2$ firm i's best response is denoted by:

$$\beta_2(\beta_1) = \frac{\delta_2 \beta_1}{\delta_2 + \beta_1}$$

We omit time subscripts to simplify notation. Deriving both sides of the equation with respect to β_1:

$$\beta_2'(\beta_1) = \frac{\delta_2^2}{(\delta_2 + \beta_1)^2} > 0$$

Evaluating $\beta_1 = 0$ on the first derivative:

$$\beta_2'(0) = \frac{\delta_2^2}{(\delta_2 + 0)^2} = 1$$

Taking the second derivative we get:

$$\beta_2''(\beta_1) = \frac{-2\delta_2^2}{(\delta_2 + \beta_1)^3} < 0$$

Thus the best response function is strictly concave, and since the slope at the origin is equal to one, both functions would not intersect at any other point apart from the origin. Here, as $\beta_1 = \beta_2 = 0$, $P_2 = \frac{Q^R}{\beta_1 + \beta_2}$ explodes to infinity, which is not affordable for the auctioneer and therefore is discarded. Consequently, there is no feasible equilibrium. □

Lemma 4.2.1

Proof. Firms maximize:

$$\max_{\beta_{i2}} \left[\min \left\{ \frac{Q^R}{\beta_{i2} + B}, K \right\} \right]^2 \left(\beta_{i2} - \frac{\beta_{i2}^2}{2\delta_i} \right)$$

There are two relevant cases. The first one is when the minimum price is $\frac{Q^R}{\beta_{i2}+B}$. We know from Lemma 4.1.2 that the optimal report for this maximization is $\beta_{i2} = \frac{\delta_i B}{\delta_i + B}$.

The second case is when the minimum price is K. As the latter is constant, the firm maximization problem is the maximum of $\beta_{i2} - \frac{\beta_{i2}^2}{2\delta_i}$, which is δ_i.

It can be verified that $\delta_i > \frac{\delta_i B}{\delta_i + B}$ since $(\delta_i)^2 > 0$. Therefore, the highest value for β_{i2} is δ_i. □

Lemma 4.2.3

Proof. Firm maximizes:

$$\max_{\beta_{i2} \in [0, \delta_i]} \left\{ \max_{\beta_{i2} \geqslant \frac{Q^R - BK}{K}} \left(\frac{Q^R}{\beta_{i2} + B} \right)^2 \left(\beta_{i2} - \frac{\beta_{i2}^2}{2\delta_i} \right), \max_{\beta_{i2} \leqslant \frac{Q^R - BK}{K}} K^2 \left(\beta_{i2} - \frac{\beta_{i2}^2}{2\delta_i} \right) \right\} \tag{25}$$

Let us first focus on the right hand maximization. If the firm could decide its report freely, it would choose $\beta_{i2} = \delta_i$. But since reports are bounded from above, there are two cases to consider: when $\delta_i < \frac{Q^R - BK}{K}$ and when $\delta_i \geqslant \frac{Q^R - BK}{K}$. If the first case is true, other firms' reports, B, must satisfy:

$$B < \frac{Q^R - K\delta_i}{K}$$

The optimal report is $\beta_{i2} = \delta_i$. Indeed, this case is irrelevant as it was already covered in section 4.2 (condition (5)). Therefore, we are always in the second case where $\delta_i \geqslant \frac{Q^R - BK}{K}$. Here, the optimal report is the upper bound of the support:

$$\beta_{i2} = \frac{Q^R - BK}{K} \tag{26}$$

The reason for the latter is that the function that is being maximized, $\left(\beta_{i2} - \frac{\beta_{i2}^2}{2\delta_i}\right)$, has a unique maximum at $\beta_{i2} = \delta_i$, so it is increasing for the domain considered in the second case. Note that (26) is equivalent to:

$$K = \frac{Q^R}{\beta_{i2} + B} \tag{27}$$

Interestingly, at the optimal report, the price cap is equal to the endogenous price, making the two initial maximization problems of (25) the same. Consequently, conditional on (27), we can reduce the referred problem to:

$$\max_{\left\{\delta_i \geqslant \beta_{i2} \geqslant \frac{Q^R - BK}{K}\right\}} \left(\frac{Q^R}{\beta_{i2} + B}\right)^2 \left(\beta_{i2} - \frac{\beta_{i2}^2}{2\delta_i}\right)$$

To solve this maximization, note that if the firm could freely choose β_{i2}, the solution would be $\beta_{i2} = \frac{\delta_i B}{\delta_i + B}$ (Lemma 4.1.2). However, as reports are bounded, we need to analyze the cases in which the unrestricted optimum is contained in the support of β_{i2} and when it is not contained separately. Following the same logic as before, since the function is increasing until it reaches its unrestricted maximum, if $\frac{Q^R - BK}{K} \leqslant \frac{\delta_i B}{\delta_i + B}$, then the optimal report is $\beta_{i2} = \frac{\delta_i B}{\delta_i + B}$, while if $\frac{Q^R - BK}{K} \geqslant \frac{\delta_i B}{\delta_i + B}$, the optimal report is $\beta_{i2} = \frac{Q^R - BK}{K}$. This is equivalent to:

$$\beta_{i2}^*(B) = \max_{\left\{\delta_i \geqslant \beta_{i2} \geqslant \frac{Q^R - BK}{K}\right\}} \left\{ \left(\frac{\delta_i B}{\delta_i + B}\right), \left(\frac{Q^R - BK}{K}\right) \right\}$$

□

Proposition 4.2.1

Proof. From Lemma 4.2.3, we know that if $B \in \left[\frac{Q^R - K\delta_i}{K}, \frac{Q^R}{K}\right]$, the optimal report for the firm is:

$$\beta^*_{i2}(B) = \max_{\left\{\delta_i \geqslant \beta_{i2} \geqslant \frac{Q^R - BK}{K}\right\}} \left\{ \left(\frac{\delta_i B}{\delta_i + B}\right), \left(\frac{Q^R - BK}{K}\right) \right\} \tag{28}$$

First we want to know for which value of B the two arguments of the maximum are equal. We will define this value as $\widetilde{B}$. In this respect:

$$\left(\frac{\delta_i B}{\delta_i + B}\right) = \left(\frac{Q^R - BK}{K}\right)$$

Or equivalently:

$$B^2 K + B(2K\delta_i - Q^R) - \delta_i Q^R = 0$$

The solution to this quadratic equation is:

$$B = \frac{-(2K\delta_i - Q^R) \pm \sqrt{(2K\delta_i)^2 + (Q^R)^2}}{2K}$$

We will use the positive solution since $(-2K\delta_i + Q^R) < \sqrt{(2K\delta_i)^2 + (Q^R)^2}$, (the sum of the rest of firms' reports) cannot be negative. Consequently:

$$\widetilde{B}_{-i}(\delta_i) = \frac{(Q^R - 2K\delta_i) + \sqrt{4(\delta_i K)^2 + (Q^R)^2}}{2K}$$

Note that (28) is the maximum of two functions that depend on B. The left hand side function is increasing, concave, and converges to δ_i as B goes to infinity. Whereas the right hand side function is linear with slope -1. If we evaluate both functions on $B = \frac{Q^R - K\delta_i}{K}$ and compare them:

$$\delta_i \lesseqgtr \frac{\delta_i(Q^R - K\delta_i)}{Q^R} \quad \Leftrightarrow \quad \delta_i K \lesseqgtr 0$$

Therefore, $\left(\frac{Q^R - BK}{K}\right) > \left(\frac{\delta_i B}{\delta_i + B}\right)$ at $B = \frac{Q^R - K\delta_i}{K}$.
If we evaluate both functions on $B = \frac{Q^R}{K}$ and compare them:

$$\frac{\delta_i Q^R}{\delta_i K + Q^R} \lesseqgtr 0 \quad \Leftrightarrow \quad \delta_i Q^R \lesseqgtr 0$$

Therefore, $\left(\frac{\delta_i B}{\delta_i + B}\right) > \left(\frac{Q^R - BK}{K}\right)$ at $B = \frac{Q^R}{K}$.
From the intermediate value theorem (Carter, 2001), we know this two functions coincide in at least one point. Moreover, as one function is strictly decreasing and the other one is strictly increasing, we know this point exists and is unique. This is a sufficient condition to justify the existence of each segment of the best

response function. □

Proposition 4.2.3

Proof. Suppose every firm $i \in N$ faces an aggregate report from the rest of the firms, B_{-i}, such that it is optimal for it to respond in accordance with its decreasing section of its best response function (Proposition 4.2.1):

$$\beta_i\Big(\sum_{j\neq i}\beta_j\Big) = \frac{Q^R - K\sum_{j\neq i}\beta_j}{K}, \qquad \forall i \in N$$

For this to hold, B_{-i} must be located in the relevant domain for firm i:

$$\frac{Q^R - K\delta_i}{K} \leqslant \sum_{j\neq i}\beta_j \leqslant \widetilde{B}_{-i}(\delta_i)$$

For any arbitrary $m \neq i, m \in N$, this is equivalent to:

$$\frac{Q^R - K\delta_i}{K} \leqslant \beta_m + \sum_{l\neq i,m}\beta_l \leqslant \widetilde{B}_{-i}(\delta_i)$$

Since all firms are reporting accordingly, we can re express the middle term as:

$$\frac{Q^R - K\delta_i}{K} \leqslant \frac{Q^R - K\sum_{j\neq m}\beta_j}{K} + \sum_{l\neq i,m}\beta_l \leqslant \widetilde{B}_{-i}(\delta_i) \quad \Leftrightarrow \quad \frac{Q^R - K\delta_i}{K} \leqslant \frac{Q^R - K\beta_i}{K} \leqslant \widetilde{B}_{-i}(\delta_i)$$

We already know that the lower bound is irrelevant since from Lemma 4.2.1, $\beta_i \leqslant \delta_i$ is always true. The upper bound instead, derives the following condition:

$$\beta_i \geqslant \frac{Q^R - K\widetilde{B}_{-i}(\delta_i)}{K}, \quad \forall i \in N$$

Considering an arbitrary $j \neq i, j \in N$ and adding up through all the firms expect i:

$$\sum_{j\neq i}\beta_j \geqslant (N-1)\frac{Q^R}{k} - \sum_{j\neq i}\widetilde{B}_j$$

□

Lemma 4.3.1

Proof. Direct from Proposition 4.2.2. If $\beta_{22}(\beta_{12}) = \frac{Q^R}{2K}$, for $\beta_{12}(\beta_{22}) = \frac{Q^R}{2K}$ to be an optimal report the following must be true:

$$\beta_{12} \in \Big[\max\Big\{\frac{Q^R - \delta_2 K}{K}, \frac{Q^R - \widetilde{B}_2(\delta_1)K}{K}\Big\}, \widetilde{B}_1(\delta_2)\Big]$$

To show this, we compare the optimal report with each bound to see if it is actually contained in the interval.

$$\begin{array}{lll} (i) & \dfrac{Q^R}{2K} \lesseqgtr \dfrac{Q^R - \delta_2 K}{K} & \Leftrightarrow \quad 2\delta_2 K \lesseqgtr Q^R \\ (ii) & \dfrac{Q^R}{2K} \lesseqgtr \dfrac{Q^R - \widetilde{B}_2(\delta_1)K}{K} & \Leftrightarrow \quad Q^R \lesseqgtr 0 \\ (iii) & \dfrac{Q^R}{2K} \lesseqgtr \widetilde{B}_1(\delta_2) & \Leftrightarrow \quad 0 \lesseqgtr (Q^R)^2 \end{array}$$

The last two conditions are direct and allow us to conclude that $\frac{Q^R}{2K} > \frac{Q^R - \widetilde{B}_2(\delta_1)K}{K}$ and $\frac{Q^R}{2K} < \widetilde{B}_1(\delta_2)$ at $\beta_{22} = \frac{Q^R}{2K}$. Condition (i) can be solved with Assumption 3.1 denoted at the beginning of the paper which stands $2\delta_i K > \bar{Q}$. Since $\bar{Q} \geqslant Q^R$, it is also true that $2\delta_i K > Q^R$ and therefore $\frac{Q^R}{2K} > \frac{Q^R - \delta_2 K}{K}$. □

Lemma 4.3.2

Proof. Firm's maximize:

$$\max_{\beta_{i1}} \pi_{i1}(\beta_{i1}) = P_1^2 \left(\beta_{i1} - \frac{(\beta_{i1})^2}{2\delta_i} \right) + K^2 \left[\frac{\bar{Q} - P_1\beta_{i1} - P_1\beta_{j1}}{2K} - \frac{(\bar{Q} - P_1\beta_{i1} - P_1\beta_{j1})^2}{8\delta_i K^2} \right]$$

Taking the first derivative of the profit function we get:

$$\pi'_{i1}(\beta_{i1}) = P_1^2 \left(1 - \frac{\beta_{i1}}{\delta_i} \right) + K^2 \left[\frac{-P_1}{2K} + \frac{2P_1(\bar{Q} - P_1\beta_{i1} - P_1\beta_{j1})}{8\delta_i K^2} \right]$$

Equating it to zero, we obtain the critical value:

$$\beta_{i1} = \frac{4P_1\delta_i + \bar{Q} - P_1\beta_{j1} - 2K\delta_i}{5P_1}$$

The second derivative is:

$$\pi''_{i1}(\beta_{i1}) = \frac{-5P_1^2}{4\delta_i} < 0$$

Thus the profit function if strictly concave. □

Proposition 4.3.2

Proof. The auctioneer minimize expenses as:

$$\min_{P_1} \mathcal{E}(P_1) = P_1(P_1 - K) \left(\frac{\bar{Q} - K(\delta_1 + \delta_2) + 2P_1(\delta_1 + \delta_2)}{3P_1} \right) + K\bar{Q}$$

Note that expenditure cannot be null because K and $\bar{Q}$ are strictly positive. Taking the first derivative:

$$\mathcal{E}'(P_1) = -K(\delta_1 + \delta_2) + \frac{\bar{Q}}{3} + \frac{4}{3}P_1(\delta_1 + \delta_2)$$

Equating it to zero, we obtain the critical value:

$$P_1 = \frac{3K(\delta_1+\delta_2)-\bar{Q}}{4(\delta_1+\delta_2)}$$

The second derivative is:

$$\mathcal{E}''(P_1) = \frac{4}{3(\delta_1+\delta_2)} > 0$$

Thus the expense function is strictly convex. □

Lemma 4.4.1

Proof. Each firm maximizes solve the following:

$$\max_{\beta_{11}} \pi_{11}(\beta_{11}) = P_1^2\Big(\beta_{11} - \frac{\beta_{11}^2}{2\delta_1}\Big) + K^2\Big[\frac{-\delta_2 K - (\beta_{11}+\beta_{21})P_1 + \bar{Q}}{K} - \frac{(-\delta_2 K - (\beta_{11}+\beta_{21})P_1 + \bar{Q})^2}{2\delta_1 (K)^2}\Big]$$
$$\max_{\beta_{21}} \pi_{21}(\beta_{21}) = P_1^2\left(\beta_{21} - \frac{\beta_{21}^2}{2\delta_2}\right) + \frac{\delta_2 K^2}{2}$$

We will start analyzing firm 1. Taking first derivative we get:

$$\pi'_{11}(\beta_{11}) = (P_1)^2\Big(1-\frac{\beta_{11}}{\delta_1}\Big) + K^2\Big[\frac{-P_1}{K} + \frac{P_1(\bar{Q} - P_1\beta_{11} - P_1\beta_{21} - \delta_2 K)}{\delta_1 K^2}\Big]$$

Equating it to zero, we can express the critical value as:

$$\beta_{11}(\beta_{21}) = \frac{P_1\delta_1 - K(\delta_1+\delta_2) + \bar{Q} - \beta_{21}P_1}{2P_1}$$

Second derivative is:

$$\pi''_{11}(\beta_{11}) = \frac{-2(P_1)^2}{\delta_1} < 0$$

Now we study the problem of firm 2. Taking first derivative we get:

$$\pi'_{21}(\beta_{21}) = (P_1)^2\Big(1-\frac{\beta_{21}}{\delta_2}\Big)$$

Equating it to zero, we can express the critical value as:

$$\beta_{21} = \delta_2$$

Second derivative is:

$$\pi''_{21}(\beta_{21}) = \frac{-(P_1)^2}{\delta_2} < 0$$

□

Proposition 4.4.2

Proof. The auctioneer minimize expenses as:

$$\min_{P_1} \mathcal{E}(P_1) = P_1(P_1 - K)(\delta_2) + K\bar{Q}$$

Note that expenditure cannot be null because K and $\bar{Q}$ are strictly positive. Taking the first derivative:

$$\mathcal{E}'(P_1) = \delta_2(2P_1 - K)$$

Equating it to zero, we obtain the critical value:

$$P_1 = \frac{K}{2}$$

The second derivative is:

$$\mathcal{E}''(P_1) = 2\delta_2 > 0$$

Thus the expense function if strictly convex. □

Appendix B: About private information on parameters

The objective of this section is to give a brief overview on how the sequential auction model of this paper changes with the introduction of incomplete information. Under the same setting presented in section 3, suppose now that firms' efficiency parameters, $\delta_i \in [0, +\infty]$, are no longer common information but private information drawn independently from a common distribution, which for simplicity is assumed to be uniform between zero and one ($\delta_i \sim U[0,1]$). Following the logic of section 4.1, we consider the second stage of the auction model with a restrictive price cap and two firms, referred to as firm i and firm j. As a consequence of information asymmetry, the model turns into a Bayesian game characterized by:

$$\langle\, I,\ \{\beta_i\}_{i \in I},\ \{\delta_i\},\ \{\pi_i\}_{i \in I},\ f\, \rangle$$

Where $I = 2$ corresponds to the number of firms competing in the auction, and $\beta_i : [\underline{\delta_i}, \overline{\delta_i}] \to \mathbb{R}_+$ is the strategy played by firm i, which is a function that goes from the set of types to the set of strategies it can take. In addition, $\delta_i \in \Delta_i$ corresponds to each firm's type, which reflects their level of productivity, π_i is the profit function of firm i and f is a probability density function that describes the probability of occurrence of a specific type of realization.

We are looking for a symmetric equilibrium, which means solving the problem faced by firm i while assuming that the other firm is also playing the same strategy (i.e. same function β_j). Notice that the two firms' types are equal *ex-ante* but they become different once their types are realized and privately announced. In pursuance of simplifying the firms' profit maximization problem, we assume that they produce inventory goods with a cost function denoted by: $C_i(Q_i) = \frac{\delta_i Q_i^2}{2}$. The only difference with the cost function described in our main model is that the interpretation of δ_i is reversed, as now higher values imply lower productivity and vice-versa. From a supply function equilibrium, firms submit their marginal cost functions by announcing their strategies:

$$\widetilde{C_i'}(Q_i) = \beta(\delta_i) Q_i$$

Note that, unlike the model, firms report functions of their parameters under certainty, instead of reporting values for their parameters directly. Moreover, for a given price of the second period, firms will offer until their marginal cost equals P_2, therefore:

$$P_2 = \beta(\delta_i) Q_i \quad \Leftrightarrow \quad Q_i = \frac{P_2}{\beta(\delta_i)}$$

Since the auctioneer is committed to fix P_2 such that it manages to buy the remaining units needed to reach $\bar{Q}$, we can express this endogenous price as:

$$Q^R = Q_i + Q_j \quad \Leftrightarrow \quad Q^R = \frac{P_2}{\beta(\delta_i)} + \frac{P_2}{\beta(\delta_j)} \quad \Leftrightarrow \quad P_2 = \frac{Q^R \beta(\delta_i)\beta(\delta_j)}{(\beta(\delta_i) + \beta(\delta_j))} \tag{29}$$

That is, price can be expressed in terms of firms' strategies. Now we are in conditions to solve the firms' maximization problem. This consists on choosing the optimal strategy that enables a firm to obtain the highest feasible profit from selling inventory units to the auctioneer, given that the other firm is playing its optimal strategy. In this respect, profits perceived by firm i can be expressed as:

$$\pi_i = P_2 Q_i - \frac{\delta_i Q_i^2}{2} \quad \Leftrightarrow \quad P_2\left(\frac{P_2}{\beta(\delta_i)}\right) - \frac{\delta_i}{2}\left(\frac{P_2}{\beta(\delta_i)}\right)^2 \quad \Leftrightarrow \quad P_2^2\left[\frac{1}{\beta(\delta_i)} - \frac{\delta_i}{2\beta(\delta_i)^2}\right] \tag{30}$$

Since firm i does not know the productivity parameter of firm j, δ_j, it maximizes over its own expected utility. Replacing (29) in (30), the firms' maximization problem can be written as:

$$\max_{\beta(\delta_i)} \int_{\underline{\delta_j}=0}^{1} \left[\frac{Q^R \beta(\delta_j)}{(\beta(\delta_i) + \beta(\delta_j))}\right]^2 \left[\beta(\delta_i) - \frac{\delta_i}{2}\right] f(\delta_j) d\delta_j$$

Taking the first derivative and equating it to zero, we can characterize the optimal strategy $\beta(\delta_i)^*$ as the best response to the optimal strategy made by the other firm:

$$\int_{\underline{\delta_j}=0}^{1} \left[\frac{(Q^R \beta(\delta_j))^2}{(\beta(\delta_i)^* + \beta(\delta_j))^3}\right] \big(\beta(\delta_j) - \beta(\delta_i)^* + \delta_i\big) f(\delta_j) d\delta_j = 0 \tag{31}$$

Since Q^R is taken as a constant we can omit it from the integral equation. This happens because, as we are solving the second period and the amount sold in the previous period is known, firms are aware of the residual demand when they solve their optimization problem. Thus, (31) can be rewritten as:

$$\int_{\underline{\delta_j}=0}^{1} \frac{(\beta(\delta_j))^3}{(\beta(\delta_i)^* + \beta(\delta_j))^3} f(\delta_j) d\delta_j = (\beta(\delta_i)^* - \delta_i) \int_{\underline{\delta_j}=0}^{1} \frac{(\beta(\delta_j))^2}{(\beta(\delta_i)^* + \beta(\delta_j))^3} f(\delta_j) d\delta_j$$

A natural way to proceed from this point forward is to guess a linear function for the symmetric equilibrium strategy $\beta(\delta_i)$ such that the integral equation holds. This method works in various contexts. For instance, in Laplace (2016) and Vives (2011). In our particular case we conjectured $\beta(\delta_i) = \alpha\delta_i$ and $\beta(\delta_i) = \alpha\delta_i + \gamma$ as linear guesses but none of them worked. Hence, we conclude that the Bayesian game of the second stage is not linear, thus it is preferable that it be solved numerically.

Appendix C: Code

Document 1: Defining variables

```
% Definitions and Notation:
% beta_it (i=firm, t=period) = report of firm i in period t.
% profit_it (i=firm, t=period) = profits perceived by firm i at period t.
% exp_auctioneer= total expenses made by auctioneer.
% B1_kink= value of aggregate reports of the rest of the firms for which
            %the best response function changes from strategic substitutes
            %to strategic complements.
% QR=residual demand.
% delta_1=efficiency parameter of firm 1.
% delta_2=efficiency parameter of firm 2.
% K=Price cap
% p1=unit unit price at period 1, announced by the auctioneer.

% Variables in their reduced form:
QR=qbar-p1.*(beta_11+beta_21)
root=sqrt((QR).^2+(2.*K.*delta_1).^2)
B1_kink=(QR-(2.*K.*delta_1)+root)./(2.*K)
beta_12=(QR-B1_kink.*K)./(K)
beta_22=(QR-beta_12.*K)./(K)
profit_12=K.^2.*((beta_12)-((beta_12).^2./(2.*delta_1)))
profit_22=K.^2.*((beta_22)-((beta_22).^2./(2.*delta_2)))
profit_11=p1.^2.*(beta_11-((beta_11).^2./(2.*delta_1)))+(profit_12)
profit_21=p1.^2.*(beta_21-((beta_21).^2./(2.*delta_2)))+(profit_22)
exp_auctioneer=p1^2.*(beta_11+beta_21)+K^2.*(beta_12+beta_22)

% Variables in their extensive form:
QR=qbar-p1.*(beta_11+beta_21)
root=sqrt((qbar-p1.*(beta_11+beta_21)).^2+(2.*K.*delta_1).^2)
B1_kink=(qbar-p1.*(beta_11+beta_21)-(2.*K.*delta_1)+root)./(2.*K)
beta_12=(qbar-p1.*(beta_11+beta_21)-((qbar-p1.*(beta_11+beta_21)-...
        (2.*K.*delta_1)+(sqrt((qbar-p1.*(beta_11+beta_21)).^2+...
        (2.*K.*delta_1).^2)))./(2.*K)).*K)./(K)
beta_22=(qbar-p1.*(beta_11+beta_21)-((qbar-p1.*(beta_11+beta_21)-...
```

```
        ((qbar-p1.*(beta_11+beta_21)-(2.*K.*delta_1)+(sqrt((qbar-p1.*...
        (beta_11+beta_21)).^2+(2.*K.*delta_1).^2))./(2.*K)).*K)./...
        (K)).*K)./(K)
profit_12=K.^2.*(((qbar-p1.*(beta_11+beta_21)-((qbar-p1.*(beta_11+...
          beta_21)-(2.*K.*delta_1)+(sqrt((qbar-p1.*(beta_11+...
          beta_21)).^2+(2.*K.*delta_1).^2))./(2.*K)).*K)./(K))-...
          (((qbar-p1.*(beta_11+beta_21)-((qbar-p1.*(beta_11+beta_21)...
          -(2.*K.*delta_1)+(sqrt((qbar-p1.*(beta_11+beta_21)).^2+...
          (2.*K.*delta_1).^2))./(2.*K)).*K)./(K)).^2./(2.*delta_1)))
profit_22=K.^2.*(((qbar-p1.*(beta_11+beta_21)-((qbar-p1.*(beta_11+...
          beta_21)-((qbar-p1.*(beta_11+beta_21)-(2.*K.*delta_1)+...
          (sqrt((qbar-p1.*(beta_11+beta_21)).^2+(2.*K.*delta_1).^2))...
          ./(2.*K)).*K)./(K)).*K)./(K))-(((qbar-p1.*(beta_11+beta_21)-...
          ((qbar-p1.*(beta_11+beta_21)-((qbar-p1.*(beta_11+beta_21)-...
          (2.*K.*delta_1)+(sqrt((qbar-p1.*(beta_11+beta_21)).^2+(2.*...
          K.*delta_1).^2))./(2.*K)).*K)./(K)).*K)./(K)).^2./(2.*delta_2))
             )
profit_11=p1.^2.*(beta_11-((beta_11).^2./(2.*delta_1)))+(K.^2.*...
          (((qbar-p1.*(beta_11+beta_21)-((qbar-p1.*(beta_11+beta_21)-...
          (2.*K.*delta_1)+(sqrt((qbar-p1.*(beta_11+beta_21)).^2+(2.*...
          K.*delta_1).^2))./(2.*K)).*K)./(K))-(((qbar-p1.*(beta_11+...
          beta_21)-((qbar-p1.*(beta_11+beta_21)-(2.*K.*delta_1)+...
          (sqrt((qbar-p1.*(beta_11+beta_21)).^2+(2.*K.*delta_1).^2))...
          ./(2.*K)).*K)./(K)).^2./(2.*delta_1))))
profit_21=p1.^2.*(beta_21-((beta_21).^2./(2.*delta_2)))+(K.^2.*(((qbar-...
          p1.*(beta_11+beta_21)-((qbar-p1.*(beta_11+beta_21)-((qbar-...
          p1.*(beta_11+beta_21)-(2.*K.*delta_1)+(sqrt((qbar-p1.*...
          (beta_11+beta_21)).^2+(2.*K.*delta_1).^2))./(2.*K)).*K)./...
          (K)).*K)./(K))-(((qbar-p1.*(beta_11+beta_21)-((qbar-p1.*...
          (beta_11+beta_21)-((qbar-p1.*(beta_11+beta_21)-(2.*K.*...
          delta_1)+(sqrt((qbar-p1.*(beta_11+beta_21)).^2+(2.*K.*...
          delta_1).^2))./(2.*K)).*K)./(K)).*K)./(K)).^2./(2.*delta_2))))
auctioneer_exp=p1.*(beta_11+beta_21)+K^2.*(beta_12+beta_22)
```

Document 2: Simulation

```
clc
clear
close all

%MODEL PARAMETERS
p1=30                   % Initial value for price of the first period.
qbar=354.55             % Target inventory
K=40                    % Unit price of outside option.
delta_1=3.28            % Efficiency parameter of firm 1.
delta_2=6.5             % Efficiency parameter of firm 2.
q_beta=1000             % Number of feasible reports from firm 1 and firm 2.
max_beta_11=3.28        % Maximum feasible report for firm 1.
max_beta_21=6.5         % Maximum feasible report for firm 2.
min_beta_11=0.5         % Minimum feasible report for firm 1.
min_beta_21=0.5         % Minimum feasible report for firm 2.
min_p1=15               % Minimum feasible price of period 1.
max_p1=39.5             % Maximum feasible price of period 1.
q_p1=1000               % Number of feasible prices for period 1.
tolerance=5;            % Tolerance between iterations of best response
                        % functions with itself for firm 1 and firm 2.
%%
%GRIDS
grid_beta_11=linspace(min_beta_11,max_beta_11,q_beta)';
%Grid, column vector of "q_beta" feasible reports for firm 1.
grid_beta_21=linspace(min_beta_21,max_beta_21,q_beta)';
%Grid, column vector of "q_beta" feasible reports for firm 2.
grid_p1=linspace(min_p1,max_p1,q_p1);
%Grid of "q_1" feasible prices for the first period.
dif=1000;
%Initial diference for firms optimization problem.

for j=1:q_p1
p1=grid_p1(j)

```

```
while dif>tolerance

% PROBLEM OF FIRM 1 IN PERIOD ONE
%This part solves the maximization of total profits of firm 1 (profit_11).
%The main idea is to find which beta_11, that belongs to its grid,
%profit_11 is maximized, given a specific value for beta_21.
%That is, for each value of beta_21 of its grid, we found the value of
%beta_11 such that total profits of firm 1 are maximized.

for i=1:q_beta;
beta_11=grid_beta_11;      %beta_11 is equal to its grid.
beta_21=grid_beta_21(i);   %beta_21 takes each value of its grid at a time
    .
profit_11=p1.^2.*(beta_11-((beta_11).^2./(2.*delta_1)))+(K.^2.*...
          (((qbar-p1.*(beta_11+beta_21)-((qbar-p1.*(beta_11+beta_21)...
          -(2.*K.*delta_1)+(sqrt((qbar-p1.*(beta_11+beta_21)).^2+...
          (2.*K.*delta_1).^2)))./(2.*K)).*K)./(K))-(((qbar-p1.*...
          (beta_11+beta_21)-((qbar-p1.*(beta_11+beta_21)-(2.*K.*...
          delta_1)+(sqrt((qbar-p1.*(beta_11+beta_21)).^2+(2.*K.*...
          delta_1).^2)))./(2.*K)).*K)./(K)).^2./(2.*delta_1))));
[value_1,position_1]=max(profit_11);
%Here we maximize profits_11. "value_1" corresponds to the value of
%beta_11 %that maximizes profit_11. "position_1" corresponds to the
%row in which "value_1" is located in the grid.
BR_firm1(i,:)=[beta_21,grid_beta_11(position_1)];
%Best response function of firm 1 given beta_21: beta_11(beta_21).
%Two column vector. First column are reports given by firm 2 and
%the second colum is the best report of firm 1 to each beta_21.
beta11op(i,1)=grid_beta_11(position_1);
%Optimal reports of beta_11 given beta_21.
end

%PROBLEM OF FIRM 2 IN PERIOD ONE
%This part solves the maximization of total profits of firm 2 (profit_21).
%The main idea is to find which beta_21, that belongs to its grid,
%profit_21 is maximized, given a specific value for beta_11.
```

```
%The difference with the problem of firm 1 solved above is that in
%this case, we take optimal reports of firm 1 in the first period as their
%initial values, intead of using the grid constructed at the begining.

for i=1:q_beta;
    beta_11=beta11op(i);
%beta_11 takes each optimal value calculated in the previous
%problem. The intuition of this is that instead of starting with
%initial values, we take the optimal reports just calculated as
%firm 2 anticipates the best response of firm 1 in its problem.
    beta_21=grid_beta_21;
%beta_21 is equal to its grid.
profit_21=p1.^2.*(beta_21-((beta_21).^2./(2.*delta_2)))+...
          (K.^2.*(((qbar-p1.*(beta_11+beta_21)-((qbar-p1.*...
          (beta_11+beta_21)-((qbar-p1.*(beta_11+beta_21)-...
          (2.*K.*delta_1)+(sqrt((qbar-p1.*(beta_11+beta_21)).^2+...
          (2.*K.*delta_1).^2)))./(2.*K)).*K)./(K)).*K)./(K))...
          -(((qbar-p1.*(beta_11+beta_21)-((qbar-p1.*...
          (beta_11+beta_21)-((qbar-p1.*(beta_11+beta_21)-...
          (2.*K.*delta_1)+(sqrt((qbar-p1.*(beta_11+beta_21)).^2+...
          (2.*K.*delta_1).^2)))./(2.*K)).*K)./(K)).*K)./(K)).^2./...
          (2.*delta_2))));
[value_2,position_2] = max(profit_21);
%Here we maximize profits_21. "value_2" corresponds to the value
%of beta_21 that maximizes profit_21. "position_2" corresponds to
%the row in which "value_2" is located in the grid.
BR_firm2(i,:)=[beta_11,grid_beta_21(position_2)];
%Best response function of firm 2 given beta_11: beta_21(beta_11).
beta21op(i,1)=grid_beta_21(position_2);
%Optimal reports of beta_21, given beta_11.
end
%Here we define the difference between the optimal reports (beta_11op,
%beta_21op) with their respectives initial grids. This are two column
%vectors of 1000 rows and one column. Then, we take the maximal value of
%the vetor of differences (for each firm) and equate that the maximum of
%those two to differences with "dif", so that the algorithm iterates until
```

```
%"dif" is below "tolerance"
dif1=max(abs(beta11op-grid_beta_11));
dif2=max(abs(beta21op-grid_beta_21));
dif=max(dif1,dif2)
%This step is crucial: we rename each firm optimum with their respective
%initial grids for them to update in each round of the loop.
grid_beta_11=beta11op;
grid_beta_21=beta21op;
end

%%
%Here we identify the domain and the range of each firm's best response
%function derived previously, defining variables for each one of those.
%In other words, we define variables for each column of the best response
%functions.
b_21=BR_firm1(:,1);
b_11_optimum=BR_firm1(:,2);
b_11=BR_firm2(:,1);
b_21_optimum=BR_firm2(:,2);
%Here we identify the point were the two best response functions
%intersect. We have beta_11 optimum for a given beta_22, and
%we also have a beta_21 optimum for a given beta_11 (these are 2
%best responses functions). To find the intersection, we look the
%position where the domain (input) of BR_firm1 is equal to the range
%(output) of BR_firm2. We find the value and position where the two
%beta_21 are equal, which is %the point where the best reponse of firm 1
%given a report of firm 2, is at %the same time the best response of
%firm 2 given a report of firm 1 (nash %equilibrium).
dif_beta_21=(abs(b_21-b_21_optimum));
%This is a column vector of the difference of the two beta_21.
[value_3,position_3]=min(dif_beta_21);
%We find the minimum of the column vector.
beta_21_STAR=b_21_optimum(position_3,1)
beta_11_STAR=b_11(position_3,1)
%These are the values for which the best response functions of each firm
%coincide.
```

```

%Now that we find the optimal reports of each firm for period one, we
%procede to solve the auctioneers minimization problem.
auctioneer_exp(j,1)=p1.*(beta_11_STAR+beta_21_STAR).*(p1-K)+(K.*qbar);
%Function of expenses of the auctioneer that depends on the optimal
%reports of period one.
end
%%
%Here we minimize the expenses function of the auctioneer.
[value_4,position_4]=min(auctioneer_exp);
p1_STAR=grid_p1(position_4);
%This is the optimal for the auctioneer to minimize its expenses.

%Figures
%Best Response figure
%We have to be very carefully on how we define the "x" and "y" variables
%in order to grapg the two best responses functions in the same axis.
%We design it for beta_21 to be in the "y" axis and "beta_11" in the "x"
%axis.
figure(1)
plot(b_11_optimum,b_21,b_11,b_21_optimum)
hold on
plot(b_11_optimum(369),b_21(369),b_11(369),b_21_optimum(369),'r*',...
    'MarkerEdgeColor','black','MarkerFaceColor','black')
ylabel('\beta_{21}(\beta_{11})')
xlabel('\beta_{11}(\beta_{21})')
title('Best Responses')
legend('\beta_{11}(\beta_{21})','\beta_{21}(\beta_{11})')
hold off
%Auctioneer minimization figure
figure (2)
plot(grid_p1,auctioneer_exp)
hold on
plot(grid_p1(205),auctioneer_exp(205),'r*',...
    'MarkerEdgeColor','black','MarkerFaceColor','black')
ylabel('Auctioneer''s total expenditure')
```

```
xlabel('P_{1}')
title('Auctioneer''s problem')
%%
beta_12_star=(qbar-p1_STAR.*(beta_11_STAR+beta_21_STAR)-((qbar-...
             p1_STAR.*(beta_11_STAR+beta_21_STAR)-(2.*K.*delta_1)+...
             (sqrt((qbar-p1_STAR.*(beta_11_STAR+beta_21_STAR)).^2+...
             (2.*K.*delta_1).^2))./(2.*K)).*K)./(K);
beta_22_star=(qbar-p1_STAR.*(beta_11_STAR+beta_21_STAR)-((qbar-...
             p1_STAR.*(beta_11_STAR+beta_21_STAR)-((qbar-p1_STAR.*...
             (beta_11_STAR+beta_21_STAR)-(2.*K.*delta_1)+(sqrt...
             ((qbar-p1_STAR.*(beta_11_STAR+beta_21_STAR)).^2+...
             (2.*K.*delta_1).^2))./(2.*K)).*K)./(K)).*K)./(K);
```