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BREATH OF FRESH AIR IN POLITICS: A MODEL OF STRATEGIC ENTRY WITH ENDOGENOUS VALENCE

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Breath of fresh air in politics: A model of strategic entry with endogenous valence characteristics

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Abstract

This work proposes a theoretical framework to address the issue of how an entrant manages to differentiate from incumbents on political competition, specifically on non-policy related attributes, despite the inherent difficulties of entry, such as economic and legal barriers, reduced capacities for campaigning, resource limitations, among others. The framework shows that when the entrant is relatively better valued than the incumbents on non-policy factors, the threat of entry forces the incumbents to allocate more resources on improving their non-policy valuation from the electorate. Also, that entrant party tends to adopt moderate policy positions when incumbents choose similar platforms. Finally, there exists a multiplicity of equilibria on policy competition among the incumbents, with only one symmetric equilibrium.

Keywords: Electoral competition, Valence, Third-party entry

JEL classification: C72, D72

1 Introduction

In last years politics, new forms of activism threaten the incumbent parties on reaching for voters that have not been represented in the political discussion. Migration, gender equality or environmentalism have been topics that dominant parties have not been able to fully grasp and have created opportunities for new parties to compete against them. New parties will naturally face difficulties when facing incumbent parties in an electoral competition, e.g. entry costs, reduced capacities to reach the electorate, limitation on resources, among others, but they can take advantage on the fact that they do not share the same political background as incumbents.

Intuitively, and anticipating how voters will conduct themselves, parties should lead to prudential withdrawals whenever their chances of winning are low enough, and therefore reducing the number of competitors in the electoral contest. More specifically, parties foreseeing that their lists will face desertion if they are not the stronger parties are more likely to avoid mounting a hopeless campaign and prefer to support more viable candidates. Nonetheless, why do new parties keep emerging and winning votes in current democracies? As Tavits (2008) affirms "new entries and their success is not an exception but rather an integral part of electoral competition in new democracies".

In fact, current representative democracies face the following problem that arises from the electoral competition itself: parties will place their candidates on the governmental offices only when they can appropriately convince a sufficient number of voters. As countries grow and develop, social, economic and culturally, already established parties need to adapt to new challenges and aim to reconnect with citizens, otherwise, there is a window of opportunity for the formation of new parties that would be able to do so. New parties bring breaths of "fresh air" to political competition and thus can give them an appropriate advantage when running their political campaigns. Nowadays, politically motivated young people tend to look to non-mainstream political arenas populated by NGOs and new social movements, which are alternative forms of political activism that work at the margin of the established political sphere. As Loader (2007) affirms,

“Young people are not necessarily any less interested in politics than previous generations, but traditional political activity no longer appears appropriate to address the concerns associated with contemporary youth cultures.”

The literature requires understanding the conditions under which new parties emerge to compete in political elections and whether non-policy characteristics, such as campaign skills, name recognition, integrity, competence, or dedication to public service, play a favorable role in their entry decisions or not. The purpose of this work is to present a formal model to answer how non-policy characteristics of new parties influence their strategical behavior on a political competition against already established parties while providing insights on how would these incumbents react to the entry of new competitors.

2 Literature Review

The literature on electoral competition began with the classical Downsian model (Downs, 1957) that predicts that when two parties compete on electoral competition, both of them would converge to the position of the median voter. Empirical evidence however has found that parties diverge on the election of political platforms (Ansolabehere, Snyder, and Stewart, 2001; Burden, 2004). The downsian model is based on several assumptions, whose results have been proven to be not robust to perturbations. For example, uncertainty on candidates positions and risk-averse behaviours (Berger, Munger, and Potthoff, 2000), party activists (Aldrich, 1983b; Aldrich, 1983a) or candidate policy preferences and uncertainty (Wittman, 1983; Calvert, 1985).

In the literature, three families of models have been developed to explain electoral competition with endogenous candidacy (Bol, Dellis, and Oak, 2016), based on candidates motivation (office-motivated or policy-motivated) and timing of entry (sequential or simultaneous). Models of entry deterrence are those in which candidates seek to maximize electoral performance (i.e. vote shares or probabilities of winning) and act sequentially, thus incumbent parties will choose strategies *ex-ante* to deter entry from a third party. Then, there are the Hotelling-Downs models with strategical candidacy, which differ from the previous kind by letting all parties decide their candidacy simultaneously. And finally, the third type of models are citizen-candidate models, which consider simultaneous entry decisions, but in contrast, candidates are policy-motivated and care intrinsically about the policy outcome of the election. Palfrey (1984) and Weber (1992, 1997) are the seminal contributions to the first type of model, whereas Osborne (1993) is for the second type, and Osborne and Slivinski (1996) and Besley and Coate (1997) are the seminal contributions to the third one.

Particularly, Palfrey (1984) examines spatial equilibrium in political competition when established parties choose their policy platforms, anticipating the entry of a vote-maximizing third party. In his work, the author considers a three-party non-cooperative game, in which incumbent parties compete over policy location simultaneously on a first stage, anticipating the potential threat of entry of a third party that would choose its own policy location in the subsequent stage. Palfrey finds that the threat of entry forces the incumbents to differentiate on the policy space, preventing the entry of the third party. Under standard assumptions, he finds that incumbent parties locate at one-fourth and three-fourths. Posteriorly, Weber (1992) proposes a model similar to Palfrey (1984), relaxing some of the assumptions over the distribution of voters' ideal policies; whereas Weber (1997) considers a setting with vote-maximizing parties in which the entrant decides to participate if and only if she expects to receive some fraction of the votes. Callander (2005a, 2005b) and Callander and Wilson (2007) consider settings in which candidates seek to maximize their winning probability and an entrant contest the election if and only if she has positive probabilities of winning the elections.

This literature has been complemented by the inclusion of non-policy or valence characteristics into the analysis, whereas now voters do not only value the location of the party on the policy space, but also take into consideration non-policy related factors such as campaign skills, name recognition, integrity, competence, or dedication to public service when casting their votes (Calvert, 1985; Wittman, 1983; Groseclose, 2001; Adams, Merrill III, Simas, and Stone, 2011). In Palfrey's model, voters are only concerned about the policy and vote so accordingly, but in reality, major parties have large advantages over new minor parties. Groseclose (2001) proposes a model of candidate location in which one of the candidates presents a valence advantage over the other, showing that this advantage alters the equilibrium in two ways. First, it causes the disadvantaged candidate to move away from the center, and second, it causes the advantaged candidate to move towards the center. In his setting, the advantaged candidate chooses to moderate his platform, whereas the disadvantaged chooses an extreme position. Adams et al. (2011) examine a two-candidate model, in which the incumbent and the entrant locate along a one-dimensional policy space and voters' decisions are based on the political location of the parties and on the candidates' valence attributes. Under those conditions, they show that entrants that are superior to the incumbent in valence characteristics have incentives to moderate their policy positions, and move towards the center. Finally, Zhakarov (2009) analyses a model of spatial policy competition with endogenous second-stage valence competition between two-office-motivated candidates under general assumptions about the distribution of voter policy preferences, voter disutility, and cost of valence functions. He finds that candidates will expect to spend more to improve their valence attributes in equilibrium and will choose differentiated policy platforms in order to reduce the cost of campaigning.

Even though the previous literature has analyzed the equilibria of political competition games by including the influence of non-policy characteristics, it has not been able yet to explain the entry on new competitors through the flexibilities they possess from grasping issues that incumbents have not been able to fully do. This work differs from its predecessors on the following aspects. The model corresponds to a three-stage game among three benefits-maximizing parties, consisting of a two-stage spatial party competition and an endogenous valence competition, in which the entrant party has a relative valence advantage over the two incumbents. Most previous models in the literature have considered exogenous valence characteristics, whereas this work, as in Zhakarov (2009), considers a costly valence function. As in Adams et. al (2011), we consider the existence of relative valuations among the valence characteristics of each party, which means that even having the same traits, a party has an advantage when valued by the electorate.

3 The Model

There is a continuum of citizens in society, each characterized by a single-peaked political preference over the interval $[0, 1]$. Citizens have a utility cost when voting for a candidate whose representative policy is different from their ideal policy. We will assume that this cost is given by a quadratic function of the distance between the party's policy and the citizen's ideal policy. Additionally, they have preferences over the consumption of a public good $g \in \mathbf{R}_+$. Therefore, a citizen's total preferences can be represented by an additive utility function, that depends on the amount of public good g that a party promises and the representative policy $x_i \in [0, 1]$ of the party. Note that this function is increasing and twice differentiable on its support. The utility function of a voter with a single-peak preference x is as follows:

$$U_x(x_i, g_i) = g_i - k\,(x - x_i)^2$$

where x_i and g_i are the policy location and public good provision of a party i, respectively. The parameter $k > 0$ represents the utility cost-per-unit of displacement on the policy space that the individual would incur when voting for a party with a different political position than its own.

There is a set $I = \{l, r, e\}$ of political parties in society, in which l and r are the incumbent parties ("Left" and "Right") and party e is a potential entrant party. The parties are represented by a candidate, but for the purpose of the model, I will refer to them interchangeably. Each party is office-motivated, thus they are only concerned about maximizing their expected benefits. Incumbent parties have to make two decision when competing: the election of a policy allocation and an amount of public good they will compromise. However, the entrant party has to additionally decide whether or not to compete against the incumbents. If the party decides to participate, it must incur in an entry cost c, which for the simplicity of the analysis is taken to be zero[1].

Voters value differently the public good depending on whom is providing it. Let us consider that the provision of a public good for any party is represented by a function $\phi_i(g) = \lambda_i g$, where the parameter $\lambda_i \in \mathbf{R}_+$ represents the capacity of the party to provide a public good. We will assume $\lambda_i = 1$ if the incumbent parties provide the public good and $\lambda_i > 1$ for all $i \in I$. This assumption reflects that the entrant party has a broader set of choices for the provision of a public good, which means that the party has higher flexibility and is able to provide public goods that are better valued by the society. Note that the entrant party with $\lambda > 1$ will have an advantage over the incumbents because it will be able to provide a higher amount of public good with the same level of investment than that of its opponents[2]. This means that incumbent parties that invest one unit in the provision of the public good will produce one unit of it, while the entrant will be able to provide λ units with the same level of investment.

The party that reaches the office obtains the budget normalized to one and must also provide the committed amount of public good. Given this, we will understand g_i as the investment on public good per-unit of the budget. We assume that parties commit to providing the promised amount of public good

[1] Further analysis should consider the strategic behavior of the entrant when it faces positive entry costs. When there are entry barriers, incumbents can change their strategic behaviors in order to deter the entry of the third party. Under this assumption, it is not possible to observe deterrence of entry, so the analysis will focus on the localization of the entrant when it has valence advantages.

[2] The opposite happens if $\lambda < 1$, where the entrant would have a disadvantage over the public good compromise relative to the incumbent parties. This case is not of interest for the current research because the entrant party would neither have an advantage over the timing of entry nor on non-policy attributes that would incentivize its entry.

justified on the grounds that politicians project that, in future elections, retrospective voters will punish office-holders who violate their pre-election promises (Adams et al., 2011). Hence, the payoffs for a party i that invest an amount g_i on the provision of the public good will be $\pi_i = 1 - g_i$ if it wins and $\pi_i = 0$ if it loses. We assume that the candidates, at the time they select their strategies, are uncertain about the election outcome because they cannot perfectly forecast the distribution of voters' ideal policy positions.

The timing of the electoral competition takes place in three strategical stages. In the first one, incumbent parties must select their policy platforms in the interval $[0, 1]$, anticipating the entry threat. After the incumbents have selected their policy platforms, the entrant party observes their policy allocation and decides whether to enter or not to the electoral competition. If it participates, it must choose its own policy allocation, and if it does not, he receives its reservation utility and the incumbents parties would be the only ones running for holding office. The decision to enter depends on whether the expected benefits of winning the election is higher than the reservation utility of the entrant party. The third stage takes place after the policy allocations have been chosen, but before the votes are cast. In this stage, all participant parties must decide simultaneously the investment on public good to which they will commit. Finally, each voter observes the political offers of each participant candidate and then vote sincerely for whoever provides them the highest utility.

For an incumbent party, a strategy will be to choose a policy platform $x_i \in [0, 1]$ and the promised investment on public good, represented by a pair $\mathbf{g}_i = (g_0, g_1) \in \mathbf{R}^2_+$, in which the first component is the provision if the entrant does not participate and the second when it does. This quantities will depend on the policy location of the participant parties, where $g_0 = g_0(x_L, x_R)$ and $g_1 = g_1(x_L, x_R, x_E)$.

As for the entrant party strategy, first it shall decide its entry, which we denote as $\epsilon \in \{0, 1\}$ where 0 is the action of not participating in the election and 1 otherwise. If he decides to participate, and observing the policy location of the incumbents (x_L, x_R), he must choose a political location $x_E = x_E\left(x_L, x_R\right)$ and thereafter, his public good commitment as a function of these locations, $g_E = g_E\left(x_L, x_R, x_E\right)$.

4 Results

We are interested in finding Subgame PerfectEquilibria (SPE), which we will refer to as *Entry Perfect Subgame Equilibrium*, in which each player chooses a pure action over their available strategies set. The sub-games are solved through backward induction to find the PSE. First, we describe the equilibrium of the sub-games in which the entrant decides whether to enter or not. This first subgame will be referred as the "Public Good Competition without Entry Subgame" and the second as the "Public Good Competition with Entry Subgame". Afterwards, and considering the best response on the public good provision subgame, we look for strategies of the entrant party to maximize its expected benefits by choosing the entry location, whenever it has incentives to do so. We refer to this sub-game as the "Entry Subgame". Finally, the last sub-game is the choice of the incumbents policy location to maximize their expected benefits, which will be referred as the complete game.

4.1 Public Good Competition Subgames

Parties on this stage seek to maximize their expected profit by choosing the amount of public good to promise, subject to the chosen policy location in previous stages. Hence, they face the following maximization problem:

$$\max_{g_i} \qquad E[\pi_i(g_i|\mathbf{g}_{-i},\mathbf{x})] = (1-g_i)\cdot p_i(g_i|\mathbf{g}_{-i},\mathbf{x})$$

Any strategy of provision of public good on the interval $[0,1]$ will strictly dominate those greater than 1, given that any provision of public good greater than one would yield a negative expected return. Hence, we can restrict the strategies for this sub-game to the subset $[0,1]$. The benefits of each party are determined by the expected net income of providing the public good, which depends on the probability of winning the election. For example, a party can commit to providing higher amounts of public good that would compensate for the ideological distance to the furthermost voters, thus increasing its probability of winning, but at the same time, it would reduce its surplus in the case of winning.

Therefore, it is necessary to start by characterizing the probabilities of winning for each party. Without loss of generality, let us assume that parties have chosen policy locations $\mathbf{x} = (x_1, x_2, x_3) \in [0,1]^3$, where parties are labeled according to their ordering on the policy spectrum. This means that party 1 is the leftmost party, party 3 is the rightmost party and party 2 is the one located in between. For example, if $(x_L, x_R, x_E) = (0.4, 0.7, 0.2)$, then party E would be labelled as 1, L as 2 and R as 3. Also, that they have committed to provide $\mathbf{G} = \{g_1, g_2, g_3\}$.

The probability that a party i wins the election is given by their expected vote share in the election. Thus, this probability is defined as:

$$p_i(g_i|\mathbf{g}_{-i},\mathbf{x}) = \int_0^1 \mathbb{1}[\forall j \neq i : U_x(x_i, g_i) \geq U_x(x_j, g_j)] f(x) dx$$

where $f(x)$ corresponds to the probability density function of the number of individuals with single-peaked preference at x. The probability corresponds to the expected number of individuals whose highest utility is given by party i.

To define the regions of the policy space where party i provides the highest utility to the voters, we define the indifferent voter position x_{ij} as the position in $[0,1]$ where a voter receives the same utility when voting for party i or j. If $x_i < x_j$, then every voter to the left of x_{ij} will prefer i over j and those at the right will prefer j over i. The contrary happens otherwise.

Abusing notation where U_{ij} is the utility of the individual with policy preference at x_{ij}, this individual is located at:

$$\begin{aligned}
U_{ij}(x_i, g_i) &= U_{ij}(x_j, g_j) \\
g_i - k\left(x_{ij} - x_i\right)^2 &= g_j - k\left(x_{ij} - x_j\right)^2 \\
g_i - k\left(x_{ij}^2 - 2x_i x_{ij} + x_i^2\right) &= g_j - k\left(x_{ij}^2 - 2x_j x_{ij} + x_j^2\right) \\
2k x_{ij}\left(x_i - x_j\right) &= g_j - g_i - k\left(x_j^2 - x_i^2\right) \\
\therefore \qquad x_{ij} &= \frac{g_j - g_i}{2k(x_i - x_j)} + \frac{x_i + x_j}{2}
\end{aligned}$$

We will define $p_i^{(k)}$ as the probability of winning for party i when k opponents have been overthrown of the electoral competition[3]. As we will see, the winning probability curve of a party i is the lower envelope of all probability curves $p_i^{(k)}$.

Assuming that for each pair of parties the indifferent voter exists in X[4], if an individual is located to the left of x_{12}, then he get the highest utility when voting for party 1. If he is between x_{12} and x_{23}, then he obtains the highest utility from party 2 and if he is to the right of x_{23}, he gets the highest utility from party 3.

Let us begin by describing the winning probability when none of the parties has been overthrown of the competition. Assuming a uniform distribution of the number of voters on the policy space, the probability function will be:

$$\begin{aligned}
p_1^{(0)} &= \int_0^{x_{12}} dx = \frac{g_2 - g_1}{2k(x_1 - x_2)} + \frac{x_1 + x_2}{2} \\
p_2^{(0)} &= \int_{x_{12}}^{x_{23}} dx = \frac{g_3 - g_2}{2k(x_2 - x_3)} - \frac{g_2 - g_1}{2k(x_1 - x_2)} + \frac{x_3 - x_1}{2} \\
p_3^{(0)} &= \int_{x_{23}}^{1} dx = 1 - \frac{g_3 - g_2}{2k(x_2 - x_3)} - \frac{x_2 + x_3}{2}
\end{aligned}$$

We can see that the probabilities $p_i^{(0)}$ are linear in g. Let us define $g_i^{(0)}$ as the minimum level of public good that the party has to offer to compete against the opponents. If party i offers an amount lower than $g_i^{(0)}$, then it is dominated by the other parties and will have zero probabilities of winning the election, whereas any amount higher than that gives it a positive probability of winning. Starting from that threshold, the party can increase its provision while competing against the other two parties, until it

[3]This means that there exist k opponent parties with an expected vote share equal to zero

[4]This correspond to the condition: $\forall (i,j) \in \{L, R, E\}, i \neq j : \exists \theta \in X \mid U(x_i, g_i|\theta) = U(x_j, g_j|\theta)$.

reaches a threshold $g_i^{(1)}$ for which it can *ex-ante* defeat one of its competitors by convincing all of his voters. From that level of provision, he competes for voters only against the lasting party. Then, the probability $p_i^{(0)}$ does not hold anymore, so we will consider probability $p_i^{(1)}$, where now one of the competitors has been defeated. The new probability will still be increasing in the provision of public good, but with a flatter slope than before. By further increasing its provision, the party can reach another threshold $g_i^{(2)}$ where it dominates the leftover party by capturing all his voters, thus having a certain win. Any further increase in the offer of the public good would result in no additional probability gain for the party because it has already convinced all voters by reaching a provision $g_i^{(2)}$. Let us define g_i^j as the amount of public good that party i needs to offer to convince the voters of party j, assuming fixed locations on the policy spectrum. Generalizing, we would have $g_i^{(1)} = \min_{j\neq i}\{g_i^j\}$ and $g_i^{(2)} = \max_{j\neq i}\{g_i^j\}$.

For example, assume that party 2 steadily increases its provision of public good starting from $g_2^{(0)}$. It could defeat party 1 or 3, depending on which party would lose first his voters against the provision increase of party 2. Whenever one of the indifferent voters between 2 and his opponents are located at the extremes, then we state that party 2 dominated one of the others parties. Assuming on behalf of the example, that $g_2 > g_2^{(1)} = g_2^1$ and $g_2 < g_2^{(2)} = g_2^3$, the probability of winning would now become:

$$p_2^{(1)} = \int_0^{x_{23}} dx = x_{23} = \frac{g_3 - g_2}{2k(x_2 - x_3)} + \frac{x_2 + x_3}{2}$$

And whenever $g_2 > g_2^{(1)} = g_2^1$ and $g_2 > g_2^{(2)} = g_2^3$, then it becomes $p_2^{(2)} = 1$.

The function that represents the probability of winning of a party i can be summarized as follows

$$p_i(g_i|\mathbf{g}_{-i}, \mathbf{x}) = \begin{cases} 0 & g_i \leq g_i^{(0)} \\ p_i^{(0)} & g_i \in \left[g_i^{(0)}, g_i^{(1)}\right] \\ p_i^{(1)} & g_i \in \left[g_i^{(1)}, g_i^{(2)}\right] \\ 1 & g_i \geq g_i^{(2)} \end{cases}$$

where $p_i^{(j)} = p_i^{(j)}(g_i|\mathbf{g}_{-i}, \mathbf{x})$.

To further clarify the winning probability curve, we show in Figure 1 the winning probability of a party i as a function of his own provision, given a fixed level of public good commitment from its competitors. Overthrowing one of the other parties occurs at the kinks on the probability curve, where the change in the slope is due to the competition against the only remaining party, instead of both opponents. The probability functions for each party are explicitly developed in Appendix A.

The next proposition follows from the piece-wise nature of the probability function and will serve us to characterize the equilibria of the game in the next section.

Proposition 1. *The probability that party i wins the election is quasi-concave in the provision of public*

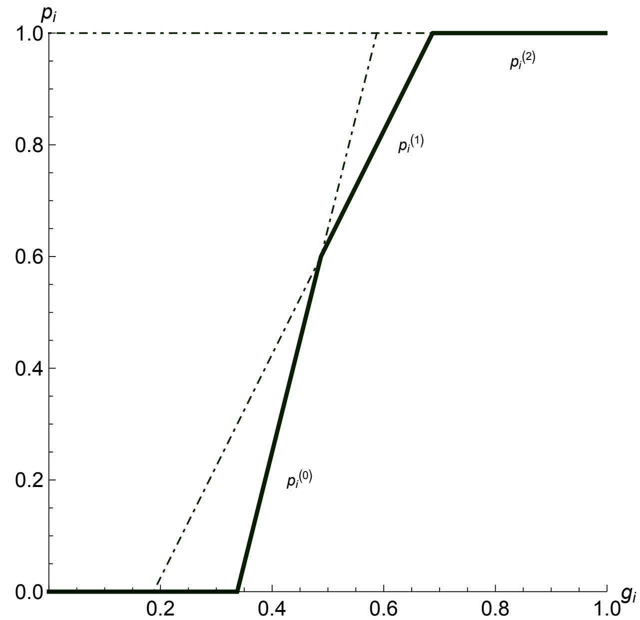


Figure 1: Probability function of party $p_i(\mathbf{G})$

good g_i.

Proof. Refer to Appendix C.1. □

Returning to the maximization of benefits of the parties, and considering expected benefits functions to be the product of a monotonically decreasing linear function and a linear piecewise increasing function, then we can expect this function to be quasi-concave on g_i.

Proposition 2. *The expected benefit function of party i is quasi-concave on the provision of public good g_i.*

Proof. Refer to Appendix C.2. □

Considering that the probability function is not continuously differentiable on its whole support, we can find first order conditions for a party i whenever it is not at one of the kinks. These are:

$$\frac{\partial E[\pi_i(g_i|\mathbf{g}_{-i},\mathbf{x})]}{\partial g_i} = 0$$

$$\underbrace{\frac{\partial p_i}{\partial g_i}}_{\text{Marginal benefits of increase in } g_i} = \underbrace{g_i\frac{\partial p_i}{\partial g_i} + p_i(g_i|\mathbf{g}_{-i},\mathbf{x})}_{\text{Marginal cost of increasing } g_i}$$

In Figure 2, we present the marginal returns and costs for a party i on the electoral competition against the other parties. Increasing the provision of public good makes the party more competitive, thus increasing its probabilities of winning, so the expected benefits would increase. The marginal return is higher in the interval $\left[g_i^{(0)}, g_i^{(1)}\right]$ than in $\left[g_i^{(1)}, g_i^{(2)}\right]$, because an increase in the provision of public good allows the party to convince voters from both opponents with the same marginal increase. In $\left[g_i^{(1)}, g_i^{(2)}\right]$, one of the opponents has already been beaten, so a marginal increase in public good only allows the party to convince voters from the remaining party. As for the marginal cost, we can identify two effects. A direct effect is the increase of the costs of provision of the committed public good at constant probability, given by $p_i(g_i|\mathbf{g_{-i}}, \mathbf{x})$, and an indirect effect from the increase on the probabilities of winning, which would increase the expected cost of providing the promised public good. As we can see in the figure, amounts lower than $g_i^{(0)}$ would be in equilibrium but are weakly dominated by choosing any greater amount, and promises that are higher than $g_i^{(2)}$ are not at equilibrium given that marginal costs would always be greater than marginal income.

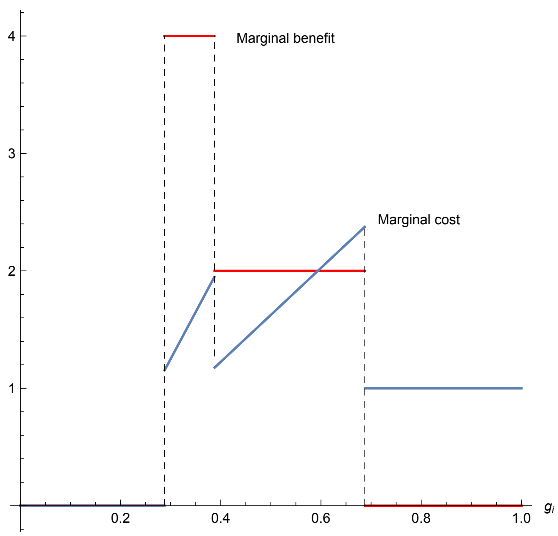


Figure 2: Marginal return and cost of party i

The results from Proposition 2 indicates that parties present decreasing marginal benefits in the provision of the public good. This would imply that the net benefits that a party can obtain by providing an additional unit of public good is higher when it is promising a low quantity of public good rather than a higher amount, explained by the fact that the marginal increase in the chances of winning is higher when promising a low amount of public good, than for a high amount. For small provisions of public good, the party can not successfully convince voters that are ideologically further away, reducing its capacity to compete against parties that are closer to those voters. When the party is promising a high amount of the public good, then it has reached out to voters that are further away, probably dominating one of the competitors, so providing an additional unit of the public good will not result in a higher increase in the probabilities of winning.

4.1.1 Public Good Competition without Entry

Let us consider that the entrant has decided not to participate, so the competition on this stage is only among the incumbent. These parties have chosen party platforms x_L and x_R, where $x_L < x_R$. The derivation of a probability function for when there are only two parties is analogous to that in which there are three. The only difference will be that there are only going to be two kinks: $g_i^{(0)}$ and $g_i^{(1)}$.

A voter that faces the policy locations of the parties will be indifferent between choosing for any of them would locate at x_{LR}. Assuming that each party has a positive probability of winning and assuming interior solution, party L will compromise a level of public good such that:

$$\begin{aligned}
\frac{\partial p_L(g_L, g_R)}{\partial g_L} &= g_L \frac{\partial p_L(g_L, g_R)}{\partial g_L} + p_L(g_L, g_R) \\
\frac{1}{2k(x_R - x_L)} &= \frac{g_L}{2k(x_R - x_L)} + \left(\frac{g_L - g_R}{2k(x_R - x_L)} + \frac{x_R + x_L}{2} \right) \\
\therefore \quad g_L^* &= \frac{1}{2} \left[1 + g_R - k(x_R^2 - x_L^2) \right]
\end{aligned}$$

Analogously for party R:

$$\begin{aligned}
\frac{\partial p_R(g_L, g_R)}{\partial g_R} &= g_R \frac{\partial p_R(g_L, g_R)}{\partial g_R} + p_R(g_L, g_R) \\
\frac{1}{2k(x_R - x_L)} &= \frac{g_R}{2k(x_R - x_L)} + \left(1 - \frac{g_L - g_R}{2k(x_R - x_L)} - \frac{x_R + x_L}{2} \right) \\
\therefore \quad g_R^* &= \frac{1}{2} \left[1 + g_L - k(x_R - x_L)(2 - x_R - x_L) \right]
\end{aligned}$$

Both parties will be in equilibrium when none has an incentive to deviate from the chosen promise of public good. This will be achieved for a pair of public good investments $\mathbf{G}_0^* = (g_L^*(x_L, x_R), g_R^*(x_L, x_R))$ for which both first-order conditions hold simultaneously.

Proposition 3. *Without the entry of third party, the incumbent parties will compromise* $g_L^*(x_L, x_R) = 1 - \frac{1}{3}k(x_R - x_L)(2 + x_R + x_L)$ *and* $g_R^*(x_L, x_R) = 1 - \frac{1}{3}k(x_R - x_L)(4 - x_R - x_L)$ *in equilibrium.*

Proof. Replacing the optimal level of provision for party L on party R's first order condition yields the following equation

$$g_R = \frac{1}{2} \left[1 + \frac{1}{2} \left[1 + g_R - k(x_R^2 - x_L^2) \right] - k(x_R - x_L)(2 - x_R - x_L) \right]$$

which solving for g_R gives $g_R^* = 1 - \frac{1}{3}k(x_R - x_L)(4 - x_R - x_L)$. Replacing back this result on party L's first order condition gives $g_L^* = 1 - \frac{1}{3}k(x_R - x_L)(2 + x_R + x_L)$. □

4.1.2 Public Good Competition with Entry

As discussed in 4.1, we will refer to a political position x_{ij} whenever that voter is indifferent between two parties i and $j \in I$. We will consider the case $0 < x_{LE} < x_{ER} < 1$, so each of the parties has a positive probability of winning. As obtained in previous sections, the expected profit function is a piece-wise concave function, so we can not directly solve the first order conditions to obtain the desired solution. Instead, we can write the equivalent maximization problem:

$$\begin{aligned} \max_{z,g_i} \quad & z \\ s.t. \quad & z \leq (1-g_i)\, p_i^{(0)} \\ & z \leq (1-g_i)\, p_i^{(1)} \\ & z \leq (1-g_i) \end{aligned}$$

This formulation allows us to obtain directly the optimal level of public good provision and the maximum expected profit for each party. To see the equivalence, we note that the optimal value will be $z^* = E[\pi_i(g_i|\mathbf{g}_{-i}, \mathbf{x})]$ whenever one of the constraints is active. The Lagrangean function associated to this optimization problem is:

$$\mathcal{L}(g_i, z|\mathbf{g}_{-i}, \mathbf{x}) = z - \mu_1(z - (1-g_i)\, p_i^{(0)}) - \mu_2(z - (1-g_i)\, p_i^{(1)}) - \mu_3(z - (1-g_i))$$

Taking first order conditions of the parties seeking to maximize their expected benefits yields:

$$[\mathcal{L}_{g_i}] \qquad \mu_1 \left((1-g_i)\, \frac{\partial p_i^{(0)}}{\partial g_i} - p_i^{(0)} \right) + \mu_2 \left((1-g_i)\, \frac{\partial p_i^{(1)}}{\partial g_i} - p_i^{(1)} \right) - \mu_3 \qquad = 0 \qquad (1)$$

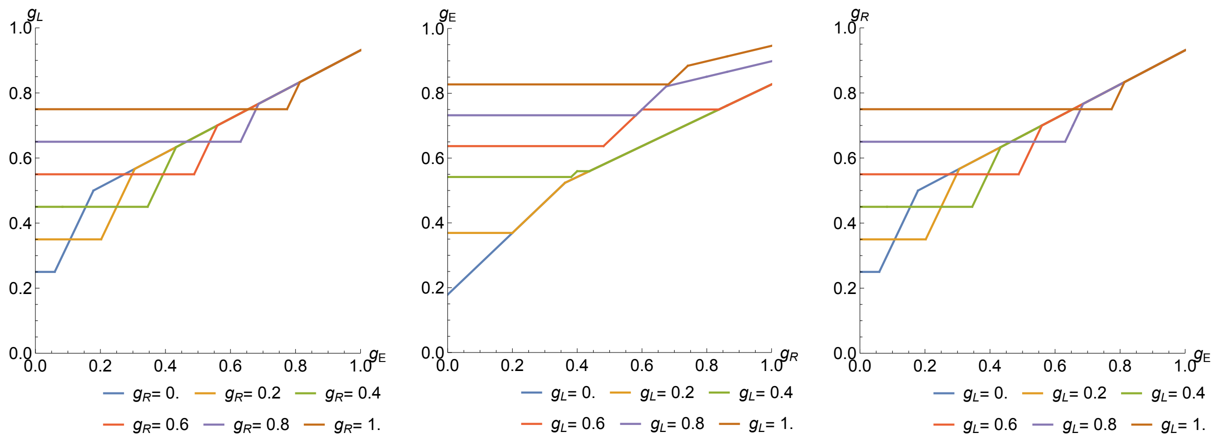
$$[\mathcal{L}_z] \qquad 1 - \mu_1 - \mu_2 - \mu_3 \qquad = 0 \qquad (2)$$

where $\mu_i \geq 0$ are the multipliers associated to each restriction. By complementary slackness:

$$\begin{aligned} \mu_1 \left(z - (1-g_i) p_i^{(0)} \right) &= 0 \\ \mu_2 \left(z - (1-g_i) p_i^{(1)} \right) &= 0 \\ \mu_3 \left(z - (1-g_i) \right) &= 0 \end{aligned}$$

These slackness conditions allows us to obtain the optimal value of the payoff function, while the first order conditions let us obtain the optimal level of promise of public good. We also note that by equation (2), at least one of the constraints has to be active. The explicit derivation for the best response of each party is obtained in Appendix B. Even though the derivation was made, for simplicity, for the case where $x_L < x_E < x_R$, this formulation allows us to derive the best responses of each party whichever order they have.

The contour curves of parties best responses are shown in Figure 3. Assuming a symmetrical location of incumbents with respect to the middle point of the spectrum, figures 3a and 3c show that



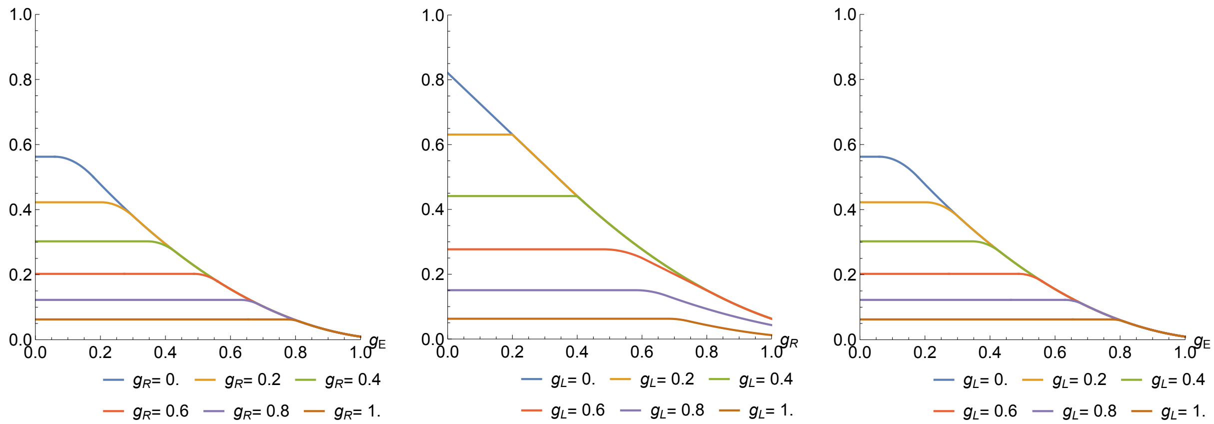
(a) Party L (b) Party E (c) Party R

Figure 3: Contour curves of parties best response functions on public good provision

incumbents' present the same best response functions. It should be noted that, when the the entrant is offering a low level of public good, the incumbents best response is to offer a constant amount, until that provision becomes a serious threat to keep their voters. From that point onward, an incumbent will start to increase its own provision in order to compete with the entrant, holding fixed the provision of the other incumbent. Eventually, they reach the threshold in which they are able to defeat the other incumbent, so the remaining parties compete just among each other. Because of this, they only need to capture the voters of each other and, therefore, they slowly increase their public good commitments than before, for each additional unit of investment on public good reduces their net payoff and does not increase their probabilities of winning as high as when all parties were competing.

From the point of view of the parties located at the extremes, we can observe that their increase in provision will only affect the party closer to it, so it would only force a direct response from the center party. This occurs as long as the provision of the extreme parties has not yielded a zero expected vote share to the center party, for extreme parties will start to compete against each other when the center party is defeated.

For the party in the center, the best response is always increasing when any of the other parties increase their provision. When extreme parties start to compete with higher provisions of public good, the only way the center party can compete against both of them is by increasing its own provision of public good. Considering party E as the center party for this situation, the first horizontal segment of the best response are levels of provision for which the entrant is not affected by the provision of incumbents, while the second flat segment is such that the entrant optimally chooses to maintain constant its provision level even though its probabilities of winning decrease, because the marginal increase in the probability of winning is not enough to compensate the increase in marginal cost.



(a) Party L (b) Party E (c) Party R

Figure 4: Contour curves of parties expected benefits on public good provision

We see in Figure 4 that payoffs are non-increasing in the provision of public good for all of the parties. The contour curves of the parties begin with a plateau, in which parties are able to compensate marginal net losses from the increase of public good provision with the marginal increase in expected vote share, resulting in a zero marginal profit zone. Competition in this region from the other parties does not have a real effect on the expected payoff of the party. Then, there is a threshold from which the party will be forced to increase its public good provision over the one that it would optimally choose without competition. From this threshold, the party is not able to compensate the marginal net loss with increments in probability, resulting in negative marginal benefits.

An equilibrium in the public good provision sub-game is a level of provision for the three parties in which none has incentives to deviate and modify their commitments. Let us define $\Phi(\mathbf{G}) = (\phi_L(\mathbf{G}), \phi_R(\mathbf{G}), \phi_E(\mathbf{G}))$ as the best response correspondence of the parties, where $\phi_i(\mathbf{G})$ is the best response of party i. An equilibrium $\mathbf{G}_1^*(\mathbf{x}) = (g_L^*(\mathbf{x}), g_R^*(\mathbf{x}), g_E^*(\mathbf{x}))$ will be reached whenever $\mathbf{G}_1^*$ is a fixed point of $\Phi(\mathbf{G})$. Friedman (1977) finds conditions that let us assure that the fixed point exists, so let us consider the following propositions.

Proposition 4. *Given party locations $\{x_L, x_R, x_E\} \in [0,1]$, each different from one another, there exists a non-cooperative equilibrium in the public good provision sub-game played for these locations.*

Proof. Refer to Appendix C.3. □

Proposition 5. *Given party locations $\{x_L, x_R, x_E\} \in [0,1]$, each different from one another, the equilibrium in the public good provision sub-game played for this locations is unique.*

Proof. Refer to Appendix C.4. □

Banach fixed-point theorem ensures existence, uniqueness of equilibrium and global stability of the best response correspondence $\Phi(\mathbf{G})$. Proving that best responses are contraction mappings (see Appendix C.4), then any sequence $\mathbf{G}^n = \Phi(\mathbf{G}^{n-1})$ will converge globally from any initial value $\mathbf{G}^0$ to the equilibrium of the sub-game $\mathbf{G}_1^*$. Even though this holds for this case, an explicit solution can be found and is shown in the following proposition.

Proposition 6. *Given party locations $\{x_L, x_R, x_E\} \in [0,1]$, each different from one another and assuming $x_L < x_E < x_R$, the equilibrium in the public good provision sub-game played for this locations is the following:*

$$g_L^* = \frac{1}{6}\left[2\lambda + 4 + k\left(\frac{x_E - x_L}{x_R - x_L}\right)(2x_E(1-(x_R - x_L)) - (x_R - x_L)(3x_L + x_R) - 2x_R)\right]$$
$$g_R^* = \frac{1}{6}\left[2\lambda + 4 + k\left(\frac{x_R - x_E}{x_R - x_L}\right)\left(3x_R^2 - 2x_E(1-(x_R - x_L)) - 2(x_L + 3)x_R - (x_L - 8)x_L\right)\right]$$
$$g_E^* = \frac{2}{3} + \frac{1}{3\lambda} - k\frac{(x_E - x_L)(x_R - x_E)(2 + (x_R - x_L))}{3\lambda(x_R - x_L)}$$

Proof. It is obtained directly by solving the linear equation system $\Phi(\mathbf{G}) = \mathbf{G}$, in which $\Phi(\mathbf{G})$ is defined in Appendix B.4. □

The optimal provision of public good is affected by the parameter λ of the public good provided by the entrant. Figure 5 shows the effect of the valuation parameter λ on the equilibrium of the public good provision, for a configuration in which incumbents have chosen symmetrical positions and the entrant has located equidistantly between them.

The provision of public good of the entrant decreases in λ in equilibrium, while it increases for the incumbents. For values of λ closer to 1, both incumbents and entrant offer similar amounts of public good to the voters for equal investments, so the entrant compete mainly on policy location and has to offer a higher amount than the entrant in order to compensate the disadvantage from choosing location after the incumbents. As λ increases, the entrant can provide a higher amount of public good than the incumbents for the same level of investment, so the entrant can reach the provision of the incumbents with lower levels of investment. In this position, incumbent parties are forced to compromise higher quantities to be able to compete with the entrant's advantage. When λ is sufficiently high, incumbents must max out their offer, obtaining zero expected profits. This situation generates incentives for incumbents to deviate from their chosen locations and select policy locations where they could compete with a lower provision. When incumbents strategies are committing to the maximum provision, each increase in the value of λ benefits the entrant by being able to reach the same valuation of the incumbents with lowers amounts of public

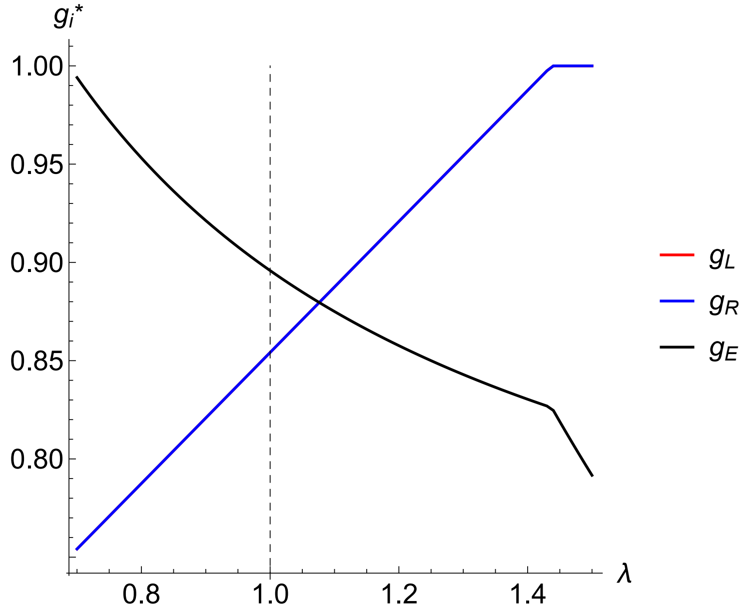


Figure 5: Public good provisions of equilibrium as function of parameter λ

good. The critical value of λ for which the incumbents would max out their offer is:

$$\lambda_L^* = 1 + k\frac{x_E - x_L}{2(x_R - x_L)}\left[2x_R + 2x_E(x_R - x_L - 1) + (x_R - x_L)(x_R + 3x_L)\right]$$
$$\lambda_R^* = 1 + k\frac{x_R - x_E}{2(x_R - x_L)}\left[x_L^2 + 2x_E(1 - x_R + x_L) + 2x_L(x_R - 4) - 3x_R(x_R - 2)\right]$$

In this situation, the entrant has the advantage on the provision of public good and can compensate the disadvantage of choosing location after the incumbents have done so.

4.2 Entry Subgame

The entrant party, anticipating the amounts of public good that the incumbents will compromise in the next stage, must decide whether to participate in the competition or not, and where to locate in the political spectrum in case it does. Given that there are no fixed costs of entry, the entrant will always decide to enter the competition. The decision of entry without entry costs is a positive sum game: if it enters the competitions, it might win the competition and obtain the budget, which would yield him a non-negative expected benefit as long as it has a positive probability of winning, while not participating has a certain zero benefit. Thus, the incentive for the entrant is always participate, even if the chances of winning the election is small.

Proposition 7. *Assuming that participating weakly dominates not participating, the entrant party will participate in the election whenever the subset* $X_E = \{x_e \in [0,1] : E[\pi_E(x_E|x_{-E})] = (1 - g_E^*(\mathbf{x})) \cdot p_E^*(\mathbf{x}) \geq$

$c\}$ is non-empty, in which $E[\pi_E(x_E|x_{-E})]$ are the expected benefits of the entrant at given locations of the incumbents. If there are not fixed entry costs, then the entrant always participate in the election.

Proof. Given that participating at zero expected profits weakly dominates not participating, let us consider the subset $X_E = \{x_E \in [0,1] : E[\pi_E(x_E|x_{-E})] \geq c\}$ of all positions where the entrant participates. We need to prove that there exists at least one position where the entrant would decide to participate, or equivalently, that the subset X_E is non-empty. The expected benefits in the entry location sub-game correspond to $E[\pi_E(x_E|x_{-E})] = (1 - g_E^*) \cdot p_E^*(\mathbf{x})$. Given that entry costs are fixed, they do not affect the optimal decision of provision on the public good subgame. Therefore, the entrant does not participate whenever the expected profits are lesser than c. If for every position $x_E \in [0,1]$, the expected profit of the entrant has an upper bound on c, then there does not exists a feasible location where the entrant would have non-negative expected benefits, thus, deterring its entry from the political competition.

If the entry cost is zero, and that probabilities are bounded in $[0,1]$, then entry would only be deterred whenever g_E^* were greater than 1, which would yield the party a negative expected profit. However, considering that choosing $g_E > 1$ are strictly dominated strategies by the strategy $g_E = 1$, then the expected benefits have a lower bound on zero for every position x_E in $[0,1]$. Therefore, the subset $X_E = [0,1]$ of all feasible entry locations is a non-empty subset of $[0,1]$. □

When considering where to locate, the entrant party must solve the following optimization problem:

$$\begin{aligned} \max_{x_E} \quad & E[\pi_E(x_E|x_{-E})] = (1 - g_E^*(x_E|x_{-E})) \cdot p_E^*(x_E|x_{-E}) \\ s.t. \quad & x_E \in X_E = \{x_E \in [0,1] : (1 - g_E^*(x_E|x_{-E})) \cdot p_E^*(x_E|x_{-E}) \geq c\} \end{aligned}$$

Where differentiable, the first order condition for this maximization problem with respect to the policy location corresponds to:

$$\begin{aligned} & \underbrace{\frac{\partial p_E}{\partial x_E}}_{\text{Direct increase in probability}} + \underbrace{\sum_{i \in N} \frac{\partial p_i}{\partial g_i} \frac{\partial g_i^*}{\partial x_E}}_{\text{Increase in probability from provision change}} = \\ & \underbrace{p_E \frac{\partial g_E^*}{\partial x_E}}_{\text{Increase in provision at given probability}} + \underbrace{g_E^* \left[\frac{\partial p_E}{\partial x_E} + \sum_{i \in N} \frac{\partial p_i}{\partial g_i} \frac{\partial g_i^*}{\partial x_E} \right]}_{\text{Increase in probability at given provision}} \end{aligned}$$

The left-hand side of the equation represents the marginal expected benefit from moving marginally to the right on the policy spectrum[5]. The marginal benefit for a displacement is composed by two effects.

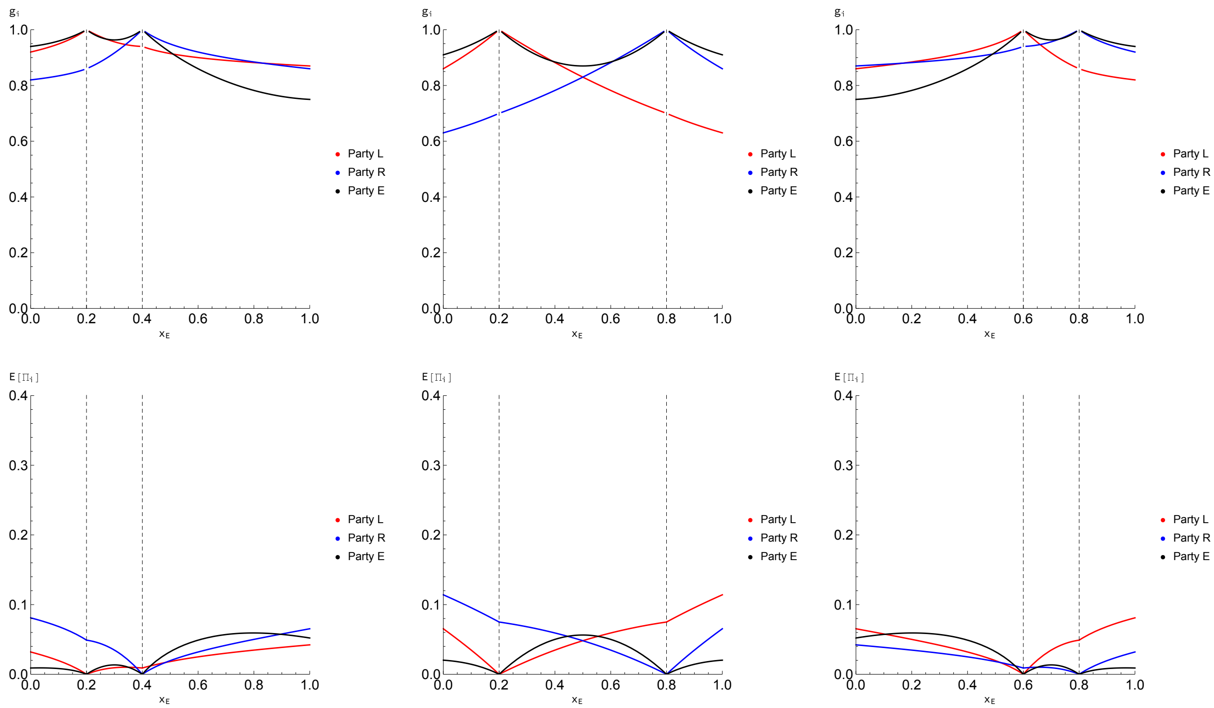
[5]We are considering that a displacement $\partial x_E > 0$ is a movement from a policy location to the right of the spectrum. When $\partial x_E < 0$, it should be considered a displacement to the left.

The first is a direct effect on the probabilities of winning the election from deviating marginally ($\frac{\partial p_E}{\partial x_E}$). Moving in one direction allows the entrant to compete closer to the party that is in that direction. The sign of this effect is uncertain and will depend if the probability increase from nearing one party compensates the loss by leaving space to the other. The second effect is an indirect change in probability given that parties will react to a displacement by also changing the offer of public good in equilibrium. On the left side of the equation, it is the marginal cost of the displacement in the spectrum. It is also composed of two effects. The first is a direct change in the optimal provision of public good in the following stage, which directly affects the provision that the party commits to. The second effect is the same change in probability, which makes the provision of the offered public good more likely, increasing the expected cost of provision.

The optimal policy location would be such that the increase in the probability of winning would compensate the marginal change in the costs of providing the public good. It should be expected that, when incumbent parties are closer to the extremes, the entrant party would prefer to locate as far as possible from them, in order to avoid competing on the public good provision which would directly reduce its net benefits. In this sense, being closer to one of the incumbent pushes the entrant to differentiate through policy location, because otherwise, they would have to costly compete on the provision of public goods. On the other side, whenever incumbent parties are closer to the center, then we would expect the entrant to chose a policy platform closer to the extremes, to again avoid a costly competition on public good with the incumbents.

Figure 6 shows the equilibrium provisions of public good and the expected profits for each party as a function of the policy location of the entrant, considering different locations for the incumbents. The entrant must choose whether to locate to the left of L, between both incumbents or to the right of R. The location decision depends on the choice of policy made by the incumbents in prior stages. Let us consider the case presented in (a), in which both incumbents have adopted similar moderate policies to the left. If he enters to the left, it can easily convince voters of the extreme left, but must compete with the incumbents for the right and moderate left voters. This means that his probabilities of winning are low even with high provisions of public good, which yields him low expected benefits. When locating in the middle, the entrant competes for the same voters than the incumbents, having better chances at convincing voters from both sides of the spectrum but having to commit to a high provision, which also yields him a low expected benefit. But when it locates to the right of both incumbents, it has the advantage to convince all right voters with lower public good commitment than the incumbents. The advantage confers the entrant a higher winning probability with a lower provision than the incumbents, so it is optimal for him to choose a policy position that differentiates him from the other parties.

When both incumbents are adopting similar positions to the right, such as in (c), then the case is

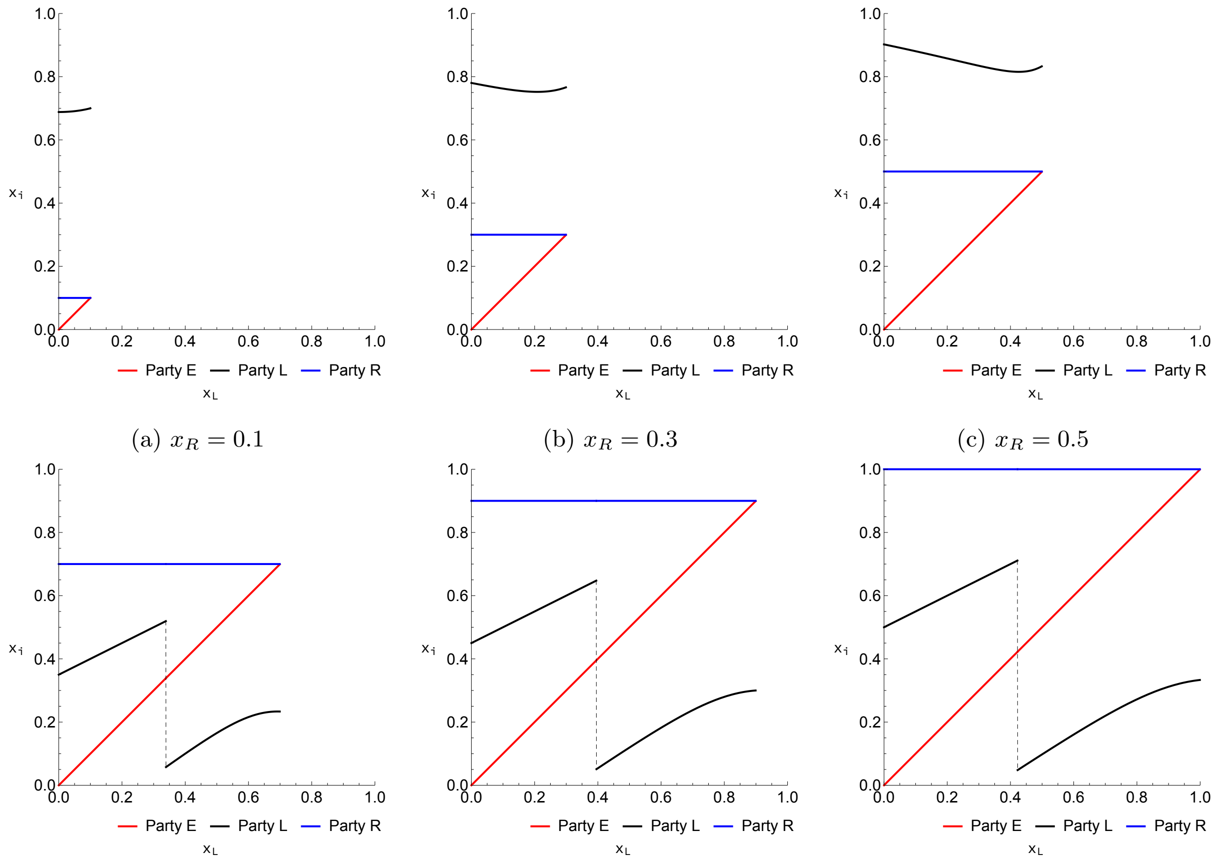


(a) $x_L = 0.2$ and $x_R = 0.4$ (b) $x_L = 0.2$ and $x_R = 0.8$ (c) $x_L = 0.6$ and $x_R = 0.8$

Figure 6: Public good provision (top) and expected pay-offs (bottom) as function of x_E.

analogous and the entrant locates optimally at a moderate left. When both incumbents choose differentiated policies, as the case in (b), in which they locate towards the extremes, then we have the following situations. If the entrant locates close to one of the incumbents, both need to provide high quantities of public good: the first to maintain his voters and the second to capture them. At the same time, the entrant has difficulties to capture the voters closer to the other incumbents, so it has incentives to deviate towards it, reducing its public good commitment and increasing the likelihood of winning. Its optimal choice of location will be at the midpoint of incumbents.

Figure 7 presents the optimal choice of location of the entrant, for given locations of the incumbents. We fix the position of party R and plot in the vertical axis the location of the parties as a function of the policy location of party L. Incumbent parties with similar platforms leave open entry opportunities for the third party to enter on those locations that incumbents are not able to reach. When incumbents locate closely to each other, the entrant will seek to differentiate himself from the incumbents, reaching extreme positions. If incumbents have differentiated policy locations towards the extreme, the opportunity for the entrant is to locate at moderate positions and capture the center votes. When policy locations of the incumbents are symmetrical and close to the center, leaving open the extremes, then a multiplicity of equilibria for the entrant could occur, for which the entrant will randomize between choosing the right or



(a) $x_R = 0.1$ (b) $x_R = 0.3$ (c) $x_R = 0.5$

(d) $x_R = 0.7$ (e) $x_R = 0.9$ (f) $x_R = 1.0$

Figure 7: Optimal entry decision of party E

the left extreme.

4.3 Incumbent Localization Subgame

In the last stage of the game, incumbent parties must choose their policy location to maximize their expected pay-off, given the possible entry of the third party. To do so, incumbents anticipate the policy location of the entrant and public good commitment that all parties would make on the public good provision stage. Therefore, when choosing policy location, incumbent i faces the following maximization problem:

$$\max_{x_i} \quad E[\pi_i(x_i|x_{-i})] = (1 - g_i^*(x_i|x_{-i}) \cdot p_i^*(x_i|x_{-i})$$

where x_{-i} is the policy position of the other incumbent. In this stage, the public good is the optimal provision, considering the location of the other incumbent and the best response on the following stages, which is $g_i^*(x_i|x_{-i}) = g_i^*(x_i|x_{-i}, x_E^*(x_i, x_{-i}))$. The case is analogous for the probability of winning of the incumbent, where $p_i(x_i|x_{-i})$ is the probability considering best response of the parties on the following stages of the game.

The solution for these maximization problems yield best response functions $\psi_L(x_R)$ and $\psi_R(x_L)$. Hence, an equilibrium will be reached when neither of the incumbents has an incentive to deviate from the political position selected. The equilibrium in the sub-game is a pair (x_L^*, x_R^*) such that they are mutual best responses to the policy platform chosen by the other incumbent. Formally, the position (x_L^*, x_R^*) corresponds to a fixed point of the correspondence $\Psi(x_L, x_R) = (\psi_L(x_R), \psi_R(x_L))$, which is $(x_L^*, x_R^*) = \Psi(x_L^*, x_R^*)$.

When differentiable, the first order condition for the incumbents location problem is the following:

$$\underbrace{\underbrace{\frac{\partial p_i}{\partial x_i}}_{\text{Direct increase in probability}} + \underbrace{\frac{\partial p_i}{\partial x_E}\frac{\partial x_E^*}{\partial x_i}}_{\text{Indirect increase by entrants reaction}} + \underbrace{\sum_{j\in I}\frac{\partial p_i}{\partial g_j}\left[\frac{\partial g_j^*}{\partial x_i} + \frac{\partial g_j^*}{\partial x_E}\frac{\partial x_E^*}{\partial x_i}\right]}_{\text{Increase in prob. due to change in provision}}}_{\text{Total increase in probability of winning}} =$$

$$\underbrace{p_i\left(\frac{\partial g_i^*}{\partial x_i} + \frac{\partial g_i^*}{\partial g_E}\frac{\partial x_E^*}{\partial x_i}\right)}_{\text{Increase in public good provision}} + \underbrace{g_i^*\left(\frac{\partial p_i}{\partial x_i} + \frac{\partial p_i}{\partial x_E}\frac{\partial x_E^*}{\partial x_i} + \sum_{j\in I}\frac{\partial p_i}{\partial g_j}\left[\frac{\partial g_j^*}{\partial x_i} + \frac{\partial g_j^*}{\partial x_E}\frac{\partial p_i}{\partial g_j}\frac{\partial x_E^*}{\partial x_i}\right]\right)}_{\text{Increase in expected cost due an increase in probability}}$$

The benefits from a marginal displacement in the policy spectrum are due to the increase in the probabilities of winning that it can achieve. This increase is composed of three effects. The first effect is an increase in the probability given that the displacement allows him to attract ideologically closer voters to that position. The second is the change on the probability due to a movement from party E in the later stage, which allows the incumbent to attract the voters that are close to the indifferent voter between the incumbent and the entrant. And finally, a third effect due to the change in the optimal provision of public good that parties would have to make in the final stage of the election. The change in optimal public good provision considers both the direct change due to the party's own displacement and the indirect change due to an optimal response of the entrant. These effects reflect the anticipation of the third party entry threat and seek to minimize the impact of the entry on its expected profits.

On the other side, the marginal expected costs due to a displacement are composed of two effects. The first is the direct effect on the amount of public good that has committed to provide. If the party changes its optimal public good provision due to the marginal displacement, then it must also commit to affording this change if it wins. If it is positive, the party must commit to providing an additional amount of public good, raising the costs of providing it if it wins, whether if it is negative, the party decreases its promised provision, reducing its expected costs. The second effect is the increase in the likelihood of winning. If the party is committing to a given amount of provision, an increase in the probability of winning leads to an increase in the likelihood that the party will have to provide it, thus, increasing its expected cost of provision.

When marginal benefits are higher than marginal costs, the incumbents still has incentives to move closer to the center because the increase in the total probabilities of winning more than compensate the increase in the costs of provision of the public good, whether the opposite occurs if marginal costs are higher, in which case the incumbent would have incentives to move further away towards the extreme. The optimal location is such that there is no net marginal gain from deviating from that given position.

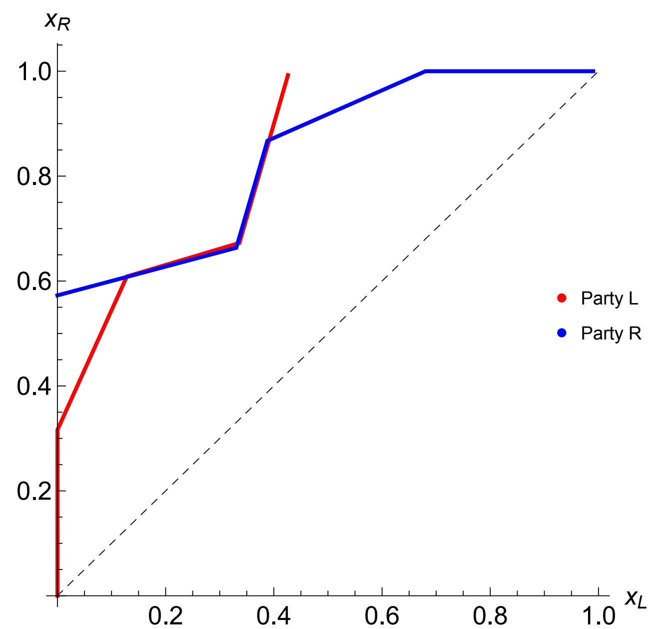


Figure 8: Best response locations of parties L and R

The best responses on location were numerically obtained, given the analytical complexity of the previous stages best responses on location of the entrant and the provision of public good. The results are shown in Figure 8, in which the optimal location of an incumbent is plotted as a function of the other incumbents' location. As it can be observed, there are multiple equilibrium locations for the incumbents. We can classify the equilibria into two: symmetrical equilibria, in which parties choose symmetrical location respect to the center of the spectrum, or asymmetrical equilibria, in which one incumbent candidate moderates his policy and the other goes towards the extreme. Incumbents have symmetrical payoff functions and they do not have any advantage on provision with respect to each other, thus, we expect both types of equilibria to occur.

There exists a unique symmetrical equilibrium located at $x_L^* = \frac{1}{3}$ and $x_R^* = \frac{2}{3}$. This result is consistent to that obtained by Palfrey (1984), whose optimal locations were $x_L^* = \frac{1}{4}$ and $x_R^* = \frac{3}{4}$, in which parties differentiate on policies in order to maximize their benefits anticipating the entry of a third party, thus not converging to the median voter location. In this model, positions are more moderated than in Palfreys, which can be explained by the fact that parties influence voters through non-policy characteristics, such as the provision of a public good, which concedes them a complementary lever to compete in an election. Considering that the entrant has an advantage on these non-policy characteristics, the incumbents tend to moderate their policies and move closer to the median voter. For this position, the entry decision of the entrant is to locate among both incumbents to compete, choosing a centered policy location. In this equilibrium, no party has incentives to deviate from their chosen location. For each of the incumbents, moving closer to the center forces them to costly compete with the entrant, thus reducing their expected profits; otherwise, moving to one of the extremes avoids reduces their expected vote share, because the entrant is able to capture the marginally moderate voters. Let us consider the following example. Assume that party L and party R are in symmetrical locations. Without loss of generality, if party L moves towards the center, then the entry threat goes closer to the right party. To avoid the effect that the entrant would have on its expected profits, party R has the incentive to move towards the center to a symmetrical position than L. If the opposite occurs, and L displaces towards the extreme to reduce its expected costs on the provision, then party E would enter closer to L. Given that party L and E would compete among them for the left voters, party R has an incentive to move towards the right, where it can reduce its expected costs and still maintain the right moderate voters that are closer to him.

As for asymmetrical equilibria, they can be explained as follows. When one of the incumbents choose an extreme position and the other a moderate one in equilibrium, then the entrant optimally locates in the other extreme. For this case, the extreme incumbent will not have incentives to deviate, because moving to the center would force it to increase its public good provision, thus lowering the expected benefits. Moving towards its extreme makes it lose support from the moderate voters against the centered incumbent. As for the latter, it also has no incentives to deviate from the chosen location. Moving in the direction of the other incumbent makes him win more extreme voters but losing some of them to the entrant in the extreme. Doing so induces him to increase its public good provision, lowering its final expected profit. The same happens if it moves towards the entrant. Therefore, we expect these equilibria to be not robust to perturbations on the parameters of the model. The asymmetrical equilibria are a result of the symmetry between valuation on public goods from the incumbent parties, which yield symmetric expected payoffs for the incumbents. Hence, further refinements of Nash equilibria are required to discard these unstable equilibria of the game and obtain a unique equilibrium. A possible solution is to relax the assumption that the valuations of a voter for the public good of the incumbents are equal and allow heterogeneity on the valuation among the voters.

5 Conclusions

The results of this work provide insights into recent electoral outcomes. When a newcomer candidate can influence voters through non-policy characteristics, such as the provision of public good, that is better valued than those of incumbents, he will aim to locate towards the center, adopting relatively moderated policies and harnessing the advantage of a favourable non-policy valence characteristic, as long as incumbents allow him to do so. The advantage provides him incentives to participate in the election, even when his probabilities of winning are low. Conversely, anticipating the threat of entry, incumbents tend to moderate their policy platforms and costly compete on non-policy aspect, to block the entry of the entrant party at the center and minimize the threat on their expected benefits for reaching office. As asymmetrical equilibria also occur in this model, further analysis is required in order to refine unstable equilibria that are not robust to perturbations on the valence characteristics of the parties.

Understanding electoral competition in the context of the actual socio-political changes requires further research, which should focus on the "fresh air" advantage of entrant parties. For example, examining if the entrant is able to maintain the advantage over time respect to the incumbents, or if changes as the party become recognized, or whether if this advantage is enough to overcome entry barriers that can deter the electoral participation of new parties. Also related is to analyze if incumbents have mechanisms to improve their relative valuation and how do they use their incumbency positions as an advantage, to be able to maximize their benefits anticipating the entry of a third party or even deter it.

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Appendix A Probabilities of Parties

A.1 Entrant Party (E)

The indifferent voter between the three parties would be located in the point where $x_{RE} = x_{LE}$, which is:

$$\begin{aligned}
x_{LE} &= x_{RE} \\
\frac{\lambda g_E - g_L}{2k(x_L - x_E)} + \frac{x_E + x_L}{2} &= \frac{\lambda g_E - g_R}{2k(x_R - x_E)} + \frac{x_E + x_R}{2} \\
\frac{\lambda}{2k}\left(\frac{1}{x_R - x_E} + \frac{1}{x_E - x_L}\right) g_E &= \frac{g_R}{2k(x_R - x_E)} + \frac{g_L}{2k(x_E - x_L)} - \frac{x_R - x_L}{2} \\
\frac{\lambda}{2k}\left(\frac{x_R - x_L}{(x_R - x_E)(x_E - x_L)}\right) g_E &= \frac{g_R}{2k(x_R - x_E)} + \frac{g_L}{2k(x_E - x_L)} - \frac{x_R - x_L}{2} \\
\therefore \quad g_E^{(0)} &= \left(\frac{x_E - x_L}{x_R - x_L}\right)\frac{g_R}{\lambda} + \left(\frac{x_R - x_E}{x_R - x_L}\right)\frac{g_L}{\lambda} - \frac{k}{\lambda}(x_R - x_E)(x_E - x_L) \qquad (3)
\end{aligned}$$

As was obtained in Section 4.1 and assuming that $g_E^L \leq g_E^R$, then:

$$g_E^{(1)} = \frac{g_L}{\lambda} + \frac{k}{\lambda}\left(x_E^2 - x_L^2\right) \qquad (4)$$

$$g_E^{(2)} = \frac{g_R}{\lambda} + \frac{k}{\lambda}\left(x_E - x_R\right)\left(x_E + x_R - 2\right) \qquad (5)$$

Therefore, the winning probability of party E is:

$$p_E(g_E|\mathbf{g}_{-E}, \mathbf{x}) = \begin{cases} 0 & g_E \leq g_E^{(0)} \\ \frac{\lambda g_E - g_R}{2k(x_R - x_E)} - \frac{\lambda g_E - g_L}{2k(x_L - x_E)} + \frac{x_R - x_L}{2} & g_E \in \left[g_E^{(0)}, g_E^{(1)}\right] \\ \frac{\lambda g_E - g_R}{2k(x_R - x_E)} + \frac{x_R + x_E}{2} & g_E \in \left[g_E^{(1)}, g_E^{(2)}\right] \\ 1 & g_j \geq g_E^{(2)} \end{cases}$$

Otherwise, if $g_E^L \geq g_E^R$, then the probability of winning would be:

$$p_E(g_E|\mathbf{g}_{-E}, \mathbf{x}) = \begin{cases} 0 & g_E \leq g_E^{(0)} \\ \frac{\lambda g_E - g_R}{2k(x_R - x_E)} - \frac{\lambda g_E - g_L}{2k(x_L - x_E)} + \frac{x_R - x_L}{2} & g_E \in \left[g_E^{(0)}, g_E^{(1)}\right] \\ 1 - \left(\frac{\lambda g_E - g_L}{2k(x_L - x_E)} + \frac{x_E + x_L}{2}\right) & g_E \in \left[g_E^{(1)}, g_E^{(2)}\right] \\ 1 & g_E \geq g_E^{(2)} \end{cases}$$

where $g_E^{(1)} = \frac{g_R}{\lambda} + \frac{k}{\lambda}\left(x_E - x_R\right)\left(x_E + x_R - 2\right)$ and $g_E^{(2)} = \frac{g_L}{\lambda} + \frac{k}{\lambda}\left(x_E^2 - x_L^2\right)$.

A.2 Left Incumbent Party (L)

For the case of the left party, we start by assuming that neither of the opponent parties has dominated the other. This means that $x_{RE} \in (0, 1)$. If $x_{RE} \leq 0$ then party R has strictly dominated party E, and if

$x_{RE} \geq 1$ then E has dominated R. In this case, party L would compete only with the remaining party.

In this case, the initial level of public good provision of party L, $g_L^{(0)}$, from which it would obtain a positive probability of winning is such that $x_{LE} = 0$. This is:

$$\begin{aligned} x_{LE} &= 0 \\ \frac{\lambda g_E - g_L}{2k(x_L - x_E)} + \frac{x_E + x_L}{2} &= 0 \\ \therefore \quad g_L^{(0)} &= \lambda g_E - k(x_E^2 - x_L^2) \end{aligned} \tag{6}$$

Given that L is one of the corner parties, and remembering that all parties have a positive expected vote share, the first party it dominates when increases g_L is the entrant party E and then party R. This would mean that $g_L^{(1)}$ is such that $x_{LE} = x_{RE}$ and $g_L^{(2)}$ when $x_{LR} = 1$. Therefore:

$$\begin{aligned} x_{LE} &= x_{RE} \\ \frac{\lambda g_E - g_L}{2k(x_L - x_E)} + \frac{x_E + x_L}{2} &= \frac{\lambda g_E - g_R}{2k(x_R - x_E)} + \frac{x_E + x_R}{2} \\ g_L - \lambda g_E + k(x_E^2 - x_L^2) &= \left(\frac{x_E - x_L}{x_R - x_E}\right)(\lambda g_E - g_R) + k(x_R + x_E)(x_E - x_L) \\ \therefore \quad g_L^{(1)} &= \lambda \left(\frac{x_R - x_L}{x_R - x_E}\right) g_E - \left(\frac{x_E - x_L}{x_R - x_E}\right) g_R + k(x_E - x_L)(x_R - x_L) \end{aligned} \tag{7}$$

and,

$$\begin{aligned} x_{LR} &= 1 \\ \frac{g_L - g_R}{2k(x_R - x_L)} + \frac{x_R + x_L}{2} &= 1 \\ g_L - g_R + k(x_R^2 - x_L^2) &= 2k(x_R - x_L) \\ \therefore \quad g_L^{(2)} &= g_R + k(x_R - x_L)(2 - x_R - x_L) \end{aligned} \tag{8}$$

Therefore, the winning probability of party L is:

$$p_L(g_L|\mathbf{g}_{-L}, \mathbf{x}) = \begin{cases} 0 & g_L \leq g_L^{(0)} \\ \frac{\lambda g_E - g_L}{2k(x_L - x_E)} + \frac{x_E + x_L}{2} & g_L \in \left[g_L^{(0)}, g_L^{(1)}\right] \\ \frac{g_L - g_R}{2k(x_R - x_L)} + \frac{x_R + x_L}{2} & g_L \in \left[g_L^{(1)}, g_L^{(2)}\right] \\ 1 & g_L \geq g_L^{(2)} \end{cases}$$

A.3 Right Incumbent Party (R)

Analogous to the case of L, we again assume that neither of the opponent parties has dominated the other. In this case, that would be that $x_{LE} \in (0, 1)$. If $x_{LE} \leq 0$ then party E has strictly dominated party L, and if $x_{LE} \geq 1$ then L has dominated E. The initial level of public good $g_R^{(0)}$ from which it would obtain a positive probability of winning is such that $x_{RE} = 1$. This is:

$$\begin{aligned}
x_{RE} &= 1 \\
\frac{\lambda g_E - g_R}{2k(x_R - x_E)} + \frac{x_R + x_E}{2} &= 1 \\
\lambda g_E - g_R + k(x_R^2 - x_E^2) &= 2k(x_R - x_E) \\
\therefore \quad g_R^{(0)} &= \lambda g_E + k(x_R - x_E)(x_R + x_E - 2)
\end{aligned} \tag{9}$$

Again, the first party it dominates when increases g_R is the entrant party E and then party L. This would mean that $g_R^{(1)}$ is such that $x_{LE} = x_{RE}$ and $g_R^{(2)}$ when $x_{LR} = 0$. Therefore:

$$\begin{aligned}
x_{LE} &= x_{RE} \\
\frac{\lambda g_E - g_L}{2k(x_L - x_E)} + \frac{x_E + x_L}{2} &= \frac{\lambda g_E - g_R}{2k(x_R - x_E)} + \frac{x_E + x_R}{2} \\
\frac{x_R - x_E}{x_E - x_L}(g_L - \lambda g_E) + k(x_E + x_L)(x_R - x_E) &= \lambda g_E - g_R + k(x_R^2 - x_E^2) \\
\therefore \quad g_R^{(1)} &= \lambda \left(\frac{x_R - x_L}{x_E - x_L}\right) g_E - \left(\frac{x_R - x_E}{x_E - x_L}\right) g_L + k(x_R - x_E)(x_R - x_L)
\end{aligned} \tag{10}$$

and,

$$\begin{aligned}
x_{LR} &= 0 \\
\frac{g_L - g_R}{2k(x_R - x_L)} + \frac{x_R + x_L}{2} &= 0 \\
g_R - g_L - k(x_R^2 - x_L^2) &= 0 \\
\therefore \quad g_R^{(2)} &= g_L + k(x_R^2 - x_L^2)
\end{aligned} \tag{11}$$

Therefore, the winning probability of party R is:

$$p_R(g_R|\mathbf{g}_{-R}, \mathbf{x}) = \begin{cases} 0 & g_R \leq g_R^{(0)} \\ 1 - \left(\frac{\lambda g_E - g_R}{2k(x_R - x_E)} + \frac{x_R + x_E}{2}\right) & g_R \in \left[g_R^{(0)}, g_R^{(1)}\right] \\ 1 - \left(\frac{g_L - g_R}{2k(x_R - x_L)} + \frac{x_R + x_L}{2}\right) & g_R \in \left[g_R^{(1)}, g_R^{(2)}\right] \\ 1 & g_R \geq g_R^{(2)} \end{cases}$$

Is it noteworthy to observe that for each party, the probabilities are non-decreasing for the whole support. It is direct to check that:

$$\frac{\partial p_i}{\partial g_i} \geq 0 \qquad \forall g_i \in [0,1] \quad \text{and} \quad i \in I$$

and also, that:

$$\frac{\partial p_i^{(0)}}{\partial g_i} \leq \frac{\partial p_i^{(1)}}{\partial g_i} \qquad \forall g_i \in [0,1] \quad \text{and} \quad i \in I$$

whenever $x_L < x_E < x_R$.

Appendix B Best Responses on Public Good Provision Subgame

Replacing (1) in (2), we obtain the following condition:

$$1-\mu_1-\mu_2-\left(\mu_1\left((1-g_i)\frac{\partial p_i^{(0)}}{\partial g_i}-p_i^{(0)}\right)+\mu_2\left((1-g_i)\frac{\partial p_i^{(1)}}{\partial g_i}-p_i^{(1)}\right)\right)=0$$

$$\mu_1\left(1+(1-g_i)\frac{\partial p_i^{(0)}}{\partial g_i}-p_i^{(0)}\right)+\mu_2\left(1+(1-g_i)\frac{\partial p_i^{(1)}}{\partial g_i}-p_i^{(1)}\right)=1 \tag{12}$$

There are 8 possible cases, depending on the values of μ_i. From equation (12), the cases $\{\mu_1 = 0, \mu_2 = 0, \mu_3 = 0\}$ and $\{\mu_1 = 0, \mu_2 = 0, \mu_3 \geq 0\}$ can be dismissed. We need to revise two cases of interest: $\{\mu_1 \geq 0, \mu_2 = 0, \mu_3 = 0\}$, where the optimal value is located in the interval $\left[g_i^{(0)}, g_i^{(1)}\right]$ and the case $\{\mu_1 = 0, \mu_2 \geq 0, \mu_3 = 0\}$, where the optimal is located in $\left[g_i^{(1)}, g_i^{(2)}\right]$. There other four cases are trivial because the optimal provision is at the kinks $g_i^{(j)}$: the optimal promise of public good is $g_i^{(1)}$ on the cases $\{\mu_1 \geq 0, \mu_2 = 0, \mu_3 \geq 0\}$, $\{\mu_1 \geq 0, \mu_2 \geq 0, \mu_3 = 0\}$ and $\{\mu_1 \geq 0, \mu_2 \geq 0, \mu_3 \geq 0\}$, and it is $g_i^{(2)}$ when $\{\mu_1 = 0, \mu_2 \geq 0, \mu_3 \geq 0\}$.

(1) Case $\mu_1 \geq 0, \mu_2 = 0, \mu_3 = 0$:

We can solve equation (12) for μ_1, obtaining the following value for the multiplier:

$$\mu_1=\left(1+(1-g_i)\frac{\partial p_i^{(0)}}{\partial g_i}-p_i^{(0)}\right)^{-1}$$

Considering $\mu_2 = \mu_3 = 0$, then it follows that $\mu_1 = 1$. So the optimal provision of public good will be the solution of the equation:

$$\mu_1=1\Rightarrow\left(1+(1-g_i)\frac{\partial p_i^{(0)}}{\partial g_i}-p_i^{(0)}\right)^{-1}=1$$

$$(1-g_i)\frac{\partial p_i^{(0)}}{\partial g_i}-p_i^{(0)}=0$$

which is the first order condition of the maximization problem on the interval $\left[g_i^{(0)}, g_i^{(1)}\right]$. Assuming that the optimal provision corresponds to g_i^*, then $z^* = (1-g_i^*)p_i^{*(0)}$. By primal feasibility, it holds if and only if:

$$\begin{aligned} z^* &\leq (1-g_i^*)p_i^{*(1)} \\ z^* &\leq (1-g_i^*) \end{aligned} \quad\Longleftrightarrow\quad \begin{aligned} p_i^{*(0)} &\leq p_i^{*(1)} \\ p_i^{*(0)} &\leq 1 \end{aligned}$$

(2) Case $\mu_1 = 0, \mu_2 \geq 0, \mu_3 = 0$:

The same arguments of the previous case can be applied to this case. We obtain the optimal provision of public good from the first order condition on the second interval:

$$\mu_2 = 1 \Rightarrow \left(1 + (1-g_i)\,\frac{\partial p_i^{(1)}}{\partial g_i} - p_i^{(1)}\right)^{-1} = 1$$

$$(1-g_i)\,\frac{\partial p_i^{(1)}}{\partial g_i} - p_i^{(1)} = 0$$

Assuming again that the optimal provision corresponds to g_i^*, then $z^* = (1-g_i^*)p_i^{*(1)}$. By primal feasibility, this case holds if and only if:

$$\begin{aligned} z^* &\leq (1-g_i^*)p_i^{*(0)} \\ z^* &\leq (1-g_i^*) \end{aligned} \qquad \Longleftrightarrow \qquad \begin{aligned} p_i^{*(1)} &\leq p_i^{*(0)} \\ p_i^{*(1)} &\leq 1 \end{aligned}$$

(3) Case $\mu_1 \geq 0, \mu_2 = 0, \mu_3 \geq 0$:

In this case, the first and third constraints are active. If $\mu_2 = 0$, then $\mu_3 = 1 - \mu_1$. So this situation will hold whenever:

$$\begin{aligned} \mu_1 &\geq 0 \\ 1 - \mu_1 &\geq 0 \end{aligned} \qquad \Longleftrightarrow \qquad 0 \leq \left(1 + (1-g_i)\,\frac{\partial p_i^{(0)}}{\partial g_i} - p_i^{(0)}\right)^{-1} \leq 1$$

Assuming that $\left(1 + (1-g_i)\,\frac{\partial p_i^{(0)}}{\partial g_i} - p_i^{(0)}\right)^{-1} \neq 0$, then we can write the previous condition as:

$$1 \leq \left(1 + (1-g_i)\,\frac{\partial p_i^{(0)}}{\partial g_i} - p_i^{(0)}\right)$$

Given that the first and third constraints are active, it follows that $p_i^{*(0)} = 1$. So, the condition reduces to:

$$1 \leq (1-g_i)\,\frac{\partial p_i^{(0)}}{\partial g_i}$$

Assuming g_i as the optimal solution, then $z^* = 1 - g_i^*$. At the same time, for $\mu_2 = 0$ to hold, by primal feasibility we would have:

$$\begin{aligned} z^* &\leq (1-g_i^*)p_i^{*(1)} \\ 1 &\leq p_i^{*(1)} \end{aligned}$$

If there was the case, then there does not exist a public good commitment such that the party can overthrow its opponents sequentially. Instead, when reaching the provision such that $p_i^{*(0)} = 1$, both opponents are defeated simultaneously. Therefore, the optimal provision is $g_i^* = g_i^{(2)}$.

(4) Case $\mu_1 \geq 0, \mu_2 \geq 0, \mu_3 = 0$:

In this case, $\mu_2 = 1 - \mu_1$, so solving for μ_1 yields:

$$\mu_1 \left(1 + (1 - g_i)\frac{\partial p_i^{(0)}}{\partial g_i} - p_i^{(0)}\right) + (1 - \mu_1)\left(1 + (1 - g_i)\frac{\partial p_i^{(1)}}{\partial g_i} - p_i^{(1)}\right) = 1$$

$$\mu_1 \left((1 - g_i)\left(\frac{\partial p_i^{(0)}}{\partial g_i} - \frac{\partial p_i^{(1)}}{\partial g_i}\right) - \left(p_i^{(0)} - p_i^{(1)}\right)\right) + ((1 - g_i)) = 0$$

$$\mu_1 = \left((1 - g_i)\left(\frac{\partial p_i^{(1)}}{\partial g_i} - \frac{\partial p_i^{(0)}}{\partial g_i}\right) - \left(p_i^{(1)} - p_i^{(0)}\right)\right)^{-1}\left((1 - g_i)\frac{\partial p_i^{(1)}}{\partial g_i} - p_i^{(1)}\right)$$

Given that the first two constraints are active, then $p_i^{(0)} = p_i^{(1)}$, which happens at $g_i^{(1)}$. Then, the optimal provision would be $g_i^* = g_i^{(1)}$. The multiplier reduces to:

$$\mu_1 = \left((1 - g_i)\left(\frac{\partial p_i^{(1)}}{\partial g_i} - \frac{\partial p_i^{(0)}}{\partial g_i}\right)\right)^{-1}\left((1 - g_i)\frac{\partial p_i^{(1)}}{\partial g_i} - p_i^{(1)}\right)$$

So the condition for this case is:

$$0 \leq \left((1 - g_i)\left(\frac{\partial p_i^{(1)}}{\partial g_i} - \frac{\partial p_i^{(0)}}{\partial g_i}\right)\right)^{-1}\left((1 - g_i)\frac{\partial p_i^{(1)}}{\partial g_i} - p_i^{(1)}\right) \leq 1$$

(5) Case $\mu_1 = 0$, $\mu_2 \geq 0, \mu_3 \geq 0$:

When considering this case, the probabilities of winning are $p_i^{*(1)} = 1$. So it follows that the optimal provision is $g_i^* = g_i^{(2)}$. This will hold if and only if:

$$\begin{aligned} \mu_2 &\geq 0 \\ 1 - \mu_2 &\geq 0 \end{aligned} \quad \Longleftrightarrow \quad 0 \leq \left(1 + (1 - g_i)\frac{\partial p_i^{(1)}}{\partial g_i} - p_i^{(1)}\right)^{-1}\Bigg|_{g_i = g_i^{(2)}} \leq 1$$

Similar to case (3), and given that the second and third constraints are active, it follows that $p_i^{*(1)} = 1$. Then, the condition reduces to:

$$1 \leq (1 - g_i)\frac{\partial p_i^{(1)}}{\partial g_i}$$

Assuming g_i as the optimal solution, then $z^* = 1 - g_i^*$. At the same time, for $\mu_1 = 0$ to hold, by primal feasibility we would have:

$$\begin{aligned} z^* &\leq (1 - g_i^*)p_i^{*(0)} \\ 1 &\leq p_i^{*(0)} \end{aligned}$$

If there was the case, then there does not exist a public good commitment such that the party can overthrow its opponents sequentially. Instead, when reaching the provision such that $p_i^{*(0)} = 1$, both opponents are defeated simultaneously. Therefore, the optimal provision is $g_i^* = g_i^{(2)}$.

(6) Case $\{\mu_1 \geq 0, \mu_2 \geq 0, \mu_3 \geq 0\}$:

Finally, if the case holds, then all constraints are active. Hence, by primal feasibility:

$$\begin{aligned} z^* &= (1-g_i^*)p_i^{*(0)} \\ z^* &= (1-g_i^*)p_i^{*(1)} \qquad \iff \qquad p_i^{*(0)} = p_i^{*(1)} = 1 \\ z^* &= (1-g_i^*) \end{aligned}$$

From the previous condition, it follows necessarily that the optimal provision is $g_i^* = g_i^{(1)} = g_i^{(2)}$.

Considering all previous cases, the best response of a party i to the public good provision of its opponents can be summarized in the following function:

$$\phi_i(g_i|\mathbf{g}_{-i},\mathbf{x}) = \begin{cases} \underset{g_i}{\arg\max}(1-g_i)p_i^{(0)} & \text{if} \quad p_i^{(0)*} \leq p_i^{(1)*} \wedge p_i^{(0)*} \leq 1 \\ \underset{g_i}{\arg\max}(1-g_i)p_i^{(1)} & \text{if} \quad p_i^{(1)*} \leq p_i^{(0)*} \wedge p_i^{(1)*} \leq 1 \\ g_i^{(1)} & \text{if} \quad 0 \leq \left((1-g_i)\left(\frac{\partial p_i^{(1)}}{\partial g_i} - \frac{\partial p_i^{(0)}}{\partial g_i}\right)\right)^{-1}\left((1-g_i)\frac{\partial p_i^{(1)}}{\partial g_i} - p_i^{(1)}\right) \leq 1 \\ g_i^{(2)} & \text{if} \quad \left(1 \leq (1-g_i)\frac{\partial p_i^{(0)}}{\partial g_i} \wedge p_i^{(1)*} \geq 1\right) \vee \left(p_i^{*(0)} = p_i^{*(1)} = 1\right) \vee \left(1 \leq (1-g_i)\frac{\partial p_i^{(1)}}{\partial g_i} \wedge p_i^{(0)*} \geq 1\right) \end{cases}$$

B.1 Best Response of Party E

For party E, we will have the following best response, assuming without loss of generality that party L can be beaten with a lower promise of public good than party R:

$$\phi_E(\mathbf{G}) = \begin{cases} \frac{1}{2}\left[1 + \left(\frac{x_R - x_E}{x_R - x_L}\right)\frac{g_L}{\lambda} + \left(\frac{x_E - x_L}{x_R - x_L}\right)\frac{g_R}{\lambda} - \frac{k}{\lambda}(x_R - x_E)(x_E - x_L)\right] \\ \qquad \text{if} \quad \frac{\lambda g_E^* - g_L}{2k(x_L - x_E)} + \frac{x_E + x_L}{2} \geq 0 \wedge \frac{\lambda g_E^* - g_R}{2k(x_R - x_E)} - \frac{\lambda g_E^* - g_L}{2k(x_L - x_E)} + \frac{x_R - x_L}{2} \leq 1 \\ \frac{1}{2}\left[1 + g_R - k(x_R^2 - x_E^2)\right] \\ \qquad \text{if} \quad \frac{\lambda g_E^* - g_L}{2k(x_E - x_L)} + \frac{x_E + x_L}{2} \leq 0 \wedge \frac{\lambda g_E^* - g_R}{2k(x_R - x_E)} + \frac{x_R + x_E}{2} \leq 1 \\ \frac{g_L}{\lambda} + \frac{k}{\lambda}\left(e^2 - l^2\right) \\ \qquad \text{if} \quad 0 \leq \frac{2k(x_L - x_E)}{\lambda(x_R - x_E)} - \left(\frac{x_L - x_E}{x_R - x_E}\right)\frac{\lambda g_E^* - g_R}{\lambda(1 - g_E^*)} + \frac{k(x_L - x_E)(x_R + x_E)}{1 - g_E^*} \leq 1 \\ \frac{g_R}{\lambda} + \frac{k}{\lambda}(x_E - x_R)(x_E + x_R - 2) \\ \qquad \text{if} \quad \left(\frac{\lambda(x_R - x_L)(1 - g_E^*)}{2k(x_E - x_L)(x_R - x_L)} \geq 1 \wedge \frac{\lambda g_E^* - g_R}{2k(x_R - x_E)} + \frac{x_R + x_E}{2} \geq 1\right) \vee \\ \qquad \frac{\lambda g_E^* - g_L}{2k(x_L - x_E)} + \frac{x_E + x_L}{2} = 0 \vee \\ \qquad \left(\frac{\lambda(1 - g_E^*)}{2k(x_R - x_E)} \geq 1 \wedge \frac{\lambda g_E^* - g_R}{2k(x_R - x_E)} - \frac{\lambda g_E^* - g_L}{2k(x_L - x_E)} + \frac{x_R - x_L}{2} \geq 1\right) \end{cases}$$

B.2 Best Response of Party L

Assuming that all parties have positive probabilities of winning, the best response function of party L is:

$$\phi_L(\mathbf{G}) = \begin{cases} \frac{1}{2}\left[1+\lambda g_E - k(x_E^2 - x_L^2)\right] \\ \qquad \text{if} \quad \frac{\lambda g_E - g_L^*}{2k(x_L - x_E)} - \frac{g_L^* - g_R}{2k(x_R - x_L)} - \frac{x_R - x_E}{2} \leq 0 \wedge \frac{\lambda g_E - g_L^*}{2k(x_L - x_E)} + \frac{x_E + x_L}{2} \leq 1 \\ \frac{1}{2}\left[1+ g_E - k(x_R^2 - x_L^2)\right] \\ \qquad \text{if} \quad \frac{\lambda g_E - g_L^*}{2k(x_L - x_E)} - \frac{g_L^* - g_R}{2k(x_R - x_L)} - \frac{x_R - x_E}{2} \geq 0 \wedge \frac{g_L^* - g_R}{2k(x_R - x_L)} + \frac{x_R + x_L}{2} \leq 1 \\ \lambda\left(\frac{x_R - x_L}{x_R - x_E}\right) g_E - \left(\frac{x_E - x_L}{x_R - x_E}\right) g_R + k(x_R - x_L)(x_E - x_L) \\ \qquad \text{if} \quad 0 \leq \left(\frac{x_E - x_L}{x_R - x_E}\right)(1 - g_L^* + gR - k(x_R^2 - x_L^2)) \leq 1 \\ g_R + k(x_E - x_L)(2 - x_E - x_L) \\ \qquad \text{if} \quad \left(\frac{1 - g_L^*}{2k(x_E - x_L)} \geq 1 \wedge \frac{g_L^* - g_R}{2k(x_R - x_L)} + \frac{x_R + x_L}{2} \geq 1\right) \vee \\ \qquad \quad \left(\frac{g_L^* - g_R}{2k(x_R - x_L)} + \frac{x_R + x_L}{2} = 1 \wedge \frac{\lambda g_E - g_L^*}{2k(x_L - x_E)} + \frac{x_E + x_L}{2} = 1\right) \vee \\ \qquad \quad \left(\frac{1 - g_L^*}{2k(x_R - x_L)} \geq 1 \wedge \frac{\lambda g_E - g_L^*}{2k(x_L - x_E)} + \frac{x_E + x_L}{2} \geq 1\right) \end{cases}$$

B.3 Best Response of Party R

Assuming that all parties have positive probabilities of winning, the best response function of party R is:

$$\phi_R(\mathbf{G}) = \begin{cases} \frac{1}{2}\left[1+\lambda g_E - k(x_R - x_E)(2 - x_R - x_E)\right] \\ \qquad \text{if} \quad \frac{g_L - g_R^*}{2k(x_R - x_L)} - \frac{\lambda g_E - g_R^*}{2k(x_R - x_E)} - \frac{x_E - x_L}{2} \leq 0 \wedge \frac{\lambda g_E - g_R^*}{2k(x_R - x_E)} + \frac{x_R + x_E}{2} \geq 0 \\ \frac{1}{2}\left[1+ g_L - k(2 - x_R - x_L)\right] \\ \qquad \text{if} \quad \frac{g_L - g_R^*}{2k(x_R - x_L)} - \frac{\lambda g_E - g_R^*}{2k(x_R - x_E)} - \frac{x_E - x_L}{2} \geq 0 \wedge \frac{g_L - g_R^*}{2k(x_R - x_L)} + \frac{x_R + x_L}{2} \geq 0 \\ \lambda\left(\frac{x_R - x_L}{x_E - x_L}\right) g_E - \left(\frac{x_R - x_E}{x_E - x_L}\right) g_L + k(x_R - x_L)(x_R - x_E) \\ \qquad \text{if} \quad 0 \leq \left(\frac{x_E - x_R}{x_E - x_L}\right)(1 + g_L - g_R^* + k(x_R - x_L)(2 - x_R - x_L)) \leq 1 \\ g_L + k(x_R^2 - x_L^2) \\ \qquad \text{if} \quad \left(\frac{1 - g_R^*}{2k(x_R - x_E)} \geq 1 \wedge \frac{g_L - g_R^*}{2k(x_R - x_L)} + \frac{x_R + x_L}{2} \leq 0\right) \vee \\ \qquad \quad \left(\frac{\lambda g_E - g_R^*}{2k(x_R - x_E)} + \frac{x_R + x_E}{2} = 0 \wedge \frac{g_L - g_R^*}{2k(x_R - x_L)} + \frac{x_R + x_L}{2} = 0\right) \vee \\ \qquad \quad \left(\frac{1 - g_R^*}{2k(x_R - x_L)} \geq 1 \wedge \frac{\lambda g_E - g_R^*}{2k(x_R - x_E)} + \frac{x_R + x_E}{2} \geq 1\right) \end{cases}$$

B.4 Best Response Correspondence

Let us consider the best response correspondence $\Phi(\mathbf{G}) = (\phi_L(\mathbf{G}), \phi_R(\mathbf{G}), \phi_E(\mathbf{G})$. Considering that each party's best response is piece-wise defined, then the best response correspondence is also a piece-wise function. Nonetheless, there are cases that are not feasible for this setting. Suppose that an incumbent chooses to provide $g_i^{(2)}$ as its best response. As we obtained in Section 4.1, providing this level of public good would yield the other parties zero probabilities of winning, and therefore, they would be overthrown of the electoral competition. But for this case to happen, there could be no feasible amount of g_{-i} that they could provide to avoid being defeated, which contradicts the fact that there exists an amount $g_{-i}^{(0)}$ for which they could have a positive chance of winning. Then, we can disregard all cases where parties offer $g_i^{(2)}$. Similar is the case for when parties offer $g_i^{(1)}$, because again, all parties have a level $g_i^{(0)}$ that they can offer to have a positive vote share and avoid being defeated.

Disregarding all these impossible situations leaves us with just one case, where all parties have a positive vote share and they offer a positive amount of public good. Therefore, the best response correspondence is $\Phi(\mathbf{G}) = (\phi_L(\mathbf{G}), \phi_R(\mathbf{G}), \phi_E(\mathbf{G})$, where:

$$\begin{aligned}
\phi_L(\mathbf{G}) &= \frac{1}{2}\left[1 + \lambda g_E - k(x_E^2 - x_L^2)\right] \\
\phi_R(\mathbf{G}) &= \frac{1}{2}\left[1 + \lambda g_E - k(x_R - x_E)(2 - x_R - x_E)\right] \\
\phi_E(\mathbf{G}) &= \frac{1}{2}\left[1 + \left(\frac{x_R - x_E}{x_R - x_L}\right)\frac{g_L}{\lambda} + \left(\frac{x_E - x_L}{x_R - x_L}\right)\frac{g_R}{\lambda} - \frac{k}{\lambda}(x_R - x_E)(x_E - x_L)\right]
\end{aligned}$$

Appendix C Proofs

C.1 Proof of Proposition 1

The winning probability of party i is non-decreasing in g_i. Furthermore, considering that the probability is continuous and piece-wise linear in the segment $[g_i^{(0)}, g_i^{(2)}]$, then there exists an inverse probability function $p_i^{-1} : [0,1] \to [g_i^{(0)}, g_i^{(2)}]$. Let us consider the upper contour set $\succsim_{p_i}(c) = \{g \in \mathbb{R}_+ : p_i(g) \geq c\}$. Whenever $c \in [0,1]$, this set will be equivalent to $\succsim_{p_i}(c) = \{g \in \mathbb{R}_+ : g \geq p_i^{-1}(c)\}$ which is a convex subset of $\mathbb{R}$. If $c \leq 0$ then $\succsim_{p_i}(c) = \mathbf{R}_+$ and if $c > 1$ then $\succsim_{p_i}(c) = \varnothing$, which are also convex. Therefore, given that the upper contour sets are convex for every $c \in \mathbb{R}_+$, then probability p_i is quasi-concave.

C.2 Proof of Proposition 2

Let us consider the expected profit function $E\left[\pi_i(g_i|\mathbf{g}_i,\mathbf{x})\right] = (1-g_i)\cdot p_i(g_i|\mathbf{g}_i,\mathbf{x})$. The function $(1-g_i)$ is a strictly decreasing function on g_i and $p_i(g_i|\mathbf{g}_i,\mathbf{x})$ is non-decreasing on g_i. The second derivative of this is:

$$\begin{aligned}\frac{\partial^2 E\left[\pi_i(g_i|\mathbf{g}_i,\mathbf{x})\right]}{\partial g_i^2} &= \frac{\partial^2\left(1-g_i\right)}{\partial g_i^2} + 2\frac{\partial\left(1-g_i\right)}{\partial g_i}\frac{\partial p_i(g_i|\mathbf{g}_i,\mathbf{x})}{\partial g_i} + \frac{\partial^2 p_i(g_i|\mathbf{g}_i,\mathbf{x})}{\partial g_i^2}\\ &= -2\frac{\partial p_i(g_i|\mathbf{g}_i,\mathbf{x})}{\partial g_i} + \frac{\partial^2 p_i(g_i|\mathbf{g}_i,\mathbf{x})}{\partial g_i^2}\end{aligned}$$

Given that p_i is linear and non-decreasing on g_i, then $\frac{\partial p_i}{\partial g_i} \geq 0$ and $\frac{\partial^2 p_i}{\partial g_i^2} = 0$, implying that $\frac{\partial^2 E[\pi_i(g_i|\mathbf{g}_i,\mathbf{x})]}{\partial g_i^2} \leq 0$ for all $g_i \geq 0$. Therefore, the expected profit $E\left[\pi_i(g_i|\mathbf{g}_i,\mathbf{x})\right]$ is quasi-concave on g_i.

C.3 Proof of Proposition 4

To begin the demonstration, let us state the following theorem from Friedman (1977):

Theorem 1. *Friedman (1977)*
Let $\Gamma = \{\mathcal{S}_1,\ldots,\mathcal{S}_n; f_1,\ldots,f_n\}$ *be a n-person non-cooperative game in normal form. If*

a) *the strategy sets* $\mathcal{S}_1,\ldots,\mathcal{S}_n$ *are non-empty, compact convex subsets of finite dimensional Euclidean spaces;*

b) *all payoff function* $f_1,\ldots,f_n$ *are continuous on* $\mathcal{S} = \mathcal{S}_1 \times \ldots \times \mathcal{S}_n$*;*

c) *every* f_i *is a quasi-concave function of* s_i *over* $\mathcal{S}_i$ *if all the other strategy vectors are held fixed,*

then Γ *has at least one Nash-equilibrium point.*

Let $\Gamma = \{\mathcal{S}_1,\ldots,\mathcal{S}_3; E[\pi_1],\ldots,E[\pi_3]\}$ be the three-party non-cooperative game on provision of public goods, where $\mathcal{S}_i$ are the strategy setsand $E[\pi_i]$ are the expected benefits functions from each party. Let us consider that,

a) The public good provision strategy sets correspond to $\mathcal{S}_i = [0,1]$. These intervals are closed intervals of $\mathbf{R}$, thus they are non-empty, compact and convex subsets of a finite Euclidean space.

b) The payoff functions are $E[\pi_i(g_i|\mathbf{g}_i,\mathbf{x})] = (1-g_i)\cdot p_i(g_i|\mathbf{g}_i,\mathbf{x})$, for any given locations of the party, all different from each other. The probability function $p_i(g_i|\mathbf{g}_i,\mathbf{x})$ is continuous on $\mathcal{S} = \mathcal{S}_1 \times \ldots \times \mathcal{S}_3$ for being the lower envelope of continuous linear functions. From the fact that the product of continuous functions is continuous, then pay-off functions $E[\pi_i(g_i|\mathbf{g}_i,\mathbf{x})]$ are continuous on $\mathcal{S} = \mathcal{S}_1 \times \ldots \times \mathcal{S}_n$.

c) From Proposition 2, payoff functions are quasi-concave are quasi-concave on their own strategy $\mathcal{S}_i$, if all the opponent strategy vectors are held fixed.

From Theorem 1, it follows that there exists a non-cooperative Nash equilibrium in the public good provision sub-game, played for given party locations $\{x_L, x_R, x_E\} \in [0, 1]$ each different from one another.

C.4 Proof of Proposition 5

To prove that the equilibrium in the public good provision sub-game is unique, it is sufficient to demonstrate that the best response correspondence satisfies Banach's fixed point theorem. Banach's fixed point theorem states:

Theorem 2. *Banach Fixed Point Theorem*
Let (X, d) be a non-empty complete metric space with a contraction mapping $T : X \to X$. Then T admits a unique fixed-point x^* in X. Furthermore, x^* can be found as follows: start with an arbitrary element x_0 in X and define a sequence $\{x_n\}$ by $x_n = T(x_n - 1)$, then $x_n \to x^*$.

Let $X = [0, 1]^3$ and d the usual metric on $\mathbf{R}^3$. Considering that X is a subset of $\mathbf{R}^3$, then it is non-empty and complete subset. Therefore, (X, d) is a non-empty complete metric space. It suffices to show that the best response correspondence is a contraction mapping on X. To prove this, we will use the following theorem:

Theorem 3. *Assume the set $D \subset \mathbf{R}^n$ is convex and the function $g : D \to \mathbf{R}^n$ has continuous partial derivatives $\frac{\partial g_i}{\partial x_i}$ in D. If for $q < 1$ the matrix norm of the Jacobian satisfies:*

$$\forall x \in D : \qquad ||J(g)|| \leq q$$

the mapping g is a contraction in D.

Proof. Let $x, y \in D$. Then the points on the straight line from x to y are given by $x + t(y - x)$ for $t \in [0, 1]$. As D is convex all these points are contained in D. Let $G(t) = g(x + t(y - x))$, then by the chain rule we have $G'(t) = J(g)(x + t(y - x)) \cdot (y - x)$ and

$$g(y) - g(x) = G(1) - G(0) = \int_0^1 G'(t)dt = \int_0^1 J(g)(x + t(y - x)) \cdot (y - x)dt$$

As an integral of a continuous function is a limit of Riemann sums the triangle inequality implies

$|| \int_a^b F(t)dt|| \leq \int_a^b ||F(t)||dt$:

$$
\begin{aligned}
||g(y) - g(x)|| &\leq \int_0^1 ||J(q)(x + t(y - x)) \cdot (y - x)||dt \\
&\leq \int_0^1 \underbrace{||J(q)(x + t(y - x))||}_{\leq q} ||(y - x)||dt \\
&\leq q||y - x||
\end{aligned}
$$

$\square$

Let us consider matrix 1-norm, defined as $||A||_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}|$, where the element a_{ij} is the element (i, j) of matrix A. Then, to prove that the best response correspondence $\Phi(\mathbf{G})$ is a contraction, it suffices to show that $||J(\Phi)||_1 \leq q < 1$.

As obtained in Appendix B.4, the Jacobian matrix of the best response correspondence is the following:

$$
J(\Phi) = \begin{pmatrix} 0 & 0 & \frac{\lambda}{2} \\ 0 & 0 & \frac{\lambda}{2} \\ \frac{1}{2\lambda}\left(\frac{x_R - x_E}{x_R - x_L}\right) & \frac{1}{2\lambda}\left(\frac{x_E - x_L}{x_R - x_L}\right) & 0 \end{pmatrix}
$$

Therefore, $||J(\Phi)||_1 = \max\left\{\frac{\lambda}{2}, |\frac{1}{2\lambda}\left(\frac{x_R - x_E}{x_R - x_L}\right)| + |\frac{1}{2\lambda}\left(\frac{x_E - x_L}{x_R - x_L}\right)|\right\} = \frac{\lambda}{2}$, whenever $\lambda \geq 1$. Assuming that this is the case, then we can choose $q = \frac{\lambda}{2}$, with $\lambda < 2$, such that:

$$
||J(\Phi)||_1 \leq \frac{\lambda}{2} < 1
$$

Hence, the best response correspondence is a contraction mapping with Lipschitz constant $\frac{\lambda}{2}$. Given that the best response correspondence is a contraction mapping on X, then the conditions of the Banach Fixed Point Theorem hold and there exists a unique fixed point on X. Therefore, the non-cooperative Nash equilibrium in the public good provision sub-game is unique.