INSTITUTO DE ECONOMÍA

TESIS de DOCTORADO

2019

Essays on Knightian Uncertainty

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PONTIFICIA UNIVERSIDAD CATOLICA
INSTITUTO DE ECONOMIA
DOCTORADO EN ECONOMIA

TESIS DE GRADO

DOCTORADO EN ECONOMIA

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August, 2019

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Santiago, August 2019

To my daughters Agustina and Amanda

Acknowledgments

I thank the members of my dissertation committee, professors Klaus Schmidt-Hebbel (advisor), Jaime Casassus (co-advisor), Rodrigo Fuentes (reviewer), and Martin Uribe (reviewer). I thank my co-authors professors Diederik Aerts and Sandro Sozzo. I am also grateful to professors Sofía Bauducco, Rhys Bidder, Jaroslav Borovicka, Jesus Fernández-Villaverde, Javier García-Cicco, Lars Peter Hansen, and Massimo Marinacci for their very helpful comments and discussions. Finally, I thank Martín Carrasco and Hans Schlechter for their valuable comments.

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Chapter 1

Introduction

Frank Knight (1921) analyzed the problem of an investor, establishing - for the first time in economics - the key difference between risk and uncertainty. He defined the term “risk” to describe situations with stochastic outcomes that are described by known probabilities, while using the term “uncertainty” to refer to situations with outcomes described by unknown probabilities. The denomination “Knightian uncertainty” comes from this distinction. After Knight (1921), Savage (1954) contributed an axiomatic treatment of decision making in which preferences over gambles could be represented by maximizing expected utility under subjective probabilities. Savage’s work extended the earlier justification of expected utility by von Neumann and Morgenstern (1944) that had assumed known objective probabilities. Savage’s axioms justify subjective assignments of probabilities when they are not known in advance (Hansen and Sargent, 2010).

Daniel Ellsberg (1961) presents his famous experiment showing empirically the existence of ambiguity aversion, when agents prefer to face known probabilities rather than unknown or vague probabilities, which is not compatible with Savage’s model. Daniel Ellsberg refers to the term ambiguity instead of Knightian uncertainty following Arrow (1951), who distinguishes between two sources of uncertainty: (i) “Risk within a model”, where uncertainty is about the outcomes that emerge in accordance to a probability model that specifies fully the outcome probabilities, and (ii) “Ambiguity among models”, where uncertainty is about which alternative model should be used to assign probabilities. Recently, Hansen and Marinacci (2016) have extended the distinction presented by Arrow adding a third type of uncertainty named as “model misspecification” (also known as model uncertainty). This uncertainty arises when the true model is not assumed to be among the original set of models under consideration; then uncertainty is induced by the approximate nature of the models under consideration used in assigning probabilities.

This thesis is comprised by two essays related to decision-making under uncertainty in the sense of Frank Knight (1921). In a nutshell, the first essay introduces model uncertainty in an otherwise standard small open economy real business cycle model, and the second essay presents an alternative model of decision-making under ambiguity characterizing uncertainty by quantum probabilities.

The first essay extends the canonical small open-economy model of Schmitt-Grohé and Uribe

(2003), which is an expected-utility model for classical uncertainty (or risk), to non-expected utility considering model uncertainty. Domestic households have multiplier preferences (Hansen and Sargent 2008), which leads them to take robust decisions in response to possible model misspecification for the economy's aggregate productivity. Using perturbation methods, this essay extends the literature on RBC models by deriving a closed-form solution for the combined welfare effect of the two sources of uncertainty (risk and model uncertainty). While classical risk has an ambiguous effect on welfare (as shown by Cho et al. 2015), the addition of model uncertainty is unambiguously welfare-deteriorating. Hence the overall effect of uncertainty on welfare is ambiguous, depending on consumer preferences and model parameters. The paper provides numerical results for the welfare effects of uncertainty measured by units of consumption equivalence. At moderate (high) levels of risk aversion, the effect of risk on household welfare is positive (negative). The addition of model uncertainty - for all levels of concern about model uncertainty and most risk aversion values - turns the overall effect of uncertainty on household welfare negative. My first result for numerical simulations shows that for slightly risk-averse households, productivity shocks are in general welfare improving. However, if they are intensely risk averse, the welfare cost of productivity risk is estimated at 0.007% of long-term consumption, similar to the value of 0.008% found by Lucas (1987). The second finding is that by introducing model uncertainty, even when the agent is only mildly risk averse, the overall welfare cost of uncertainty becomes welfare-deteriorating. The third result, based on numerical calculations, shows that the total effect of uncertainty on welfare depends on the interaction of risk and model uncertainty. Welfare costs are increasing in both risk aversion and the concern for model uncertainty. Considering a moderately high degree of risk aversion and a reasonable concern for model uncertainty, the overall effect of uncertainty on welfare amounts to a loss of 0.4% of long-term consumption, which is two orders of magnitude higher than the finding by Lucas. If I consider the case of an agent with very high concern for model misspecification, the welfare cost reaches a staggering 2.3% of long-term consumption. Finally, I consider the existence of financial frictions, reflected in a high debt sensitivity of the sovereign risk premium, which has been proven to increase the welfare cost of productivity shocks in a small open economy, for the case of classical risk aversion (Ericson and Liu 2012). In a risk aversion context, the largest simulated welfare cost of business cycles is 0.013%, which is significantly smaller than the largest value obtained when including preferences for robustness. Therefore, with or without financial frictions, households with a preference for robustness are willing to sacrifice a lot to live in an economy without fear of misspecification. It is important to remark that the analytical decomposition and combination of the effects of the two types of uncertainty considered here and the resulting effect on overall welfare have not been derived in the previous literature on small open economies.

The second essay, co-authored with professors Diederik Aerts and Sandro Sozzo[1], elaborates a quantum model to describe the Ellsberg paradox, showing that the mathematical formalism of quantum mechanics is capable to represent the ambiguity present in this kind of situations, in terms of the appearance of typical quantum effects as interference and superposition. Then we analyze

[1]Aerts, Sozzo and Tapia (2014): Identifying Quantum Structures in the Ellsberg Paradox. *International Journal of Theoretical Physics*, Vol. 53(10), pp 3666-3682.

the data collected in a concrete experiment performed on the Ellsberg paradox and work out a representation of them in a complex Hilbert space. We are able to represent elections in Ellsberg's urn by considering superposition of priors, even when each individual prior cannot explain the empirically observed behavior. In our model, even when agents are neutral to risk, they can be ambiguity-averse.

Chapter 2

Welfare Cost of Model Uncertainty in a Small Open Economy

Abstract

This paper extends the canonical small open-economy model of Schmitt-Grohé and Uribe (2003), which is an expected-utility model for classical uncertainty (or risk), to non-expected utility considering model uncertainty. Domestic households have multiplier preferences (Hansen and Sargent 2008), which leads them to take robust decisions in response to possible model misspecification for the economy's aggregate productivity. Using perturbation methods, the paper extends the literature on RBC models by deriving a closed-form solution for the combined welfare effect of the two sources of uncertainty (risk and model uncertainty). While classical risk has an ambiguous effect on welfare (as shown by Cho et al. 2015), the addition of model uncertainty is unambiguously welfare-deteriorating. Hence the overall effect of uncertainty on welfare is ambiguous, depending on consumer preferences and model parameters. The paper provides numerical results for the welfare effects of uncertainty measured by units of consumption equivalence. At moderate (high) levels of risk aversion, the effect of risk on household welfare is positive (negative). The addition of model uncertainty – for all levels of concern about model uncertainty and most risk aversion values – turns the overall effect of uncertainty on household welfare negative. It is important to remark that the analytical decomposition and combination of the effects of the two types of uncertainty considered here and the resulting ambiguous effect on overall welfare have not been derived in the previous literature on small open economies.

2.1 Introduction

There is a long tradition in assessing the welfare cost of consumption variability, due to different sources of uncertainty. Lucas (1987) started this research area, considering a representative agent who obtains utility only from consumption, assuming expected utility, constant relative risk aversion, and a trend-stationary stochastic process of the log of consumption. Lucas finds a small effect of uncertainty on welfare: the percentage increase in the mean of consumption required to leave consumers indifferent between consumption fluctuations and a perfectly smooth consumption path is 0.008%[1].

This paper contributes to the literature on the welfare cost of business-cycle fluctuations in a small open economy (SOE) by introducing model uncertainty. Households face two types of uncertainty: classical uncertainty about the state of future aggregate productivity[2] and model uncertainty about the probability model that describes the movements of productivity. The characterization of the small open economy follows Schmitt-Grohé and Uribe (2003), augmented by multiplier preferences (Hansen and Sargent 2008). These preferences correspond to a non-expected utility approach that reflects a concern for model misspecification (model uncertainty). Households are endowed with a reference probability model about the stochastic variable (aggregate productivity), but they distrust its accuracy, considering the possibility that the model is misspecified in a way that is difficult to detect statistically. Consequently, households want to take decisions that are robust to model misspecification, by solving a max-min problem. Households maximize utility with respect to consumption, labor supply, and investment. Households face a malevolent nature that minimizes the expected utility choosing an unfavorable probabilistic distribution, called the worst case distribution. Nevertheless, the malevolent nature is penalized for the deviations from the reference model by means of conditional relative entropy between the reference and the distorted models. The solution of the minimization leads to a recursive value function, similar to Epstein and Zin (1989) and Weil (1990) preferences (succinctly EZW preferences) and the risk adjustment in the continuation value for risk sensitive preferences. This paper decomposes the effects of doubts and volatility on total welfare, deriving a closed-form solution for the cost due to stochastic consumption fluctuations and the cost derived from uncertainty about the stochastic representation of productivity shocks. While consumption fluctuations (volatility) have an ambiguous effect on welfare (as shown by Cho et al., 2015), the additional model uncertainty is unambiguously welfare-deteriorating. Hence the overall effect of uncertainty on welfare is ambiguous, depending on consumer preferences and model parameters. Then, the paper provides numerical results for the welfare effects of uncertainty. The simulations show that when classical risk has a positive effect on welfare (due to a low risk aversion), addition of model uncertainty can reverse the sign of the overall welfare effect. The total effect of uncertainty, including both sources, is shown to be negative for a wide range of parameter values.

The relevance of this paper's results lies in the broad discussion about macroeconomic stabilization, as highlighted by Lucas (1987, 2003). If the welfare cost of business cycles is negligible or if households prefer economic uncertainty, there is no space for counter-cyclical policy in small open economies. My first result shows that for slightly risk-averse households, productivity shocks are in general welfare improving. However, if they are intensely risk averse, the welfare cost of productivity risk is estimated at 0.007% of long-term consumption, similar to the value of 0.008% found by

[1]Lucas finds that the cost of consumption variability is proportional to risk aversion and to the variance of shocks. His 0.008% estimate corresponds to unitary risk aversion and a shock variance equal to 0.013.

[2]García-Cicco, Pancrazi, and Uribe (2010) identify productivity as the main driving factor of business cycles in small open economies.

Lucas (1987). The second finding is that by introducing model uncertainty, even when the agent is only mildly risk averse, the overall welfare cost of uncertainty becomes welfare-deteriorating. My third result, based on numerical calculations, shows that the total effect of uncertainty on welfare depends on the interaction of risk and model uncertainty. Welfare costs are increasing in both risk aversion and the concern for model uncertainty. Considering a moderately high degree of risk aversion and a reasonable concern for model uncertainty, the overall effect of uncertainty on welfare amounts to a loss of 0.4% of long-term consumption, which is two orders of magnitude higher than the finding by Lucas. If I consider the case of an agent with very high concern for model misspecification, the welfare cost reaches a staggering 2.3% of long-term consumption. Finally, I consider the existence of financial frictions, reflected in a high debt sensitivity of the sovereign risk premium, which has been proven to increase the welfare cost of productivity shocks in a SOE, for the case of classical risk aversion (Ericson and Liu 2012). In a risk aversion context the largest simulated welfare cost of business cycles is 0.013%[3], which is significantly smaller than the largest value obtained when including preferences for robustness. Therefore, in both cases households with a preference for robustness are willing to sacrifice a lot to live in an economy without fear of misspecification.

I discuss the literature related to my research next. Since Lucas' seminal work, the small cost of consumption fluctuations estimated by him has motivated many authors to extend his model in different ways. One branch of these extensions assumes traditional household preferences characterized by expected utility and time separability. Imrohoroglu (1989), Krusell and Smith (1999), and Krusell et al. (2009) extend Lucas' model by including uninsurable individual risk and incomplete markets, finding larger welfare costs of consumption variability. Alvarez and Jermann (2004) consider fluctuations in asset prices reporting a small cost of variability on welfare. Subsequent research shows that consumption volatility is not always welfare deteriorating when output is endogenous and there are multiplicative productivity shocks. Cho et al. (2015) identify two effects of productivity risk on welfare in production economies: the fluctuation effect and the mean effect. Whereas the first effect is always detrimental to welfare of risk averse agents, the second may increase welfare. When households respond to larger productivity uncertainty by working harder and investing more, the mean values of output and consumption could increase with higher uncertainty. Hence the final effect of uncertainty on welfare is ambiguous. In an open economy, where adjusting capital is easier than in the closed economy, the positive mean effect is larger than in a closed economy. In the same vein, Lester et al. (2014) show that welfare can be increasing in volatility of exogenous shocks in production economies where factor supply is sufficiently elastic.

Closely related to this investigation, a branch of literature specifies recursive utility without time-separable preferences. In this context, there are three main modeling approaches: EZW preferences, risk sensitivity, and robust control. Obstfeld (1994), Dolmas (1998), and Epaulard and Pommeret (2003) use EZW preferences to separate between the effect of risk aversion and the intertemporal elasticity of substitution to derive the welfare cost of business-cycle fluctuations. In a nutshell, they underscore the importance of shock persistence and intertemporal substitutability: the welfare cost of uncertainty increases with both risk aversion and the intertemporal elasticity

[3]Table 4 in Appendix C presents welfare calculations for a broad set of parameters.

of substitution. Xu (2017) examines the effects of stochastic volatility on welfare for a continuous-time extension of EZW preferences and shows that volatility risk can improve welfare, depending on model parameters.

Tallarini (2000) introduces risk sensitivity with the aim to match financial asset prices, while not affecting his model's ability to account for aggregate fluctuations. Assuming a unitary intertemporal elasticity of substitution[4], Tallarini shows that the risk-free rate and the market price of risk are better matched with augmented risk aversion, and this does not make consumption smoother. Moreover, the welfare cost of business-cycle fluctuations increases when preference parameters are chosen to match financial data, and the welfare implications are larger for the production economy than for the endowment economy.

In general, the advantage of introducing recursive utility as EZW preferences and risk sensitivity is that it allows to match models with financial data. Expected utility models with high risk aversion deliver a high market price of risk (addressing the equity premium puzzle), as observed in macro-financial data, but also raise the risk-free interest rate (not addressing the risk-free rate puzzle). On the other hand, recursive utility can reconcile the high market price of risk with reasonable values of the risk-free rate, because it allows to separate between risk aversion and the intertemporal elasticity of substitution. However, the value for the risk aversion parameter for explaining the behavior of consumption and asset prices observed in the data is very high. Lucas (2003) discusses the latter findings, stating that no one has found risk aversion parameters of the required size to match the data, and concludes: "It would be good to have the equity premium puzzle resolved, but I think we need to look beyond high estimates of risk aversion to do it."

Barillas et al. (2009) determine the welfare benefits of removing model uncertainty for a closed economy, reinterpreting part of the market price of risk as a market price of model uncertainty. Using the model of error detection to calibrate the concern for model misspecification, they show that modest amounts of model uncertainty can substitute for large amounts of risk aversion in terms of choices and asset prices. Therefore this approach allows to reconcile the high market price of risk with reasonable values of the risk-free rate without resorting to high risk aversion, and the welfare cost of business cycle calculation is as big as that presented by Tallarini (2000). In the same vein, Ellison and Sargent (2015) asses the combined effect of idiosyncratic risk and robustness. They find that individual risk has a larger impact on the cost of business cycles if agents have preferences for robustness, and the combined effect exceeds the sum of individual effects. As opposed to Barillas et al. (2009) and Ellison and Sargent (2015), who consider a closed economy and an exogenous stochastic process for consumption, this paper focuses on a small open production economy with optimal endogenous consumption.

Regarding solution of the model, this paper is related to the literature on perturbation methods for recursive utility. I perform a second-order approximation of the value function and equilibrium conditions (which depend on the value function). Up to a first-order approximation, the equilibrium conditions produced by recursive utility or expected utility are strictly equivalent, generating

[4]This hypothesis allows Tallarini to solve the production economy model by approximation, using the linear-quadratic method developed by Hansen et al. (1999).

what Levin et al. (2008) define as macroeconomic equivalence. But the underlying microeconomic differences between first-order equivalent models become important when optimal policy is derived (microeconomic dissonance). Furthermore, according to Schmitt-Grohé and Uribe (2004), first-order techniques are not well suited to handle welfare comparisons across alternative stochastic or policy environments, even without recursive preferences. Caldara et al. (2012) compare different solution methods for computing equilibria of dynamic stochastic general equilibrium (DSGE) models with recursive preferences. Their main finding is that perturbation methods are competitive with projection methods and value function iterations, in terms of accuracy, while being several orders of magnitude faster to run.

The paper is organized as follows. Section 2 presents the theoretical RBC model for the small open economy, describing in detail agent preferences and the characterization of the agent's alternative probabilistic models. Section 3 describes the solution method, derives the model solution, and presents the analytical welfare analysis. Section 4 performs numerical calculations for the welfare cost of classical and model uncertainty. Section 5 concludes.

2.2 The model

The model extends on the canonical Small-Open-Economy Real-Business-Cycle (SOE-RBC), model introduced by Schmitt-Grohé and Uribe (2003). In the SOE-RBC model, productivity shocks drive business cycles in a single-good and single-asset production economy. Here, the SOE-RBC model is extended by considering household preferences for robustness to possible model misspecification.

Extending the distinction put forward by Arrow (1951), Hansen and Marinacci (2016) distinguish between three sources of uncertainty: "(i) Risk within a model, where uncertainty is about the outcomes that emerge in accordance to a probability model that specifies fully the outcome probabilities. (ii) Ambiguity among models, where uncertainty is about which alternative model should be used to assign the probabilities. (iii) Model misspecification, where uncertainty is induced by the approximate nature of the models under consideration used in assigning probabilities."

This research is related to the third source of uncertainty, which is termed model uncertainty. Households have multiplier preferences (Hansen and Sargent 2001, 2008, 2010)[5] regarding productivity (represented by a Hicks neutral technical process): they have a benchmark model for this stochastic variable but they do not fully trust it. Households acknowledge that the benchmark model is an approximation of the true data-generating process and could be misspecified in some way. This leads them to take robust decisions that perform well across a variety of probability models "near" the benchmark. Agents locate their approximating model within a set of alternative models that are statistically nearby and which are probabilistic models characterized in terms of the distortions from the benchmark model. The distortions can be represented in terms of martingales that twist the benchmark measure in order to obtain absolutely continuous measures that represent

[5]Multiplier preferences are a particular case of variational preferences, as discussed by Maccheroni et al. (2006a, 2006b).

the alternative models.

Let F_t be the information available at time t, $\pi(\epsilon_{t+1})$ the benchmark density of shocks ϵ_{t+1} and $\hat{\pi}(\epsilon_{t+1} \mid F_t)$ an admissible alternative density conditioned on available information. The likelihood ratio between the alternative density of shocks and the benchmark probabilistic model is $m_{t+1} = \frac{\hat{\pi}(\epsilon_{t+1}|F_t)}{\pi(\epsilon_{t+1})}$, is non-negative and $E(m_{t+1} \mid F_t) = 1$[6]. The characterization of alternative models using the likelihood ratio allows to calculate the distorted expectation of a stochastic variable W_{t+1} in period t as $\hat{E}[W_{t+1} \mid F_t] = \hat{E}_t[W_{t+1}] = E_t[m_{t+1}W_{t+1}]$[7]. The conditional relative entropy $E[m_t \log m_t \mid F_t]$, known in econometrics as the Kullback-Leibler divergence, measures discrepancies between the alternative and reference models.

Multiplier preferences, represented by the Bellman equation (2.1), are defined in terms of a parameter θ that penalizes discrepancies between the distorted and reference models. These preferences present a max-min problem, where the households maximize utility choosing consumption, labor supply, and investment, while an evil agent minimizes utility by his choice of the worst probabilistic scenario. The max-min optimization is subject to the agents' flow budget constraint (2.2), technology (2.3), the law of motion of capital (2.4), the stochastic process for aggregate productivity (2.6), and a restriction that imposes a unitary expected value of the likelihood ratio between the reference and distorted probability measures (2.7).

$$V(d_t, k_t, a_t) = \max_{c_t, h_t, k_{t+1}, d_{t+1}} \min_{m_{t+1}} u(c_t, h_t) + \beta E_t[m_{t+1}V(d_{t+1}, k_{t+1}, a_{t+1}) + \theta m_{t+1} \log m_{t+1}] \tag{2.1}$$

$$s.t. \quad d_{t+1} = (1 + r_t)d_t - y_t + c_t + i_t + \frac{\phi}{2}(k_{t+1} - k_t)^2 \tag{2.2}$$

$$y_t = a_t k_t^{\alpha} h_t^{1-\alpha} \tag{2.3}$$

$$k_{t+1} = (1 - \delta)k_t + i_t \tag{2.4}$$

$$r_t = r^* + \psi(e^{\bar{d}_t - \bar{d}} - 1) \tag{2.5}$$

$$a_{t+1} = (1 - \rho)a_{ss} + \rho a_t + \tilde{\eta}\epsilon_{t+1} \tag{2.6}$$

$$E_t[m_{t+1}] = 1 \tag{2.7}$$

where c_t represents consumption, h_t denotes labor supply, β is the subjective discount factor, d_t denotes the households' debt position, r_t represents the interest rate at which domestic residents can borrow abroad, y_t denotes domestic output, i_t represents gross investment, k_t is physical capital, and $u(c_t, h_t)$ is the concave period utility function, which is increasing in c_t and decreasing in h_t.

The term $\frac{\phi}{2}(k_{t+1} - k_t)^2$ in equation (2.2) captures capital adjustment costs that avoid excessive investment volatility in response to shocks in productivity or in the foreign interest rate. The production function (2.3) is a Cobb-Douglas function, implying a unitary elasticity of substitution between labor and capital. The stock of physical capital evolves according to (2.4), where δ represents the rate of depreciation of capital.

The domestic interest rate r_t define in equation (2.5) is imperfectly arbitraged to the world

[6] $E(m_{t+1} \mid F_t) = \int \frac{\hat{\pi}(\epsilon_{t+1}|F_t)}{\pi(\epsilon_{t+1})}\pi(\epsilon_{t+1})d\epsilon_{t+1} = 1$, integrated with respect to the Lebesgue measure.

[7] Similar to the change of measure for the transformation to risk-neutral probabilities used in asset pricing.

interest rate r^*, with a country interest rate premium that is debt elastic and takes the form $\psi(e^{\bar{d}_t - \bar{d}} - 1)$[8]. Here $\bar{d}_t$ is the domestic cross-sectional average level of debt which is considered exogenous by households, $\bar{d}$ represents the steady-state level of average debt and $\psi > 0$ denotes the sensitivity of the risk premium to deviations of average debt from its steady-state value.

The law of motion of productivity is given by a first-order autoregressive process (equation (2.6)), where ϵ_t is the stochastic productivity shock, which is normally distributed with zero mean and unit standard deviation. The parameter $\tilde{\eta}$ defines the standard deviation of the innovation, a_{ss} is the steady state value of productivity, and coefficient ρ reflects the persistence of shocks. The exogenous process for productivity is specified in levels and not in logs (in contrast to the original SOE-RBC model by Schmitt-Grohé and Uribe), in order to focus exclusively on the implications of volatility for welfare[9]. It is also possible to use logs, as Cho et al. (2015), correcting the process mean in order to obtain a mean-preserving spread[10], but this procedure introduces right skewness to the distribution of productivity, which tends to be welfare increasing (Lester et al., 2014).

The symbol E_t represents the conditional expectations operator, associated to the reference probability distribution for ϵ_{t+1}. The likelihood ratio m_{t+1} between a distorted density versus the reference density allows to perform a change in the probability measure. The symbol $\hat{E}_t$ denotes the distorted conditional expectations operator, then $E_t[m_{t+1}V(d_{t+1}, k_{t+1}, a_{t+1})] = \hat{E}_t[V(d_{t+1}, k_{t+1}, a_{t+1})]$.

The penalty parameter θ measures the decision makers' concern about robustness to misspecification. This parameter enters the value function multiplying the relative entropy of distortion. Hence if an alternative probability model is particularly unfavorable in terms of future expected utility, it may not solve the minimization due to the countervailing effect of relative entropy. The penalty has a lower bound θ, called neurotic breakdown point by Hansen and Sargent (2008). Below this point it is useless to seek more robustness; the minimizing agent is sufficiently unconstrained that he can push the criterion function to minus infinity. On the other hand, if θ goes to infinity, the concern for model misspecification vanishes, so that the agent's preferences are characterized only by classical risk aversion. In the inner minimization the objective function is convex in m_{t+1} and the constraint is linear which allows to find the solution (as discussed by Hansen and Sargent 2008). The need for tractability of the minimization leads to the choice of relative entropy to measure model discrepancies.

The solution of the minimization characterizes the worst-case distortion[11]:

$$m^*_{t+1} = \frac{\exp\left(\frac{-V(d_{t+1},k_{t+1},a_{t+1})}{\theta}\right)}{E_t\left[\exp\left(\frac{-V(d_{t+1},k_{t+1},a_{t+1})}{\theta}\right)\right]} \tag{2.8}$$

[8]Schmitt-Grohé and Uribe (2003) and Schmitt-Grohé and Uribe (2017) discuss several ways to close SOE models and enable their convergence to a meaningful steady-state equilibrium. This equation is one of the different alternatives of model closure and is very popular in international macroeconomics.

[9]Following Lester et al. (2014) and Basu and Bundick (2017).

[10]If $x \sim N(\mu, \sigma^2)$, then if $X = e^x$ the expected value of X is a function of the variance of x, $E(X) = e^{\mu + \frac{\sigma^2}{2}}$.

[11]The worst probabilistic scenario puts larger probability weights on bad productivity shocks.

Substituting m^* into the original problem implies the following Bellman equation for the household problem:

$$V(d_t, k_t, a_t) = \max_{c_t, h_t, k_{t+1}, d_{t+1}} u(c_t, h_t) - \beta\theta \log E_t \exp\left(\frac{-V(d_{t+1}, k_{t+1}, a_{t+1})}{\theta}\right) \tag{2.9}$$

$$s.t. \quad d_{t+1} = (1+r_t)d_t - a_t k_t^\alpha h_t^{1-\alpha} + c_t + k_{t+1} - (1-\delta)k_t + \frac{\phi}{2}(k_{t+1} - k_t)^2$$

$$a_{t+1} = (1-\rho)a_{ss} + \rho a_t + \tilde{\eta}\epsilon_{t+1}$$

where the first restriction is the households' budget constraint after replacing equations (2.3) to (2.4) in equation (2.2).

The first-order conditions associated to the household's maximization problem are:

$$\lambda_t = \beta(1+r_{t+1})E_t\left\{m^*_{t+1}\lambda_{t+1}\right\} = \beta(1+r_{t+1})\hat{E}_t\left\{\lambda_{t+1}\right\} \tag{2.10}$$

$$\lambda_t(1+\phi(k_{t+1}-k_t)) = \beta E_t\left\{m^*_{t+1}\lambda_{t+1}\left[\alpha a_{t+1}k_{t+1}^{\alpha-1}h_{t+1}^{1-\alpha} + 1 - \delta + \phi(k_{t+2}-k_{t+1})\right]\right\}$$

$$= \beta\hat{E}_t\left\{\lambda_{t+1}\left[\alpha a_{t+1}k_{t+1}^{\alpha-1}h_{t+1}^{1-\alpha} + 1 - \delta + \phi(k_{t+2}-k_{t+1})\right]\right\} \tag{2.11}$$

$$-u_h(c_t, h_t) = \lambda_t(1-\alpha)a_t k_t^\alpha h_t^{-\alpha} \tag{2.12}$$

where λ_t is the marginal utility of consumption ($\lambda_t = u_c(c_t, h_t)$), (2.10) is the Euler equation, (2.11) is the first-order condition related to capital accumulation, and (2.12) is the first-order condition for labor supply. In equations (2.10) and (2.11), as opposed to expected utility models, the equilibrium conditions under multiplier preferences include the value function itself. Households are assumed to be identical; therefore the average debt in equilibrium is equal to the households' level of debt, $d_t = \bar{d}_t$.

2.3 Model solution

I start by referring to the perturbation approach to solve the model presented in the previous section. Perturbation algorithms build a Taylor series expansion of the agents' decision rules. I implement a second-order approximation because the standard linearization method is useless for this model, as discussed above in section 1. Perturbation methods were introduced by Judd and Guu (1992) and intuitively explained by Schmitt-Grohé and Uribe (2004). Perturbation methods are very fast and, despite their local nature, highly accurate for a large range of values of the state variables (Arouba et al., 2006). Caldara et al. (2012) present an algorithm to use perturbation methods, extending the work of Schmitt-Grohé and Uribe (2004) for recursive utility, in particular, EZW preferences. Finally, Bidder and Smith (2012b) introduce Caldara et al. (2012) algorithm in to the robust control literature. I adopt this method to solve my model.

2.3.1 Second-order approximation of the value function

The first step to calculate the second-order approximation of the value function is to write the process of aggregate productivity in terms of a perturbation parameter χ, in the following way:

$$a_{t+1} = (1-\rho)a_{ss} + \rho a_t + \chi\tilde{\eta}\epsilon_{t+1} \tag{2.13}$$

When the perturbation parameter value is set at $\chi = 1$ ($\chi = 0$), the model is stochastic (deterministic). The parameter scaling the variance of the shocks is included in the set of vector of state variables: $s_t = (d_t, k_t, a_t; \chi)$.

The second step is specifying the model's equilibrium conditions augmented by the definition of the value function, considering that all endogenous variables are functions of state variables:

$$V(s_t) = u(c_t(s_t), h_t(s_t)) - \beta\theta \log E_t \exp\left[\frac{-V((s_{t+1}))}{\theta}\right] \tag{2.14}$$

$$\lambda_t(s_t) = \beta(1 + r^*_{t+1} + \psi(e^{d_{t+1}(s_t)-\bar{d}} - 1))E_t\left\{\frac{\exp(\frac{-V_{t+1}(s_{t+1})}{\theta})}{E_t\exp(\frac{-V_{t+1}(s_{t+1})}{\theta})}\lambda_{t+1}(s_{t+1})\right\} \tag{2.15}$$

$$\begin{aligned}\lambda_t(s_t)(1+\phi(k_{t+1}(s_t)-k_t)) = \beta E_t\Bigg\{&\frac{\exp(\frac{-V_{t+1}(s_{t+1})}{\theta})}{E_t\exp(\frac{-V_{t+1}(s_{t+1})}{\theta})}\lambda_{t+1}(s_{t+1})\\ &\cdot\left[\alpha a_{t+1}(s_t)k_{t+1}(s_t)^{\alpha-1}h_{t+1}(s_{t+1})^{1-\alpha} + 1 - \delta + \phi(k_{t+2}(s_{t+1}) - k_{t+1}(s_t))\right]\Bigg\}\end{aligned} \tag{2.16}$$

$$-u_h(c_t(s_t), h_t(s_t)) = \lambda_t(s_t)(1-\alpha)a_t k_t^\alpha h_t(s_t)^{-\alpha} \tag{2.17}$$

$$\lambda_t(s_t) = u_c(c_t(s_t), h_t(s_t)) \tag{2.18}$$

$$\begin{aligned}d_{t+1}(s_t) = {}&(1 + r^*_t + \psi(e^{d_t-\bar{d}} - 1))d_t - a_t k_t^\alpha h_t(s_t)^{1-\alpha} + c_t(s_t) + k_{t+1}(s_t) - (1-\delta)k_t\\ &+ \frac{\phi}{2}(k_{t+1}(s_t) - k_t)^2\end{aligned} \tag{2.19}$$

$$a_{t+1}(s_t) = (1-\rho)a_{ss} + \rho a_t + \chi\tilde{\eta}\epsilon_{t+1} \tag{2.20}$$

The third step is to take the first derivatives of (2.14) to (2.19) with respect to the states $s_t = (d_t, k_t, a_t; \chi)$ and evaluate them at the non-stochastic steady state. This leads to 24 equations (6 equilibrium conditions times 4 state variables) and 24 unknowns (the first derivatives of the 6 endogenous c_t, d_{t+1}, h_t, k_{t+1}, λ_t and V_t variables with respect to the states evaluated at the non-stochastic steady state). After solving the system of equations, the next step is to take the derivatives of the first derivatives of (2.14) to (2.19) again with respect to the states: This step arrives at a new system of 96 equations (6 first derivatives of the equilibrium conditions times 4 state variables) and 96 unknowns (the second derivatives of the 6 endogenous variables). The results are reported in Appendix A.2.

Since the equilibrium conditions depend on the value function, it is necessary to derive its approximation around the non-stochastic steady state, which allows to compute welfare calculations.

In order to simplify the exposition, I only report the approximation of the value function; but the rest of the policy functions could be determined following the same procedure.

The second-order approximation of the value function is:

$$\begin{aligned} V(d_t, k_t, a_t; \chi) \simeq & V_{ss} + V_{d,ss}(d_t - d_{ss}) + V_{k,ss}(k_t - k_{ss}) + V_{a,ss}(a_t - a_{ss}) + V_{\chi,ss}\chi \\ & + V_{dk,ss}(d_t - d_{ss})(k_t - k_{ss}) + V_{da,ss}(d_t - d_{ss})(a_t - a_{ss}) + V_{d\chi,ss}(d_t - d_{ss})\chi \\ & + \frac{1}{2}V_{kk,ss}(k_t - k_{ss})^2 + V_{ka,ss}(k_t - k_{ss})(a_t - a_{ss}) + V_{k\chi,ss}(k_t - k_{ss})\chi \\ & + \frac{1}{2}V_{aa,ss}(a_t - a_{ss})^2 + V_{a\chi,ss}(a_t - a_{ss})\chi + \frac{1}{2}V_{\chi\chi,ss}\chi^2 \end{aligned} \tag{2.21}$$

Each term $V_{i,ss}$ and $V_{ij,ss}$ is a scalar equal to the corresponding first and second-order derivative of the value function for $i, j = \{d, k, a; \chi\}$, evaluated at the non-stochastic steady state. The value function evaluated at the non-stochastic steady state, $V(d_{ss}, k_{ss}, a_{ss}; 0) = V_{ss}$. Hence, evaluating the approximation of the value function at the non-stochastic steady state assuming $\chi = 1$, yields the following:

$$V(d_{ss}, k_{ss}, a_{ss}, 1) = V_{ss} + \frac{1}{2}V_{\chi\chi,ss} \tag{2.22}$$

Van Binsbergen et al. (2009), show that $V_{\chi\chi,ss}$ is the only coefficient that is affected by uncertainty aversion at a second-order approximation and that $V_{\chi\chi,ss} \neq 0$. $V_{\chi\chi,ss}$ reflects the change in the value function when the variance of productivity is $\tilde{\eta}$ instead of zero.

For the robust SOE-RBC model described by equations (2.14) to (2.20), I derive the following equation for the coefficient $V_{\chi\chi,ss}$, the second derivative of the value function with respect to the coefficient that scales the variance of productivity shocks (this coefficient is the only that depends on robustness parameter θ):

$$V_{\chi\chi,ss} = \frac{\beta\tilde{\eta}^2}{1-\beta}\left[V_{aa,ss} - \frac{V_{a,ss}^2}{\theta}\right] \tag{2.23}$$

2.3.2 Welfare analysis

The value function perturbation reflects that, up to a first-order approximation, the policy functions of the model with multiplier preferences are equivalent to the expected utility model considering the same instantaneous utility. But, at a second-order approximation, the value function approximation differs between the two models by a term that is a function of the parameter that governs robustness and the variance of the productivity shock. If the parameter that governs robustness (θ) goes to infinity, reflecting that the concern for model misspecification goes to zero, the second part of equation (2.23) goes to zero, which reflects the case when the household is only risk averse. Therefore, expression (2.23) is the sum of two components: $V_{\chi\chi,ss}^{risk}$ is related to risk and

$V^{robust}_{\chi\chi,ss}$ is related to model uncertainty.

$$V^{risk}_{\chi\chi,ss} = \frac{\beta\tilde{\eta}^2}{1-\beta} V_{aa,ss} \tag{2.24}$$

$$V^{robust}_{\chi\chi,ss} = -\frac{\beta\tilde{\eta}^2}{1-\beta} \frac{V^2_{a,ss}}{\theta} \tag{2.25}$$

The term $\frac{\beta\tilde{\eta}^2}{1-\beta}$ that appears in both equations (2.24) and (2.25) is unambiguously positive. Hence the sign of $V^{risk}_{\chi\chi,ss}$ depends only on $V_{aa,ss}$. As discussed by van Binsbergen et. al (2009), the latter term has an ambiguous sign in a real business cycle model with endogenous labor and capital. Cho et al. (2015) have shown that considering production economies under classical uncertainty generates two effects on welfare: the fluctuation effect and the mean effect. The fluctuation effect is always negative for the welfare of risk averse agents. The mean effect may increase or reduce welfare, depending on the calibration of the model's parameters. Therefore the total effect of risk on the value function is ambiguous a priori.

On the other hand, the sign of the effect of the concern for model uncertainty $V^{robust}_{\chi\chi,ss}$ is unambiguously negative and its absolute value is decreasing in θ. A larger θ implies a smaller concern for model misspecification. Accordingly, the overall effect of risk and model uncertainty on the value function is ambiguous[12].

It is important to note that the analytical decomposition and combination of the effects of the two types of uncertainty considered here - classical risk and model uncertainty - and the resulting ambiguous effect on overall welfare have not been derived in the previous literature.

2.3.3 Welfare cost calculations

This section presents calculations of the welfare cost of classical and model uncertainty, reflected in consumption equivalent units. This implies computing the percentage loss in long-term mean consumption (compensating variation) τ that would make the household indifferent between consuming $(1-\tau)c_{ss}$ per period under certainty and c_{ss} under uncertainty, where c_{ss} is the steady-state value of consumption. As shown by van Binsbergen et al. (2009), the coefficient $V_{\chi\chi,ss}$ of the value function approximation could be used to measure the welfare cost of uncertainty. The term $\frac{1}{2}V_{\chi\chi,ss}$ represents how much indirect utility changes when the variance of the productivity shock is $\tilde{\eta}^2$ instead zero.

For the robust SOE-RBC model developed in this paper, equation (2.23) decomposes the effects of volatility and doubts on total welfare effect, deriving a closed-form solution for the cost to face well-understood stochastic consumption fluctuations and the cost to face uncertainty about the stochastic representation of productivity shocks. For welfare comparison, this expression is transformed into consumption equivalent units. In order to do this, I introduce an explicit form of households' instantaneous utility function, which is due to by Greenwood, Hercowitz, and Huffman

[12]The intertemporal discount factor β and the standard deviation of productivity shocks $\tilde{\eta}$ amplify the overall effect of risk and robustness on welfare in absolute value.

(1988):

$$u(c_t, h_t) = \frac{\left(c_t - \frac{h_t^\omega}{\omega}\right)^{1-\sigma} - 1}{1-\sigma} \tag{2.26}$$

Greenwood, Hercowitz, and Huffman (GHH) preferences, with a wage elasticity of labor supply equal to $\frac{1}{1-\omega}$, have the advantage that the labor supply is insensitive to wealth effects because it is independent of the level of consumption. This prevents persistent shocks affecting the level of employment[13]. The parameter σ measures risk aversion, and its reciprocal is the intertemporal elasticity of substitution.

Considering GHH preferences (2.26), I compute τ the welfare loss due to aggregate uncertainty from the following:

$$\frac{\left(c_{ss} - \frac{h_{ss}^\omega}{\omega}\right)^{1-\sigma} - 1}{1-\sigma} + \frac{1}{2}V_{\chi\chi,ss} = \frac{\left(c_{ss}(1-\tau) - \frac{h_{ss}^\omega}{\omega}\right)^{1-\sigma} - 1}{1-\sigma} \tag{2.27}$$

Replacing $V_{\chi\chi,ss}$ from (2.23) into (2.27) an solving for τ:

$$\tau = \frac{\left(c_{ss} - \frac{h_t^\omega}{\omega}\right)}{c_{ss}} - \frac{1}{c_{ss}}\left(\frac{\beta\eta^2}{2}(1-\sigma)\left[V_{aa,ss} - \frac{V_{a,ss}^2}{\theta}\right] + \left(c_{ss} - \frac{h_{ss}^\omega}{\omega}\right)^{1-\sigma}\right)^{\frac{1}{1-\sigma}} \tag{2.28}$$

It is not possible to continue advancing analytically as the coefficient $V_{aa,ss}$ can only be determined numerically, nevertheless, $V_{a,ss}$ as all first order coefficients of value function's approximation are determined analytically and presented in Appendix A.2.

2.4 Model calibration and numerical calculations

This section presents numerical calculations for the welfare cost by means of consumption equivalent units in order to perform welfare comparisons of the model's two sources of uncertainty: risk and model uncertainty. The calibration of the main parameters of the SOE-RBC model follows Schmitt-Grohé and Uribe (2003) and is summarized in Table 2.1[14]:

Table 2.1: Calibration of model parameters

δ	r^*	α	$\bar{d}$	ω	ϕ	ψ	ρ	$\tilde{\eta}$	β	a_{ss}
0.1	0.04	0.32	0.7442	1.455	0.028	0.000742	0.42	0.0129	0.9615	1

where δ is the depreciation rate, r^* is the international interest rate, α is the capital share, $\bar{d}$ is steady-state debt, ω is the wage elasticity of labor supply, ϕ is the capital adjustment cost coefficient, ψ is the sensitivity of sovereign risk to debt, ρ reflects the persistence of productivity shocks, $\tilde{\eta}$ is the volatility of productivity shocks, β is the subjective discount factor, and a_{ss} is the steady-state value of productivity.

[13] By eliminating the wealth effect, these preferences raise the likelihood that welfare improves with uncertainty, as shown by Lester et al. (2014).

[14] Parameter calibration by Schmitt-Grohé and Uribe (2003) follows Mendoza (1991) original calibration for Canada.

For the key robustness parameter θ, for my extended model, I use three different values, which span the range of possible values. The first value is infinity, which represents the classical uncertainty case, where the concern for model misspecification is absent. The second is $\theta = 8$, which comes from Bidder and Smith (2012b), who estimated this value by using error-detection probabilities[15] that express how difficult it is for a decision-maker, with limited data, to distinguish between the worst-case probabilistic scenario and the benchmark model. The third value is set at $\theta = 1.2$, to consider a case of extremely high concern for model misspecification. This is the smallest value for which model convergence is still achieved.

For the risk-aversion coefficient σ, I also use three different values: $\sigma = 1$ corresponding to logarithmic preferences; $\sigma = 2$, used by Schmitt-Grohé and Uribe (2003); and $\sigma = 5$, which is a high value for risk aversion. Appendix A.3 presents welfare calculations for a broader set of values for the risk aversion parameter to check for consistency of the results.

The following Table 2.2 reports the percentage loss in long-term mean consumption, that makes the household indifferent between consuming $(1 - \tau)c_{ss}$ per period under certainty and c_{ss} under uncertainty, for different values of risk aversion (σ). If the value of τ, determined by equation (2.28), is positive (negative), facing uncertainty implies a welfare cost (premium) for the representative agent. The table reports the compensating variation τ for nine combinations of values for the agents' degree of risk aversion σ and their concern for robustness θ.

Table 2.2: Welfare loss estimations due to aggregate uncertainty, for different parameter values of risk aversion and concern for model uncertainty (Percentage loss in long-term mean consumption, τ)

Model	Risk Aversion		
	σ =1	σ =2	σ =5
Risk ($\theta = \infty$)	-0.0083	-0.0042	0.008
Risk + Model Uncertainty ($\theta = 8$)	0.0048	0.0282	0.4082
Risk + Model Uncertainty ($\theta = 1.2$)	0.0786	0.2021	2.3322

Let's consider first the case when households are only risk averse (first line in Table 2.2). At relatively low levels of risk aversion ($\sigma = 1$ and 2), the agent reaps a welfare premium. But when risk aversion is larger (in the table, at a value of $\sigma = 5$), the presence of risk is welfare deteriorating. This reflects the well-established result that risk has an ambiguous effect on welfare.

Now consider adding the households' concern about model misspecification, which represents an unambiguous welfare loss (second and third lines in the table). The presence of this welfare loss turns the welfare gain, obtained at relatively low levels of risk aversion ($\sigma = 1$ and 2), into an overall welfare loss. And at a high level of risk aversion (σ= 5), the overall welfare loss is significantly raised by the concern about model specification.

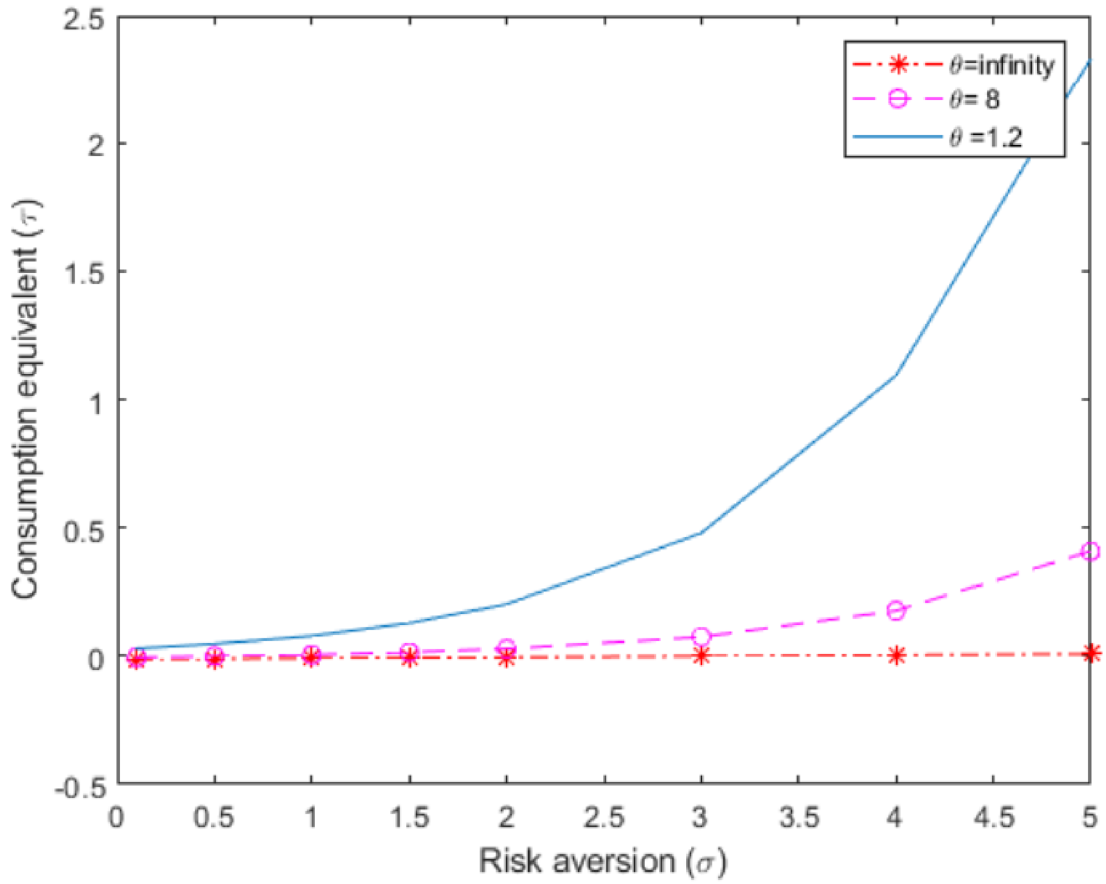
The latter is observed even at a moderate degree of concern for model uncertainty ($\theta = 8$, line 2). At a very high level of concern for model uncertainty ($\theta = 1.2$, line 3), the overall welfare

[15] This methodology is also applied by Hansen and Sargent (2008).

costs of uncertainty are one order of magnitude larger than at a moderate level. When a high level of concern for model uncertainty (θ= 1.2) is combined with high risk aversion (σ= 5), the overall welfare loss due to uncertainty reaches a very high level, equivalent to 2.3% of long-term consumption.

In Figure 1, I expand the calculations for the compensating variation of consumption reported in Table 2 for a wider range of risk aversion values, extending from $\sigma = 0$ to σ= 5. The three lines are drawn for the corresponding three values of θ that I set above. The figure reflects how the overall welfare loss due to uncertainty grows exponentially with risk aversion and, in particular, with the concern about model uncertainty.

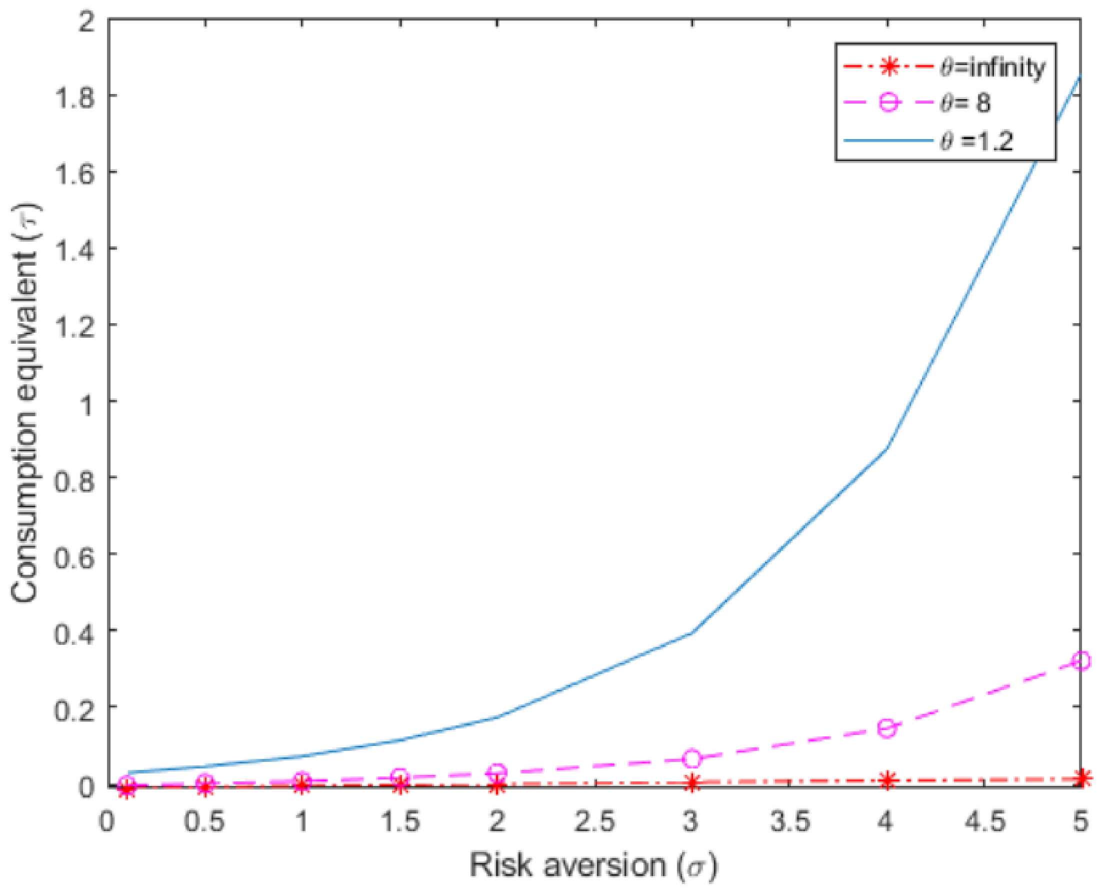
Figure 2.1: Welfare loss estimations due to aggregate uncertainty, for different parameter values of risk aversion and concern for model uncertainty; no financial frictions case (Percentage loss in long-term mean consumption, τ)



Up to here I have considered a low sensitivity of the debt risk premium to the level of sovereign debt, reflected in ψ=0.000742, following Schmitt-Grohé and Uribe (2003). Now I consider a high value for the debt sensitivity parameter, consistent with the existence of financial frictions. Ericson and Liu (2012) examine the welfare cost of business cycles introducing financial frictions in a SOE with a risk-averse agent, without any other concern for uncertainty. They find that productivity shocks are welfare improving in the absence of financial frictions but they become welfare deteriorating under financial frictions.

For the financial frictions case, I set ψ at a value of 5, which is the median value estimated by García-Cicco et al. (2010). Figure 2 presents the welfare cost estimations for the financial frictions case, analogous to Figure 1 where financial frictions were absent.

Figure 2.2: Welfare loss estimations due to aggregate uncertainty, for different parameter values of risk aversion and concern for model uncertainty; financial frictions case (Percentage loss in long-term mean consumption, τ)



Comparison of Figures 1 and 2 shows that in the absence of concern for model uncertainty ($\theta = \infty$), the introduction of financial frictions reduces the positive effect of uncertainty at low levels of risk aversion and raises the negative effect diminished at high levels of risk aversion. This replicates the results reported by Ericson and Liu (2012), in the context of their SOE-RBC model, limited to classical risk. My model adds model uncertainty to classical risk about productivity. Analyzing my model without financial frictions, uncertainty becomes costly and the cost is increasing in the level of concern for model misspecification. However, when adding financial frictions the effect of uncertainty on the highest welfare cost by about 20%. The explanation for this result is due to the link between consumption, foreign debt, and the sovereign risk premium. Model uncertainty generates an increase in precautionary savings, then sovereign risk declines, and the domestic interest rate falls. The opportunity cost of consumption decreases, which softens the fall in consumption in response to the increase in precautionary savings.

2.5 Conclusions

This paper contributes to the literature on the welfare cost of business-cycle fluctuations in a SOE by introducing model uncertainty. Households face two types of uncertainty: classical uncertainty about the state of future aggregate productivity and model uncertainty about the probability model that describes the movements of productivity. The characterization of the small open economy follows the canonical SOE-RBC of Schmitt-Grohé and Uribe (2003), augmented by multiplier preferences (Hansen and Sargent 2008). These preferences correspond to a non-expected utility approach that reflects a concern for model misspecification (model uncertainty). Households are endowed with a reference probability model about the stochastic variable (aggregate productivity), but they distrust its accuracy, considering the possibility that the model is misspecified in a way that is difficult to detect statistically. Barillas et al. (2009) and Ellison and Sargent (2015) highlight that if agents care about robustness to model misspecification, business cycle fluctuations are more costly in terms of agents' welfare than if only risk aversion is considered.

Using perturbation methods, the paper derives a closed-form solution for the combined welfare effect of the two sources of uncertainty. While consumption fluctuations have an ambiguous effect on welfare (as shown by Cho et al., 2015 and Lester et al., 2014), the additional model uncertainty is unambiguously welfare-deteriorating. Hence the overall effect of uncertainty on welfare is ambiguous, depending on consumer preferences and model parameters. Then, the paper provides numerical results for the welfare effects of uncertainty. The simulations show that when classical risk has a positive effect on welfare (due to a low risk aversion), addition of model uncertainty can reverse the sign of the overall welfare effect. The total effect of uncertainty, including both sources, is shown to be negative for a wide range of parameter values.

The relevance of this paper's results lies in the broad discussion about macroeconomic stabilization, as highlighted by Lucas (1987, 2003). If the welfare cost of business cycles is negligible or if households prefer economic uncertainty, there is no space for counter-cyclical policy in small open economies. My first result for numerical simulations shows that for slightly risk-averse households, productivity shocks are in general welfare improving. However, if they are intensely risk averse, the welfare cost of productivity risk is estimated at 0.007% of long-term consumption, similar to the value of 0.008% found by Lucas (1987). The second finding is that by introducing model uncertainty, even when the agent is only mildly risk averse, the overall welfare cost of uncertainty becomes welfare-deteriorating. The third result, based on numerical calculations, shows that the total effect of uncertainty on welfare depends on the interaction of risk and model uncertainty. Welfare costs are increasing in both risk aversion and the concern for model uncertainty. Considering a moderately high degree of risk aversion and a reasonable concern for model uncertainty, the overall effect of uncertainty on welfare amounts to a loss of 0.4% of long-term consumption, which is two orders of magnitude higher than the finding by Lucas. If I consider the case of an agent with very high concern for model misspecification, the welfare cost reaches a staggering 2.3% of long-term consumption. Finally, I consider the existence of financial frictions, reflected in a high debt sensitivity of the sovereign risk premium, which has been proven to increase the welfare cost of productivity shocks in a SOE, for the case of classical risk aversion (Ericson and Liu 2012). In a risk aversion context the largest simulated welfare cost of business cycles is 0.013, which is significantly smaller than the largest value obtained when including preferences for robustness. Therefore, with or without financial frictions households with a preference for robustness are willing to sacrifice a lot to live in an economy without fear of misspecification.

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Chapter 3

Identifying Quantum Structures in the Ellsberg Paradox

Abstract

Empirical evidence has confirmed that quantum effects occur frequently also outside the microscopic domain, while quantum structures satisfactorily model various situations in several areas of science, including biological, cognitive and social processes. In this paper, we elaborate a quantum mechanical model which faithfully describes the *Ellsberg paradox* in economics, showing that the mathematical formalism of quantum mechanics is capable to represent the *ambiguity* present in this kind of situations, because of the presence of *contextuality.* Then, we analyze the data collected in a concrete experiment we performed on the Ellsberg paradox and work out a complete representation of them in complex Hilbert space. We prove that the presence of quantum structure is genuine, that is, *interference* and *superposition* in a complex Hilbert space are really necessary to describe the conceptual situation presented by Ellsberg. Moreover, our approach sheds light on 'ambiguity laden' decision processes in economics and decision theory, and allows to deal with different Ellsberg-type generalizations, e.g., the *Machina paradox.*

3.1 Introduction

Traditional approaches in economics follow the hypothesis that agents' behavior during a decision process is mainly ruled by *expected utility theory* (EUT) [1, 2]. Roughly speaking, in presence of uncertainty, decision makers choose in such a way that they maximize their expected utility. Notwithstanding its mathematical tractability and predictive success, the structural validity of EUT at the individual level is questionable. Indeed, systematic empirical deviations from the predictions of EUT have been observed which are usually referred to as *paradoxes* [3, 4].

EUT was formally elaborated by von Neumann and Morgenstern [1]. They presented a set of axioms that allow to represent decision–maker preferences over the set of *acts* (functions from the set of states of the nature into the set of consequences) by a suitable functional $E_p u(.)$, for

Act	red	yellow	black
f_1	12$	0$	0$
f_2	0$	0$	12$
f_3	12$	12$	0$
f_4	0$	12$	12$

Table 3.1: The payoff matrix for the Ellsberg paradox situation.

some Bernoulli utility function u on the set of consequences and an objective probability measure p on the set of states of the nature. An important aspect of EUT concerns the treatment of uncertainty. Knight had pointed out the difference between *risk* and *uncertainty* reserving the term *risk* for situations that can be described by known (or physical) probabilities, and the term *uncertainty* to refer to situations in which agents do not know the probabilities associated with each of the possible outcomes[5]. Von Neumann and Morgenstern modeling did not contemplate the latter possibility, since all probabilities are *objectively*, i.e. physically, given in their scheme. For this reason, Savage extended EUT allowing agents to construct their own subjective probabilities when physical probabilities are not available [2]. According to Savage's model, the distinction proposed by Knight seems however irrelevant. Ellsberg instead showed that Knightian's distinction is empirically meaningful [3]. In particular, he presented the following experiment. Consider one urn with 30 red balls and 60 balls that are either yellow or black, the latter in unknown proportion. One ball will be drawn from the urn. Then, free of charge, a person is asked to bet on one of the acts f_1, f_2, f_3 and f_4 defined in Table **??**. When asked to rank these gambles most of the persons choose to bet on f_1 over f_2 and f_4 over f_3. This preference cannot be explained by EUT. Indeed, individuals' ranking of the sub–acts [12 on *red*; 0 on *black*] versus [0 on *red*; 12 on *black*] depends upon whether the event *yellow* yields a payoff of 0 or 12, contrary to what is suggested by the Sure–Thing principle, an important axiom of Savage's model.[1] Nevertheless, these choices have a direct intuition: f_1 offers the 12 prize with an *objective probability* of 1/3, and f_2 offers the same prize but in an element of the *subjective partition* $\{black, yellow\}$. In the same way, f_4 offers the prize with an objective probability of 2/3, whereas f_3 offers the same payoff on the union of the unambiguous event *red* and the ambiguous event *yellow*. Thus, in both cases the unambiguous bet is preferred to its ambiguous counterpart, a phenomenon called *ambiguity aversion* by Ellsberg.

Many extensions of EUT have been worked out to cope with Ellsberg–type preferences, which mainly consist in replacing the Sure–Thing Principle by weaker axioms. We briefly summarize the most known, as follows.

(i) *Choquet expected utility.* This model considers a subjective non–additive probability (or, capacity) over the states of nature rather than a subjective probability. Thus, decision–makers could underestimate or overestimate probabilities in the Ellsberg experiment, and ambiguity aversion is

[1]The Sure–Thing principle was stated by Savage by introducing the *businessman example*, but it can be presented in an equivalent form, the *independence axiom*, as follows: if persons are indifferent in choosing between simple lotteries L_1 and L_2, they will also be indifferent in choosing between L_1 mixed with an arbitrary simple lottery L_3 with probability p and L_2 mixed with L_3 with the same probability p.

equivalent to the convexity of the capacity (pessimistic beliefs) [6].

(ii) *Max–Min expected utility.* The lack of knowledge about the states of nature of the decision–maker cannot be represented by a unique probability measure but, rather, by a set of probability measures. Then, an act f is preferred to g iff $\min_{p \in P} E_p u(f) > \min_{p \in P} E_p u(f)$, where P is a convex and closed set of additive probability measures. Ambiguity aversion is represented by the pessimistic beliefs of the agent which takes decisions considering the worst probabilistic scenario [7].

(iii) *Variational preferences.* In this dynamic generalization of the Max–Min expected utility, agents rank acts according to the criterion $\inf_{p \in \triangle} \{E_p u(f) + c(p)\}$, where $c(p)$ is a closed and convex penalty function associated with the probability election [8].

(iv) *Second order probabilities.* This is a model of preferences over acts where the decision–maker prefers act f to act g iff $E_\mu \phi(E_p u(f)) > E_\mu \phi\ (E_p u(g))$, where E is the expectation operator, u is a von Neumann–Morgenstern utility function, ϕ is an increasing transformation, and μ is a subjective probability over the set of probability measures p that the decision–maker thinks are feasible. Ambiguity aversion is here represented by the concavity of the transformation ϕ [9].

Despite approaches (i)–(iv) have been widely used in economic and financial modeling, none of them is immune of critics [4, 10]. And, worse, none of these models can satisfactorily represent more general Ellsberg–type situations (e.g., the *Machina paradox* [4, 11]). As a consequence, the construction of a unified perspective representing ambiguity is still an unachieved goal in economics and decision making.

We have recently inquired both conceptually and structurally into the above approaches generalizing EUT [6, 7, 8, 9] to cope with ambiguity. The latter is defined as a situation without a probability model describing it as opposed to *risk*, where a classical probability model on a σ–algebra of events is presupposed. The generalizations in (i)–(iv) consider more general structures than a single classical probability model on a σ–algebra. We are convinced that this is the point: ambiguity, due to its contextuality, structurally need a non–classical probability model. To this end we have elaborated a general framework, based on the notion of *contextual risk* and inspired by the probability structure of quantum mechanics, which is intrinsically different from a classical probability on a σ-algebra, the set of events is indeed *not* a Boolean algebra [13, 14, 15, 16]. Inspired by this approach, we work out in the present article a complete mathematical representation of the Ellsberg paradox situation (states, payoffs, acts, preferences) in the standard mathematical formalism of quantum mechanics, hence by using a complex Hilbert space, and representing the probability measures by means of projection valued measures on this complex Hilbert space [17] (Sect. 3.2). This analysis leads us to claim that the structure of the probability models is essentially different from the ones of known approaches – projection valued measures instead of σ–algebra valued measures [12]. But, more important, also the way in which states are represented in quantum mechanics, i.e. by unit vectors of the Hilbert space, introduces a fundamentally different aspect, coping both mathematically and intuitively with the notion of ambiguity as introduced in economics. Successively, we analyze the experimental data we collected in a *statistically relevant* experiment we performed, where real decision–makers were asked to bet on the different acts defined by the Ellsberg paradox situation [13, 14] (Sect. 3.3). We show that our quantum mechanical model faithfully represents the subjects' preferences together with experimental statistics. Furthermore, we describe the choices between acts f_1/f_2 and between acts f_3/f_4 by quantum observables represented by compatible spectral families (Sect. 3.4). We finally prove that the requirement of compatibility of the latter observables makes it necessary to introduce a Hilbert space over *complex* numbers, namely that *imaginary* numbers are needed since our experimental data cannot be modeled in a real Hilbert space, i.e. a vector space over *real* numbers only, in case we want the observables representing our experiment to be compatible (Sect. 3.5). Complex numbers in quantum theory stand for the quantum effect of interference, and indeed, we can identify in our modeling how interference produces the typical Ellsberg deviation leading to measured data in our experiment.

The results above strongly suggest that, more generally, 'ambiguity laden' situations can be explained in terms of the appearance of typically quantum effects, namely, contextuality – our quantum model is contextual, see the discussion on context in Sect. 3.2 –, superposition – we explicitly use superposition to construct the quantum states representing the Ellsberg bet situations in Sect. 3.2 – and interference – i.e. complex numbers –, and that quantum structures have the capacity to mathematically deal with this type of situations. Hence, our findings naturally fit within the growing 'quantum interaction research' which mainly applies quantum structures to cognitive situations [18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28].

3.2 A quantum model in Hilbert space for the Ellsberg paradox

Here we present a Hilbert space representation of the Ellsberg situation and derive new results elaborating the already obtained ones. But we first need to anticipate a discussion on the notions of 'context' and 'contextual influence' and how they are used in the present framework. The notion of context typically denotes what does not pertain to the entity under study but that can interact with it. In the foundations of quantum mechanics, context more specifically indicates the 'measurement context', which influences the measured quantum entity in a stochastic way. As a consequence of this contextual interaction, the state of the quantum entity changes, thus determining a transition from potential to actual. A quantum mechanical context is represented by a self-adjoint operator or, equivalently, by a spectral family. An analogous effect occurs in a decision process, where there is generally a contextual influence (of a cognitive nature) having its origin in the way the mind of the person involved in the decision, e.g., a choice between two bets, relates to the situation that is the subject of the decision making, e.g., the Ellsberg situation. This is why, in our analysis of the Ellsberg paradox, we use the definition and representation of context and contextual influence to indicate the cognitive interaction taking place between the conceptual Ellsberg situation and the human mind in a decision process. We are now ready to proceed with our quantum modeling.

To this end let us consider the situation illustrated in Tab. 1, Sect. 3.1. The simplest Hilbert space that can supply a faithful modeling of the Ellsberg situation is the three dimensional complex Hilbert space $\mathbb{C}^3$ whose canonical basis we denote by $\{|1,0,0\rangle, |0,1,0\rangle, |0,0,1\rangle\}$. We introduce the model in different steps: we will see that in each of them, quantum structures enable a new and satisfying way to model an aspect of ambiguity. At the last step it will be made clear that modeling the agents' decisions at a statistical level requires the full probabilistic apparatus of quantum mechanics, that is, both states and measurements.

3.2.1 The conceptual Ellsberg entity

The first part of the model consists of the Ellsberg situation without considering neither the different acts nor the person nor the bet to be taken. Hence it is the situation of the urn with 30 red balls and 60 black and yellow balls in unknown proportion (*conceptual Ellsberg entity*). Already at this stage, the presence of ambiguity can be mathematically taken into account by means of the quantum mechanical formalism. To this aim we introduce a quantum mechanical context e and represent it by means of the family $\{P_r, P_{yb}\}$, where P_r is the one dimensional orthogonal projection operator on the subspace generated by the unit vector $|1,0,0\rangle$, and P_{yb} is the two dimensional orthogonal projection operator on the subspace generated by the unit vectors $|0,1,0\rangle$ and $|0,0,1\rangle$. $\{P_r, P_{yb}\}$ is a spectral family, since $P_r \perp P_{yb}$ and $P_r + P_{yb} = \mathbb{1}$. Contexts, more specifically measurement contexts, are represented by spectral families of orthogonal projection operators (equivalently, by a self–adjoint operator determined by such a family) also in quantum mechanics. Again in analogy with quantum mechanics, we represent the states of the conceptual Ellsberg entity by means of unit vectors of $\mathbb{C}^3$. For example, the unit vector

$$|v_{ry}\rangle = |\frac{1}{\sqrt{3}}e^{i\theta_r}, \sqrt{\frac{2}{3}}e^{i\theta_y}, 0\rangle \tag{3.1}$$

can be used to represent a state describing the Ellsberg situation. Indeed, the probability for 'red' in the state $p_{v_{ry}}$ represented by $|v_{ry}\rangle$ is

$$|\langle 1,0,0|v_{ry}\rangle|^2 = \langle v_{ry}|1,0,0\rangle\langle 1,0,0|v_{ry}\rangle = \parallel P_r|v_{ry}\rangle \parallel^2 = \frac{1}{3} \tag{3.2}$$

Moreover, the probability for 'yellow or black' in the state $p_{v_{ry}}$ represented by $|v_{ry}\rangle$ is

$$\parallel P_{yb}|v_{ry}\rangle \parallel^2 = \langle 0, \sqrt{\frac{2}{3}}e^{i\theta_y}, 0|0, \sqrt{\frac{2}{3}}e^{i\theta_y}, 0\rangle = \frac{2}{3} \tag{3.3}$$

But this is not the only state describing the Ellsberg situation. For example, the unit vector

$$|v_{rb}\rangle = |\frac{1}{\sqrt{3}}e^{i\phi_r}, 0, \sqrt{\frac{2}{3}}e^{i\phi_b}\rangle \tag{3.4}$$

also represents a state describing the Ellsberg situation. We thus denote the set of all states describing the Ellsberg situation (*Ellsberg state set*) by

$$\Sigma_{Ells} = \{p_v : \ |v\rangle = |\frac{1}{\sqrt{3}}e^{i\theta_r}, \rho_y e^{i\theta_y}, \rho_b e^{i\theta_b}\rangle \mid 0 \leq \rho_y, \rho_b, \ \rho_y^2 + \rho_b^2 = \frac{2}{3}\} \tag{3.5}$$

which is associated with a subset (not necessarily a linear subspace) of $\mathbb{C}^3$. If a state belongs to Σ_{Ells}, this state delivers a quantum description of the Ellsberg situation, together with the context e represented by the spectral family $\{P_r, P_{yb}\}$ in $\mathbb{C}^3$.

3.2.2 Modeling acts and utility

In the second step of our construction, we describe the different acts f_1, f_2, f_3 and f_4. Here a second measurement context g is introduced. The context g describes the ball taken out of the urn and its color verified, red, yellow or black. Also g is represented by a spectral family of orthogonal projection operators $\{P_r, P_y, P_b\}$, where P_r is already defined, while P_y is the orthogonal projection operator on $|0,1,0\rangle$ and P_b is the orthogonal projection operator on $|0,0,1\rangle$. Thus, the probabilities $\mu_r(g,p_v)$, $\mu_y(g,p_v)$ and $\mu_b(g,p_v)$ of drawing a red ball, a yellow ball and a black ball, respectively, in a state p_v represented by the unit vector $|v\rangle = |\rho_r e^{i\theta_r}, \rho_y e^{i\theta_y}, \rho_b e^{i\theta_b}\rangle$ are

$$\mu_r(g,p_v) = \| P_r|v\rangle \|^2 = \langle v|P_r|v\rangle = \rho_r^2 \tag{3.6}$$
$$\mu_y(g,p_v) = \| P_y|v\rangle \|^2 = \langle v|P_y|v\rangle = \rho_y^2 \tag{3.7}$$
$$\mu_b(g,p_v) = \| P_b|v\rangle \|^2 = \langle v|P_b|v\rangle = \rho_b^2 \tag{3.8}$$

The acts f_1, f_2, f_3 and f_4 are observables in our modeling, hence they are represented by self-adjoint operators, built on the spectral decomposition $\{P_r, P_y, P_b\}$. More specifically, we have

$$\mathcal{F}_1 = 12\$P_r \tag{3.9}$$
$$\mathcal{F}_2 = 12\$P_b \tag{3.10}$$
$$\mathcal{F}_3 = 12\$P_r + 12\$P_y \tag{3.11}$$
$$\mathcal{F}_4 = 12\$P_y + 12\$P_b = 12\$P_{yb} \tag{3.12}$$

Let us now analyze the expected payoffs and the utility connected with the different acts. For the sake of simplicity, we identify here the utility with the expected payoff, which implies that we are considering *risk neutral agents*. Consider an arbitrary state $p_v \in \Sigma_{Ells}$ and the acts f_1 and f_4. We have

$$U(f_1,g,p_v) = \langle p_v|\mathcal{F}_1|p_v\rangle = 12\$ \cdot \frac{1}{3} = 4\$ \tag{3.13}$$
$$U(f_4,g,p_v) = \langle p_v|\mathcal{F}_4|p_v\rangle = 12\$ \cdot \frac{2}{3} = 8\$ \tag{3.14}$$

which shows that both these utilities are completely *independent* of the considered state of Σ_{Ells}, i.e. they are *ambiguity free*. Consider now the acts f_2 and f_3, and again an arbitrary state $p_v \in \Sigma_{Ells}$. We have

$$U(f_2,g,p_v) = \langle p_v|\mathcal{F}_2|p_v\rangle = 12\$\mu_b(g,p_v) \tag{3.15}$$
$$U(f_3,g,p_v) = \langle p_v|\mathcal{F}_3|p_v\rangle = 12\$(\mu_r(g,p_v) + \mu_y(g,p_v)) \tag{3.16}$$

which shows that both utilities strongly depend on the state p_v, due to the ambiguity the two acts are confronted with. This can be significantly revealed by considering two extreme cases. Let $p_{v_{ry}}$ and $p_{v_{rb}}$ be the states represented by the vectors $|v_{ry}\rangle$ and $|v_{rb}\rangle$ in Eqs. (3.1) and (3.4), respectively. These states give rise for the act f_2 to utilities

$$U(f_2, g, p_{v_{ry}}) = 12\$\mu_b(g, p_{v_{ry}}) = 12\$ \cdot 0 = 0\$ \tag{3.17}$$

$$U(f_2, g, p_{v_{rb}}) = 12\$\mu_b(g, p_{v_{rb}}) = 12\$ \cdot \frac{2}{3} = 8\$. \tag{3.18}$$

This shows that a state $p_{v_{rb}}$ exists within the realm of ambiguity, where the utility of act f_2 is greater than the utility of act f_1, and also a state $p_{v_{ry}}$ exists within the realm of ambiguity, where the utility of act f_2 is smaller than the utility of act f_1. If we look at act f_3, we find for the two considered extreme states the following utilities

$$U(f_3, g, p_{v_{ry}}) = 12\$(\mu_r(g, p_{v_{ry}}) + \mu_y(g, p_{v_{ry}})) = 12\$(\frac{1}{3} + \frac{2}{3}) = 12\$ \tag{3.19}$$

$$U(f_3, g, p_{v_{rb}}) = 12\$(\mu_r(g, p_{v_{rb}}) + \mu_y(g, p_{v_{rb}})) = 12\$(\frac{1}{3} + 0) = 4\$. \tag{3.20}$$

Analogously, namely the state $p_{v_{ry}}$ gives rise to a greater utility, while the state $p_{v_{rb}}$ gives rise to a smaller utility than the independent one obtained in act f_4.

3.2.3 Decision making and superposition states

The third step of our modeling consists in taking directly into account the role played by ambiguity. We proceed as follows [21].

We suppose that the two extreme states $p_{v_{ry}}$ and $p_{v_{rb}}$ in Sect. 3.2.1 are relevant in the mind of the person that is asked to bet. Hence, it is a superposition state of these two states that will guide the decision process during the bet. Let us construct a general superposition state p_{v_s} of these two states. Hence the vector $|v_s\rangle$ representing p_{v_s} can be written as

$$|v_s\rangle = ae^{i\alpha}|v_{rb}\rangle + be^{i\beta}|v_{ry}\rangle \tag{3.21}$$

where a, b, α and β are chosen in such a way that $\langle v_s|v_s\rangle = 1$, which means that

$$1 = a^2 + b^2 + \frac{2ab}{3}\cos(\beta - \alpha + \theta_r - \phi_r) \tag{3.22}$$

or, equivalently,

$$\cos(\beta - \alpha + \theta_r - \phi_r) = \frac{3(1 - a^2 - b^2)}{2ab} \tag{3.23}$$

The amplitude of the state p_{v_s} with the first basis vector $|1, 0, 0\rangle$ is given by

$$\langle 1, 0, 0|v_s\rangle = \frac{a}{\sqrt{3}}e^{i(\alpha+\phi_r)} + \frac{b}{\sqrt{3}}e^{i(\beta+\theta_r)} \tag{3.24}$$

Therefore, the transition probability is

$$|\langle 1,0,0|v_s\rangle|^2 = \frac{1}{3}(a^2+b^2+3-3a^2-3b^2) = \frac{1}{3}(3-2a^2-2b^2) = \mu_r(g,p_{v_s}) \tag{3.25}$$

as one can easily verify. Analogously, the amplitudes with the second and third basis vectors are given by

$$\begin{aligned}
\langle 0,1,0|v_s\rangle &= ae^{i\alpha}\langle 0,1,0|v_{rb}\rangle + be^{i\beta}\langle 0,1,0|v_{ry}\rangle = \sqrt{\frac{2}{3}}be^{i(\beta+\theta_y)} && (3.26)\\
\langle 0,0,1|v_s\rangle &= ae^{i\alpha}\langle 0,0,1|v_{rb}\rangle + be^{i\beta}\langle 0,0,1|v_{ry}\rangle = \sqrt{\frac{2}{3}}ae^{i(\alpha+\theta_b)} && (3.27)
\end{aligned}$$

respectively. Therefore, the transition probabilities are

$$\begin{aligned}
|\langle 0,1,0|v_s\rangle|^2 &= \frac{2}{3}b^2 = \mu_y(g,p_{v_s}) && (3.28)\\
|\langle 0,0,1|v_s\rangle|^2 &= \frac{2}{3}a^2 = \mu_b(g,p_{v_s}) && (3.29)
\end{aligned}$$

From the foregoing follows that a general superposition state is represented by the unit vector

$$|v_s\rangle = \frac{1}{\sqrt{3}}|ae^{i(\alpha+\phi_r)} + be^{i(\beta+\theta_r)}, \sqrt{2}be^{i(\beta+\theta_y)}, \sqrt{2}ae^{i(\alpha+\theta_b)}\rangle \tag{3.30}$$

and that the utilities corresponding to the different acts are given by

$$\begin{aligned}
U(f_1,g,p_{v_s}) &= \langle v_s|\mathcal{F}_1|v_s\rangle = 4\$(3-2a^2-2b^2) && (3.31)\\
U(f_2,g,p_{v_s}) &= \langle v_s|\mathcal{F}_2|v_s\rangle = 4\$\cdot 2a^2 && (3.32)\\
U(f_3,g,p_{v_s}) &= \langle v_s|\mathcal{F}_3|v_s\rangle = 4\$\cdot(3-2a^2) && (3.33)\\
U(f_4,g,p_{v_s}) &= \langle v_s|\mathcal{F}_4|v_s\rangle = 4\$(2a^2+2b^2) && (3.34)
\end{aligned}$$

We can see that it is not necessarily the case that $\mu_r(g,p_{v_s}) = \frac{1}{3}$, which means that choices of a and b can be made such that the superposition state $p_{v_s} \notin \Sigma_{Ells}$. The reason is that Σ_{Ells} is not a linearly closed subset of $\mathbb{C}^3$.

Let us then consider some of the extreme possibilities of superpositions. We remind that the latter superpositions are not relevant for the modeling of the Ellsberg paradox as it was originally formulated, but they come into play if one wants to represent more general Ellsberg-type situations. For example, choose $a=b=\frac{\sqrt{3}}{2}$. Then we have $\mu_y(g,p_{v_s}) = \frac{2}{3}\cdot\frac{3}{4} = \frac{1}{2}$, $\mu_b(g,p_{v_s}) = \frac{2}{3}\cdot\frac{3}{4} = \frac{1}{2}$, and $\mu_r(g,p_{v_s}) = 0$, and $\cos(\beta-\alpha+\theta_r-\phi_r) = \frac{3(1-a^2-b^2)}{2ab} = -1$, hence $\beta = \pi+\alpha-\theta_r+\phi_r$. Thus, the state represented by

$$|v_s(yb)\rangle = \frac{\sqrt{3}}{2}(e^{i\alpha}|v_{rb}\rangle + e^{i(\pi+\alpha-\theta_r+\phi_r)}|v_{ry}\rangle) \tag{3.35}$$

gives rise to probability zero for a red ball to be drawn. Another extreme choice is, when we take $a=b=\sqrt{\frac{3}{8}}$. Then we have $\mu_y(g,p_{v_s}) = \frac{2}{3}\cdot\frac{3}{8} = \frac{1}{4}$, $\mu_b(g,p_{v_s}) = \frac{2}{3}\cdot\frac{3}{8} = \frac{1}{4}$ and $\mu_r(g,p_{v_s}) = \frac{1}{2}$, and

$\cos(\beta - \alpha + \theta_r - \phi_r) = +1$, which means $\beta = \alpha - \theta_r + \phi_r$. Hence, the state

$$|v_s(r)\rangle = \sqrt{\frac{3}{8}}(e^{i\alpha}|v_{rb}\rangle + e^{i(\alpha-\theta_r+\phi_r)}|v_{ry}\rangle) \tag{3.36}$$

gives rise to probability $\frac{1}{2}$ for a red ball to be drawn. These are extreme superposition states which are not compatible with the situation as formulated by Ellsberg, but they could be useful if suitable Ellsberg–type extensions are taken into account.

To construct non–trivial superpositions that retain probability $\frac{1}{3}$ for drawing a red ball, we require that

$$\frac{1}{3} = \mu_r(g, p_{v_s}) = \frac{1}{3}(3 - 2a^2 - 2b^2) \tag{3.37}$$

or, equivalently,

$$a^2 + b^2 = 1 \tag{3.38}$$

which implies that $\cos(\beta - \alpha + \theta_r - \phi_r) = 0$, hence $\beta = \frac{\pi}{2} + \alpha - \theta_r + \phi_r$.

Let us construct now two examples of superposition states that conserve the $\frac{1}{3}$ probability for drawing a red ball, and hence are conservative superpositions, and express ambiguity as is thought to be the case in the Ellsberg paradox situation. The first state refers to the comparison for a bet between f_1 and f_2. The ambiguity of not knowing the number of yellow and black balls in the urn, only their sum to be 60, as compared to knowing the number of red balls in the urn to be 30, gives rise to the thought that 'eventually there are perhaps almost no black balls and hence an abundance of yellow balls'. Jointly, and in superposition, the thought also comes that 'it is of course also possible that there are more black balls than yellow balls'. These two thoughts in superposition, are mathematically represented by a state p_{v_s}. The state p_{v_s} will be closer to $p_{v_{ry}}$, the extreme state with no black balls, if the person is deeply ambiguity averse, while it will be closer to $p_{v_{rb}}$, the extreme state with no yellow balls, if the person is attracted by ambiguity. Hence, these two tendencies are expressed by the values of a and b in the superposition state. If we consider again the utilities, this time with $a^2 + b^2 = 1$, we have

$$U(f_1, g, p_{v_s}) = 4\$ \tag{3.39}$$
$$U(f_2, g, p_{v_s}) = 4\$ \cdot 2a^2 \tag{3.40}$$
$$U(f_3, g, p_{v_s}) = 4\$ \cdot (3 - 2a^2) \tag{3.41}$$
$$U(f_4, g, p_{v_s}) = 8\$ \tag{3.42}$$

So, for $a^2 < \frac{1}{2}$, which exactly means that the superposition state p_{v_s} is closer to the state $p_{v_{ry}}$ than to the state $p_{v_{rb}}$, we have that $U(f_2, g, p_{v_s}) < U(f_1, g, p_{v_s})$, and hence a person with strong ambiguity aversion in the situation of the first bet, will then prefer to bet on f_1 and not on f_2. Let us choose a concrete state for the bet between f_1 and f_2, and call it $p_{v_s^{12}}$, and denote its superposition state by $|v_s^{12}\rangle$. Hence, for $|v_s^{12}\rangle$ we take $a = \frac{1}{2}$ and $b = \frac{\sqrt{3}}{2}$ and hence $a^2 = \frac{1}{4}$ and $b^2 = \frac{3}{4}$. For the angles we must have $\beta - \alpha + \theta_r - \phi_r = \frac{\pi}{2}$, hence let us choose $\theta_r = \phi_r = 0$, $\alpha = 0$, and $\beta = \frac{\pi}{2}$. This gives us

$$|v_s^{12}\rangle = \frac{1}{2\sqrt{3}}|1+\sqrt{3}e^{i\frac{\pi}{2}}, \sqrt{2}\sqrt{3}e^{i\frac{\pi}{2}}, \sqrt{2}\rangle = \frac{1}{2\sqrt{3}}|1+i\sqrt{3}, i\sqrt{6}, \sqrt{2}\rangle \tag{3.43}$$

On the other hand, for $\frac{1}{2} < a^2$, which means that the superposition state is closer to the state $p_{v_{rb}}$ than to the state $p_{v_{ry}}$, we have that $U(f_3, g, p_{v_s}) < U(f_4, g, p_{v_s})$, and hence a person with strong ambiguity aversion in the situation of the second bet, will then prefer to bet on f_4 and not on f_3. Also for this case we construct an explicit state, let us call it $p_{v_s^{23}}$, and denote it by the vector $|v_s^{34}\rangle$. Hence, for $|v_s^{34}\rangle$ we take $a = \frac{\sqrt{3}}{2}$ and $b = \frac{1}{2}$ and hence $a^2 = \frac{3}{4}$ and $b^2 = \frac{1}{4}$. For the angles we must have $\beta - \alpha + \theta_r - \phi_r = \frac{\pi}{2}$, hence let us choose $\theta_r = \phi_r = 0$, $\alpha = 0$, and $\beta = \frac{\pi}{2}$. This gives us

$$|v_s^{34}\rangle = \frac{1}{2\sqrt{3}}|\sqrt{3}+e^{i\frac{\pi}{2}}, \sqrt{2}e^{i\frac{\pi}{2}}, \sqrt{2}\sqrt{3}\rangle = \frac{1}{2\sqrt{3}}|\sqrt{3}+i, i\sqrt{2}, \sqrt{6}\rangle \tag{3.44}$$

The superposition states $p_{v_s}^{12}$ and $p_{v_s}^{34}$ representing the unit vectors $|v_s^{12}\rangle$ and $|v_s^{34}\rangle$, respectively, will be used in the next sections to provide a faithful modeling of a concrete experiment in which decisions are expressed by real agents.

3.3 An experiment testing the Ellsberg paradox

We have observed in the previous sections that genuine quantum aspects intervene in the description of the Ellsberg paradox. This will be even more evident from the analysis of a *statistically relevant* experiment we performed of this paradox, whose results were firstly reported in [14]. To perform the experiment we sent out the following text to several people, consisting of a mixture of friends, relatives and students, to avoid as much as possible a statistical selection bias.

We are conducting a small-scale statistics investigation into a particular problem and would like to invite you to participate as test subjects. Please note that it is not the aim for this problem to be resolved in terms of correct or incorrect answers. It is your preference for a particular choice we want to test. The question concerns the following situation. Imagine an urn containing 90 balls of three different colors: red balls, black balls and yellow balls. We know that the number of red balls is 30 and that the sum of the the black balls and the yellow balls is 60. The questions of our investigation are about the situation where somebody randomly takes one ball from the urn.

(i) *The first question is about a choice to be made between two bets: bet f_1 and bet f_2. Bet f_1 involves winning '10 euros when the ball is red' and 'zero euros when it is black or yellow'. Bet f_2 involves winning '10 euros when the ball is black' and 'zero euros when it is red or yellow'. The question we would ask you to answer is: Which of the two bets, bet f_1 or bet f_2, would you prefer?*

(ii) *The second question is again about a choice between two different bets, bet f_3 and bet f_4. Bet f_3 involves winning '10 euros when the ball is red or yellow' and 'zero euros when the ball is black'. Bet f_4 involves winning '10 euros when the ball is black or yellow' and 'zero euros when the ball is red'. The second question therefore is: Which of the two bets, bet f_3 or bet f_4, would you prefer?*

Please provide in your reply message the following information.

For question 1, your preference (your choice between bet f_1 and bet f_2). For question 2, your preference (your choice between bet f_3 and bet f_4). By 'preference' we mean 'the bet you would take if this situation happened to you in real life'. You are expected to choose one of the bets for each of the questions, i.e. 'not choosing is no option'. You are welcome to provide a brief explanation of your preferences, which may be of a purely intuitive nature, only mentioning feelings, for example, but this is not required. It is all right if you only state your preferences without giving any explanation.

One final remark about the colors. Your choices should not be affected by any personal color preference. If you feel that the colors of the example somehow have an influence on your choices, you should restate the problem and take colors that are indifferent to yours, if this does not work, use other neutral characteristics to distinguish the balls.

Let us now analyze the obtained results.

We had 59 respondents participating in our test of the Ellsberg paradox problem, which is the typical number of participants in experiments on psychological effect of the type studied by Kahneman and Tversky, such as the conjunction fallacy, and the disjunction effect. (see, e.g., [29, 30]). We do believe that ambiguity aversion is a psychological effect within this calls of effects, which means that in case our hypothesis on the nature of ambiguity aversion is correct, our test is significant. This being said, it would certainly be interesting to make a similar test with a larger number of participants, which is something we plan for the future. We however also remark that the quantum modeling scheme is general enough to also model statistical data that are different from the ones collected in this specific test. Next to this remark, we want to point out that in the present paper we want to prove that these real data 'can' be modeled in our approach.

The answers of the participants were distributed as follows: (a) 34 subjects preferred bets f_1 and f_4; (b) 12 subjects preferred bets f_2 and f_3; (c) 7 subjects preferred bets f_2 and f_4; (d) 6 subjects preferred bets f_1 and f_3. This makes the weights with preference of bet f_1 over bet f_2 to be 0.68 against 0.32, and the weights with preference of bet f_4 over bet f_3 to be 0.69 against 0.31. It is worth to note that 34+12=46 people chose the combination of bet f_1 and bet f_4 or bet f_2 and bet f_3, which is 78%. In Sect. 3.4 we apply our quantum mechanical model to these experimental data.

3.4 Quantum modeling the experiment

As anticipated in Sect. 3.2.3, we take into account the superposition states $p_{v_s}^{12}$ and $p_{v_s}^{34}$ to put forward a description of the choices made by the participants in the test described in Sect. 3.3. We employ in this section the spectral methods that are typically used in quantum mechanics to construct self–adjoint operators.

First we consider the choice to bet on f_1 or on f_2. This is a choice with two possible outcomes, let us call them o_1 and o_2. Thus, we introduce two projection operators P_1 and P_2 on the Hilbert space $\mathbb{C}^3$, and represent the observable associated with the first bet by the self–adjoint operator (spectral decomposition) $\mathcal{O}_{12} = o_1 P_1 + o_2 P_2$. Then, we consider the choice to bet on f_3 or on f_4. This is a choice with two possible outcomes too, let us call them o_3 and o_4. Thus, we introduce two projection operators P_3 and P_4 on $\mathbb{C}^3$, and represent the observable associated with the second bet by the self–adjoint operator (spectral decomposition) $\mathcal{O}_{34} = o_3 P_3 + o_4 P_4$.

To model the empirical data in Sect. 3.3, we recall that we tested in our experiment 59 participants, and 40 preferred f_1 over f_2, while the remaining 19 preferred f_2 over f_1. This means that we should construct P_1 and P_2 in such a way that

$$\langle v_s^{12}|P_1|v_s^{12}\rangle = \frac{40}{59} = 0.68 \quad \langle v_s^{12}|P_2|v_s^{12}\rangle = \frac{19}{59} = 0.32 \tag{3.45}$$

In our experiment of the 59 participants there were 41 preferring f_4 over f_3, and 18 who choose the other way around. Hence, we should have

$$\langle v_s^{34}|P_3|v_s^{34}\rangle = \frac{18}{59} = 0.31 \quad \langle v_s^{34}|P_4|v_s^{34}\rangle = \frac{41}{59} = 0.69 \tag{3.46}$$

Both bets should give rise to no preference, hence probabilities $\frac{1}{2}$ in all cases, when there is no ambiguity, when the state is p_{v_c} represented by the vector

$$|v_c\rangle = |\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\rangle \tag{3.47}$$

Let us preliminarily denote by $\mathcal{S}_1^{12}$, $\mathcal{S}_2^{12}$, $\mathcal{S}_3^{34}$ and $\mathcal{S}_4^{34}$ the eigenspaces associated with the eigenvalues o_1, o_2, o_3 and o_4, respectively. We can assume that $\mathcal{S}_2^{12}$ and $\mathcal{S}_4^{34}$ are one dimensional, and $\mathcal{S}_1^{12}$ and $\mathcal{S}_3^{34}$ are two dimensional, without loss of generality. Then, we prove the following two theorems.

Theorem 1. *If, under the hypothesis on the eigenspaces formulated above, the two self–adjoint operators (spectral decompositions) $\mathcal{O}_{12} = o_1 P_1 + o_2 P_2$ and $\mathcal{O}_{34} = o_3 P_3 + o_4 P_4$ are such that $[\mathcal{O}_{12}, \mathcal{O}_{34}] = 0$ in $\mathbb{C}^3$, then two situations are possible, (i) an orthonormal basis $\{|e_1\rangle, |e_2\rangle, |e_3\rangle\}$ of common eigenvectors exists such that $P_1 = |e_1\rangle\langle e_1| + |e_2\rangle\langle e_2|$, $P_2 = |e_3\rangle\langle e_3|$, $P_3 = |e_2\rangle\langle e_2| + |e_3\rangle\langle e_3|$ and $P_4 = |e_1\rangle\langle e_1|$, or (ii) $\mathcal{O}_{12} = \mathcal{O}_{34}$, and hence $\mathcal{S}_2^{12} = \mathcal{S}_4^{34}$ and $\mathcal{S}_1^{12} = \mathcal{S}_3^{34}$.*

Proof: Suppose that $\mathcal{O}_{12} \neq \mathcal{O}_{34}$. Since $\mathcal{S}_2^{12}$ and $\mathcal{S}_4^{34}$ are one dimensional, we can choose $|e_1\rangle$ and $|e_3\rangle$ unit vectors respectively in $\mathcal{S}_2^{12}$ and $\mathcal{S}_4^{34}$. Since $[\mathcal{O}_{12}, \mathcal{O}_{34}] = 0$ it follows that $[P_2, P_4] = 0$, and hence $|e_1\rangle \perp |e_3\rangle$ or $|e_1\rangle = \lambda|e_2\rangle$ for some $\lambda \in \mathbb{C}$. But, if $|e_1\rangle = \lambda|e_2\rangle$, we have $\mathcal{S}_2^{12} = \mathcal{S}_4^{34}$, and hence $\mathcal{S}_1^{12} = (\mathcal{S}_2^{12})^\perp = (\mathcal{S}_4^{34})^\perp = \mathcal{S}_3^{34}$, which entails $\mathcal{O}_{12} = \mathcal{O}_{34}$. Hence, we have $|e_1\rangle \perp |e_3\rangle$. Since $\mathcal{S}_1^{12}$ and $\mathcal{S}_3^{34}$ are both two dimensional, their intersection is one dimensional, or they are equal. If $\mathcal{S}_1^{12} = \mathcal{S}_3^{34}$, then also $(\mathcal{S}_1^{12})^\perp = (\mathcal{S}_3^{34})^\perp$, which would entail again that $\mathcal{O}_{12} = \mathcal{O}_{34}$. Hence, we have that their intersection is one dimensional. We choose $|e_2\rangle$ a unit vector contained in this intersection, and hence $|e_2\rangle \perp |e_1\rangle$ and $|e_2\rangle \perp |e_3\rangle$. This means that $\{|e_1\rangle, |e_2\rangle, |e_3\rangle\}$ is an orthonormal basis, and $P_1 = |e_1\rangle\langle e_1| + |e_2\rangle\langle e_2|$, $P_2 = |e_3\rangle\langle e_3|$, $P_3 = |e_2\rangle\langle e_2| + |e_3\rangle\langle e_3|$ and $P_4 = |e_1\rangle\langle e_1|$.

Theorem 2. *For the data of our experiment exist compatible self-adjoint operators, hence, following the notations introduced, $\mathcal{O}_{12} \neq \mathcal{O}_{34}$ such that $[\mathcal{O}_{12}, \mathcal{O}_{34}] = 0$, modeling both bets. This means that,* *again following the notations introduced, that we have*

$$\langle e_1|e_1\rangle = \langle e_2|e_2\rangle = \langle e_3|e_3\rangle = 1 \tag{3.48}$$

$$\langle e_1|e_2\rangle = \langle e_1|e_3\rangle = \langle e_2|e_3\rangle = 0 \tag{3.49}$$

$$|\langle v_s^{12}|e_1\rangle|^2 + |\langle v_s^{12}|e_2\rangle|^2 = 0.68 \tag{3.50}$$

$$|\langle v_s^{12}|e_3\rangle|^2 = 0.32 \tag{3.51}$$

$$|\langle v_s^{34}|e_2\rangle|^2 + |\langle v_s^{34}|e_3\rangle|^2 = 0.31 \tag{3.52}$$

$$|\langle v_s^{34}|e_1\rangle|^2 = 0.69 \tag{3.53}$$

$$|\langle v_c|e_1\rangle|^2 + |\langle v_c|e_2\rangle|^2 = 0.5 = |\langle v_c|e_3\rangle|^2 \tag{3.54}$$

$$|\langle v_c|e_2\rangle|^2 + |\langle v_c|e_3\rangle|^2 = 0.5 = |\langle v_c|e_1\rangle|^2 \tag{3.55}$$

Proof: We can explicitly construct an orthonormal basis $\{|e_1\rangle, |e_2\rangle, |e_3\rangle\}$ which simultaneously satisfies Eqs. (3.48)–(3.55). We omit the explicit construction, for the sake of brevity, and we only report the solution, as follows. The orthonormal vectors $|e_1\rangle$ and $|e_3\rangle$ are respectively given by

$$|e_1\rangle = |0.38e^{i61.2^\circ}, 0.13e^{i248.4^\circ}, 0.92e^{i194.4^\circ}\rangle \tag{3.56}$$

$$|e_3\rangle = |0.25e^{i251.27^\circ}, 0.55e^{i246.85^\circ}, 0.90e^{i218.83^\circ}\rangle \tag{3.57}$$

One can then construct at once a unit vector $|e_2\rangle$ orthogonal to both $|e_1\rangle$ and $|e_3\rangle$, which we don't do explicitly, again for the sake of brevity.

Theorems 1 and 2 entail that within our quantum modeling approach the two bets can be represented by commuting observables such that the statistical data of the real experiment are faithfully modeled.

The following orthogonal projection operators then model the agents' decisions.

$$P_2 = |e_3\rangle\langle e_3| = \begin{pmatrix} 0.06 & 0.14e^{i4.42^\circ} & 0.23e^{i32.44^\circ} \\ 0.14e^{-i4.42^\circ} & 0.30 & 0.49e^{i28.02^\circ} \\ 0.23e^{-i32.44^\circ} & 0.49e^{-i28.02^\circ} & 0.81 \end{pmatrix} \tag{3.58}$$

and

$$P_4 = |e_1\rangle\langle e_1| = \begin{pmatrix} 0.14 & 0.05e^{-i187.2^\circ} & 0.35e^{-i133.2^\circ} \\ 0.05e^{i187.2^\circ} & 0.02 & 0.12e^{i54^\circ} \\ 0.35e^{i133.2^\circ} & 0.12e^{-i54^\circ} & 0.85 \end{pmatrix} \tag{3.59}$$

Thus, $P_1 = \mathbb{1} - P_2$ and $P_3 = \mathbb{1} - P_4$ can be easily calculated.

Let us now come to the representation of the observables. The observable associated with the preference between f_1 and f_2 is then represented by the self–adjoint operator (spectral decomposition) $\mathcal{O}_{12}$, while the observable associated with the preference between f_3 and f_4 is represented by the self–adjoint operator (spectral decomposition) $\mathcal{O}_{34}$. More explicitly, if we set $o_1 = o_3 = 1$ and

$o_2 = o_4 = -1$, we get the following explicit representations.

$$\begin{aligned} \mathcal{O}_{12} &= P_1 - P_2 = \mathbb{1} - 2P_2 \\ &= \begin{pmatrix} 0.87 & -0.28e^{i4.42^\circ} & -0.46e^{i32.44^\circ} \\ -0.28e^{-i4.42^\circ} & 0.40 & -0.98e^{i28.02^\circ} \\ -0.45e^{-i32.44^\circ} & -0.98e^{-i28.02^\circ} & -0.62 \end{pmatrix} \end{aligned} \tag{3.60}$$

$$\begin{aligned} \mathcal{O}_{34} &= P_3 - P_4 = \mathbb{1} - 2P_4 \\ &= \begin{pmatrix} 0.71 & -0.10e^{-i187.2^\circ} & -0.70e^{-i133.2^\circ} \\ -0.10e^{i187.2^\circ} & 0.97 & -0.24e^{i54^\circ} \\ -0.70e^{i133.2^\circ} & -0.24e^{-i54^\circ} & -0.69 \end{pmatrix} \end{aligned} \tag{3.61}$$

A direct calculation of the commutator operator $[\mathcal{O}_{12}, \mathcal{O}_{34}]$ reveals that the corresponding observables are indeed compatible.

3.5 A real vector space analysis

We show in this section that the possibility of representing the experimental data in Sect. 3.3 by compatible measurements for the bets relies crucially on our choice of a Hilbert space over complex numbers as a modeling space. Indeed, if a Hilbert space over real numbers is attempted, no compatible observables for the bets and the data in Sect. 3.3 can be constructed any longer, as the following analysis reveals.

Let us indeed attempt to represent the Ellsberg paradox situation in the real Hilbert space $\mathbb{R}^3$. This comes to allowing only values of 0 and π for the phases of the complex vectors in Sect. 3.2.3. It is then easy to see that the only conservative superpositions that remain are the extreme states themselves, that is,

$$|v_s^{12}\rangle = |v_{ry}\rangle = |\pm\frac{1}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}, 0\rangle \tag{3.62}$$

$$|v_s^{34}\rangle = |v_{rb}\rangle = |\pm\frac{1}{\sqrt{3}}, 0, \pm\sqrt{\frac{2}{3}}\rangle \tag{3.63}$$

This means that only superposition states such as $|v_s^{12}\rangle$ and $|v_s^{34}\rangle$ are only conservative, in case complex numbers are used for the superposition. This is a first instance of the necessity of complex numbers for a quantum modeling of the Ellsberg situation, because indeed, we should be able to represent the 'priors', hence the quantum states, by superpositions of the extreme states, not by the extreme states themselves. But, let us prove that, even if we opt for representing the quantum states by the extreme states, no real Hilbert space representation with compatible observables modeling the bets and the experimental data is possible.

Taking into account the content of Th. 1, and making use of the notations introduced in Sect. 3.4, we can state the following: For a compatible solution to exist, we need to find two unit vectors

$|x\rangle$ and $|y\rangle$ such that $|x\rangle \in \mathcal{S}_2^{12}$ and $|y\rangle \in \mathcal{S}_4^{34}$. For case (i) of Th. 1 to be satisfied, this is the case where the self-adjoint operators representing the compatible measurements are different, we have that $|x\rangle$ needs to be orthogonal to $|y\rangle$, which can be expressed as $\langle x|y\rangle = 0$. For the case (ii) of Th. 1 to be satisfied, this is the case where the self-adjoint operators representing the compatible measurements are equal, we have that $|x\rangle$ needs to be a multiple of $|y\rangle$, and since both are unit vectors, and we work in a real Hilbert space, this means that $|x\rangle = \pm|y\rangle$. This can be expressed as $\langle x|y\rangle = \pm 1$. In the following we will prove that such $|x\rangle$ and $|y\rangle$ do not exist in a real Hilbert space.

In our proof we start by supposing that these vectors exist and find a contradiction. Let us put $|x\rangle = |a, b, c\rangle$ and $|y\rangle = |d, f, g\rangle$. Since $|x\rangle$ and $|y\rangle$ are unit vectors, we have $1 = a^2 + b^2 + c^2 = d^2 + f^2 + g^2$. Generalizing Eqs. (3.51), (3.53), (3.54) and (3.55) to considering the two cases (i) and (ii) of Th. 1, we have that the vectors $|x\rangle$ and $|y\rangle$ must satisfy following equations.

$$
\begin{aligned}
\frac{1}{2} &= |\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}|a, b, c\rangle|^2 \\
&= \frac{1}{3}(a+b+c)^2 = \frac{1}{3}(a^2+b^2+c^2+2ab+2ac+2bc) && (3.64) \\
0.69 &= |\langle \frac{1}{\sqrt{3}}, 0, \sqrt{\frac{2}{3}}|a, b, c\rangle|^2 \\
&= \frac{1}{3}(a+\sqrt{2}c)^2 = \frac{1}{3}(a^2+2c^2+2\sqrt{2}ac) && (3.65) \\
\frac{1}{2} &= |\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}|d, f, g\rangle|^2 \\
&= \frac{1}{3}(d+f+g)^2 = \frac{1}{3}(d^2+f^2+g^2+2df+2dg+2fg) && (3.66) \\
0.32 &= |\langle \frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}, 0|d, f, g\rangle|^2 \\
&= \frac{1}{3}(d+\sqrt{2}f)^2 = \frac{1}{3}(d^2+2f^2+2\sqrt{2}df) && (3.67)
\end{aligned}
$$

We stress that we have considered here the ++ signs choices for $|v_s^{12}\rangle$ and $|v_s^{34}\rangle$. We will later consider the other possibilities. Elaborating we get the following set of equations to be satisfied

$$
\begin{aligned}
1 &= a^2+b^2+c^2 && (3.68) \\
1.5 &= a^2+b^2+c^2+2ab+2ac+2bc && (3.69) \\
2.07 &= a^2+2c^2+2\sqrt{2}ac && (3.70) \\
1 &= d^2+f^2+g^2 && (3.71) \\
1.5 &= d^2+f^2+g^2+2df+2dg+2fg && (3.72) \\
0.96 &= d^2+2f^2+2\sqrt{2}df && (3.73)
\end{aligned}
$$

The points (a, b, c) satisfying Eq. (3.69) lie on a cone in the origin and centred around $|v_c\rangle$, and the points (a, b, c) satisfying Eq. (3.70) lie on a cone in the origin and centred around $|v_s^{34}\rangle$. Hence Eqs. (3.69) and (3.70) can only jointly be satisfied where these two cones intersect. Further need (a, b, c)

to be the coordinates of a unit vector, which is expressed by Eq. (3.68). For two cones there are in a three dimensional real space only two possibilities, or they cut each other in two lines through the origin, or they do not have an intersection different from the origin. On two lines through the origin, four unit vectors can be found always. This means that Eqs. (3.68), (3.69) and (3.70) have four solutions, or none. We are in a situation here of four solutions, which are the following

$$1. \quad a = 0.052 \quad b = 0.192 \quad c = 0.980 \tag{3.74}$$

$$2. \quad a = -0.931 \quad b = 0.065 \quad c = -0.359 \tag{3.75}$$

$$3. \quad a = -0.052 \quad b = -0.192 \quad c = -0.980 \tag{3.76}$$

$$4. \quad a = 0.931 \quad b = -0.065 \quad c = 0.359 \tag{3.77}$$

Also the solutions of Eqs. (3.71), (3.72) and (3.73) are four points on the two intersecting lines of a cone, or none, if the cones do not intersect. In the situation corresponding to our experimental data, we also here find four solutions, which are

$$1. \quad d = 0.969 \quad f = 0.008 \quad g = 0.248 \tag{3.78}$$

$$2. \quad d = 0.154 \quad f = -0.802 \quad g = -0.577 \tag{3.79}$$

$$3. \quad d = -0.969 \quad f = -0.008 \quad g = -0.248 \tag{3.80}$$

$$4. \quad d = -0.154 \quad f = 0.802 \quad g = 0.577 \tag{3.81}$$

Our proof follows now easily, when we verify that none of these solutions allows $|x\rangle$ and $|y\rangle$ to be an orthogonal pair of vectors, or a pair of vectors equal to each other, or to ones opposite. We can verify this by calculating the number $\langle x|y\rangle$, and seeing that they are all different from 0, different from +1, and different from -1. We indeed have

$$\langle e_1|e_3\rangle^{11} = 0.296 \quad \langle e_1|e_3\rangle^{12} = -0.711 \tag{3.82}$$

$$\langle e_1|e_3\rangle^{13} = -0.296 \quad \langle e_1|e_4\rangle^{13} = 0.711 \tag{3.83}$$

$$\langle e_1|e_3\rangle^{21} = -0.990 \quad \langle e_1|e_3\rangle^{22} = 0.011 \tag{3.84}$$

$$\langle e_1|e_3\rangle^{23} = 0.990 \quad \langle e_1|e_3\rangle^{24} = -0.011 \tag{3.85}$$

$$\langle e_1|e_3\rangle^{31} = -0.296 \quad \langle e_1|e_3\rangle^{32} = 0.711 \tag{3.86}$$

$$\langle e_1|e_3\rangle^{33} = 0.296 \quad \langle e_1|e_4\rangle^{33} = -0.711 \tag{3.87}$$

$$\langle e_1|e_3\rangle^{41} = 0.990 \quad \langle e_1|e_3\rangle^{42} = -0.011 \tag{3.88}$$

$$\langle e_1|e_3\rangle^{43} = -0.990 \quad \langle e_1|e_3\rangle^{44} = 0.011 \tag{3.89}$$

To complete our proof of the non-existence of compatible observables in a real Hilbert space, we need to analyse also all the other possibilities, i.e. all possible choices of $+$ and $-$ for $|v_s^{12}\rangle$ and $|v_s^{34}\rangle$. This can be done completely along the same lines of the above, and hence we do not represent it explicitly here. We have however verified all cases carefully, and indeed, none of the possibilities lead to vectors $|x\rangle$ and $|y\rangle$ such that their inner product $\langle x|y\rangle$ equals 0, +1, or −1.

The above result is relevant, in our opinion, and we think it is worth to discuss it more in detail. The existence of compatible observables to represent the decision-makers' choice among the different acts in our experiment on the Ellsberg paradox is a direct consequence of the fact that we used a complex Hilbert space as a modeling space. As we have proved here, if one instead uses a real vector space, then the collected data cannot be reproduced by compatible observables. Hence, one has two possibilities, in this case. Either one requires that compatible observables exist that accord with an Ellsberg-type situation, and then one has to accept a complex Hilbert space representation where ambiguity aversion is coded into superposed quantum states (note that these superpositions are of the 'complex-type', hence entailing genuine interference – superpositions with complex (non-real) coefficients always entail that the quantum effect of interference is present). Alternatively, one can use a representation in a real vector space but, then, one should accept that an Ellserg-type situation cannot be reproduced by compatible observables. In either case, the appearance of quantum structures – interference due to the presence of genuine complex numbers, or incompatibility due to the impossibility to represent the data by compatible measurements – seems unavoidable in the Ellsberg paradox situation.

Our quantum theoretic modeling of the Ellsberg paradox situation is thus completed. We however want to add some explanatory remarks concerning the novelty of our approach, as follows.

(i) We have incorporated the subjective preference of traditional economics approaches in the quantum state describing the conceptual Ellsberg entity. At variance with existing proposals, the subjective preference coded in the quantum state can be different for each one of the acts f_j, since it is not derived from any prefixed mathematical rule. Therefore, we can naturally explain a situation with f_1 preferred to f_2 and f_4 preferred to f_3, without the need of assuming extra hypotheses.

(ii) In the present paper, we have detected genuine quantum aspects in the description of the Ellsberg paradox situation, namely, contextuality, superposition, and the ensuing interference. Further deepening and experimenting most probably will reveal other quantum aspects. Hence the hypothesis that quantum effects, of a conceptual nature, concretely drive decision–makers' behavior in uncertainty situations is warranted.

(iii) We have focused here on the Ellsberg paradox. But, our quantum theoretic modeling is sufficiently general to cope with various generalizations of the Ellsberg paradox, which are problematical in traditional economics approaches, such as the Machina paradox [17]. This opens the way toward the construction of a unified framework extending standard expected utility and modeling 'ambiguity laden' situations in economics and decision theory.

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Appendix A

Appendix Welfare Cost of Model Uncertainty in a SOE

A.1 The model's first-order conditions

In this section I report the first-order conditions that come from household's maximization problem. Assuming that households have a GHH Bernoulli utility function and preferences for robustness.

$$\left(c_t - \frac{h_t^\omega}{\omega}\right)^{-\sigma} = \beta(1+r_t)E_t\left\{\frac{\exp(\frac{-V_{t+1}}{\theta})}{E_t\exp(\frac{-V_{t+1}}{\theta})}\left(c_{t+1} - \frac{h_{t+1}^\omega}{\omega}\right)^{-\sigma}\right\} \tag{A.1}$$

$$\left(c_t - \frac{h_t^\omega}{\omega}\right)^{-\sigma}(1+\phi(k_{t+1}-k_t)) = \beta E_t\left\{\frac{\exp(\frac{-V_{t+1}}{\theta})}{E_t\exp(\frac{-V_{t+1}}{\theta})}\left(c_{t+1} - \frac{h_{t+1}^\omega}{\omega}\right)^{-\sigma}\left[\alpha e^{a_{t+1}}k_{t+1}^{\alpha-1}h_{t+1}^{1-\alpha} + 1 - \delta + \phi(k_{t+2}-k_{t+1})\right]\right\} \tag{A.2}$$

$$h_t^{\omega-1} = (1-\alpha)e^{a_t}k_t^\alpha h_t^{-\alpha} \tag{A.3}$$

A.1.1 Model's deterministic steady state solution

In the deterministic steady state, the productivity shock is constant and equal to one and the other state variables do not change over time. The equations characterizing the steady state comprise the definition of the value function (A.4), the first-order conditions (A.5)-(A.7) and the budget constraint(A.8):

$$V_{ss} = \frac{\left(c_{ss} - \frac{h_{ss}^\omega}{\omega}\right)^{1-\sigma} - 1}{(1-\sigma)(1-\beta)} \tag{A.4}$$

$$1 = \beta(1+r^*) \tag{A.5}$$

$$1 = \beta(\alpha e^a k_{ss}^{\alpha-1}h_{ss}^{1-\alpha} + (1-\delta)) \tag{A.6}$$

$$h_{ss}^{\omega-1} = (1-\alpha)e^a\left(\frac{k}{h}\right)^\alpha \tag{A.7}$$

$$c_{ss} = y_{ss} - r^*\bar{d} - i_{ss} \tag{A.8}$$

Then using (A.6):

$$\frac{k_{ss}}{h_{ss}} = \left[\frac{\frac{1}{\beta} - (1-\delta)}{\alpha e^{a}} \right]^{\frac{1}{\alpha-1}} = \kappa \tag{A.9}$$

From the steady-state labor supply (A.7), I obtain:

$$h_{ss} = [(1-\alpha)e^{a}\kappa^{\alpha}]^{\frac{1}{\omega-1}} \tag{A.10}$$

$$k_{ss} = \kappa h_{ss} \tag{A.11}$$

$$\tag{A.12}$$

A.2 Model solution

Here, I present the first-order derivatives of the value function with respect to the state variables, calculated using the envelope theorem with restrictions, in order to find the first-order coefficients of the Taylor approximation:

$$V_{d,t} = -\lambda_t[(1 + r_t^* + \psi(e^{d_t - \bar{d}} - 1)) + \psi d_t e^{d_t - \bar{d}}] \tag{A.13}$$

$$V_{k,t} = \lambda_t[(1-\delta) + \alpha a_t k_t^{\alpha-1} h_t^{1-\alpha} + \phi(k_{t+1} - k_t)] \tag{A.14}$$

$$V_{a,t} = \beta \frac{E_t \left\{ \exp(\frac{-V_{t+1}}{\theta}) \rho V_{a,t+1} \right\}}{E_t \exp(\frac{-V_{t+1}}{\theta})} + \lambda_t k_t^{\alpha} h_t^{1-\alpha} \tag{A.15}$$

$$V_{\chi,t} = \beta \frac{E_t \left\{ \exp(\frac{-V_{t+1}}{\theta}) (V_{a,t+1}(\tilde{\eta}\epsilon_{t+1}) + V_{\chi,t+1}) \right\}}{E_t \exp(\frac{-V_{t+1}}{\theta})} \tag{A.16}$$

Evaluating at the deterministic steady state:

$$V_{d,ss} = -\lambda_{ss}[(1 + r_t^*) + \psi d_{ss}] \tag{A.17}$$

$$V_{k,ss} = \lambda_{ss}[(1-\delta) + \alpha e^{a_{ss}} k_{ss}^{\alpha-1} h_{ss}^{1-\alpha}] \tag{A.18}$$

$$V_{a,ss} = \frac{\lambda_{ss} k_{ss}^{\alpha} h_{ss}^{1-\alpha}}{1 - \beta\rho} \tag{A.19}$$

$$V_{\chi,ss} = 0 \tag{A.20}$$

The second-order derivatives of the value function with respect to the state varibles are:

$$\begin{aligned}
V_{\chi\chi,t} = {} & \beta \frac{E_t\left\{\exp(\frac{-V_{t+1}}{\theta})\cdot -\frac{1}{\theta}(V_{d,t+1}d_{\chi,t+1}+V_{k,t+1}k_{\chi,t+1}+V_{a,t+1}\tilde{\eta}\epsilon_{t+1}+V_{\chi,t+1})(V_{a,t+1}\tilde{\eta}\epsilon_{t+1}+V_{\chi,t+1})\right\}}{E_t\exp(\frac{-V_{t+1}}{\theta})} \\
& +\beta\frac{E_t\left\{\exp(\frac{-V_{t+1}}{\theta})(V_{a,t+1}\tilde{\eta}\epsilon_{t+1}+V_{\chi,t+1})\right\}E_t\left\{\exp(\frac{-V_{t+1}}{\theta})\cdot\frac{1}{\theta}(V_{d,t+1}d_{\chi,t+1}+V_{k,t+1}k_{\chi,t+1}+V_{a,t+1}\tilde{\eta}\epsilon_{t+1}+V_{\chi,t+1})\right\}}{[E_t\exp(\frac{-V_{t+1}}{\theta})]^2} \\
& +\beta\frac{E_t\left\{\exp(\frac{-V_{t+1}}{\theta})((V_{ad,t+1}d_{\chi,t+1}+V_{ak,t+1}k_{\chi,t+1}+V_{aa,t+1}\tilde{\eta}\epsilon_{t+1}+V_{a\chi,t+1})\tilde{\eta}\epsilon_{t+1}+V_{\chi d,t+1}d_{\chi,t+1}+V_{\chi k,t+1}k_{\chi,t+1}+V_{\chi a,t+1}c\epsilon_{t+1}+V_{\chi\chi,t+1})\right\}}{E_t\exp(\frac{-V_{t+1}}{\theta})}
\end{aligned} \tag{A.21}$$

Evaluating at the deterministic steady state:

$$V_{\chi\chi,ss} = -\frac{\beta}{\theta}V_{a,ss}^2\tilde{\eta}^2 + \beta V_{aa,ss}\tilde{\eta}^2 + \beta V_{\chi\chi,ss} \tag{A.22}$$

$$\Rightarrow \quad V_{\chi\chi,ss} = \frac{\beta\tilde{\eta}^2}{1-\beta}\left[V_{aa,ss} - \frac{V_{a,ss}^2}{\theta}\right] \tag{A.23}$$

The value function second-order derivatives and equilibrium conditions first-order derivatives.

$$d_{d,t} = 1 + rf + d_t e^{d_t - d_f}\psi + \left(-1 + e^{d_t - d_f}\right)\psi + c_{d,t} - a_t h_t^{-\alpha} k_t^{\alpha}(1-\alpha)h_{d,t} + k_{d,t+1} + (-k_t + k_{t+1})\phi k_{d,t+1} \tag{A.24}$$

$$d_{k,t+1} = -1 + \delta + c_{t,k} - a_t\left(h_t^{1-\alpha}k_t^{-1+\alpha}\alpha + h_t^{-\alpha}k_t^{\alpha}(1-\alpha)h_{t,k}\right) + (-k_t + k_{t+1})\phi\left(-1 + k_{k,t+1}\right) + k_{k,t+1} \tag{A.25}$$

$$d_{a,t+1} = -h_t^{1-\alpha}k_t^{\alpha} + c_{t,a} - a_t h_t^{-\alpha}k_t^{\alpha}(1-\alpha)h_{t,a} + k_{a,t+1} + (-k_t + k_{t+1})\phi k_{a,t+1} \tag{A.26}$$

$$d_{\chi,t+1} = c_{\chi,t} - a_t h_t^{-\alpha}k_t^{\alpha}(1-\alpha)h_{\chi,t} + k_{\chi,t+1} + (-k_t + k_{t+1})\phi k_{\chi,t+1} \tag{A.27}$$

$$\begin{aligned}
& -\sigma\left(c_t - \frac{h_t^{\omega}}{\omega}\right)^{-1-\sigma}\left(c_{d,t} - h_t^{-1+\omega}h_{d,t}\right) = e^{d_t - df}\beta\psi E\left[\frac{e^{-\frac{V_{t+1}}{\theta}}\left(c_{t+1} - \frac{h_{t+1}^{\omega}}{\omega}\right)^{-\sigma}}{E\left[e^{-\frac{V_{t+1}}{\theta}}\right]}\right] \\
& + \beta\left(1 + rf + \left(-1 + e^{d_t - df}\right)\psi\right)\left(E\left[e^{-\frac{V_{t+1}}{\theta}}\left(\frac{\left(c_{t+1} - \frac{h_{t+1}^{\omega}}{\omega}\right)^{-\sigma}E\left[\frac{e^{-\frac{V_{t+1}}{\theta}}\left(V_{a,t+1}a_{d,t+1} + V_{k,t+1}k_{d,t+1} + d_{d,t+1}V_{d,t+1}\right)}{\theta}\right]}{E\left[e^{-\frac{V_{t+1}}{\theta}}\right]^2}\right.\right.\right. \\
& \left.\left. - \frac{\sigma\left(c_{t+1} - \frac{h_{t+1}^{\omega}}{\omega}\right)^{-1-\sigma}\left(c_{d,t+1} - h_{t+1}^{-1+\omega}h_{d,t+1}\right)}{E\left[e^{-\frac{V_{t+1}}{\theta}}\right]}\right)\right] \\
& \left. - E\left[\frac{e^{-\frac{V_{t+1}}{\theta}}\left(c_{t+1} - \frac{h_{t+1}^{\omega}}{\omega}\right)^{-\sigma}\left(V_{a,t+1}a_{d,t+1} + V_{k,t+1}k_{d,t+1} + d_{d,t+1}V_{d,t+1}\right)}{\theta E\left[e^{-\frac{V_{t+1}}{\theta}}\right]}\right]\right)
\end{aligned} \tag{A.28}$$

$$
\begin{aligned}
&-\sigma\left(c-\frac{h_t^{\omega}}{\omega}\right)^{-1-\sigma}\left(c_{k,t}-h^{-1+\omega}h_{k,t}\right)=\beta\left(1+rf+\left(-1+e^{d_t-df}\right)\psi\right)\\
&\left(E\left[e^{-\frac{V_{t+1}}{\theta}}\left(\frac{\left(c_{t+1}-\frac{h_{t+1}^{\omega}}{\omega}\right)^{-\sigma}E\left[\frac{e^{-\frac{V_{t+1}}{\theta}}\left(V_{a,t+1}a_{k,t+1}+k_{t+1}V_{k,t+1}+d_{k,t+1}V_{d,t+1}\right)}{\theta}\right]}{E\left[e^{-\frac{V_{t+1}}{\theta}}\right]^2}\right.\right.\right.\\
&\left.\left.-\frac{\sigma\left(c_{t+1}-\frac{h_{t+1}^{\omega}}{\omega}\right)^{-1-\sigma}\left(c_{k,t+1}-h_{t+1}^{-1+\omega}h_{k,t+1}\right)}{E\left[e^{-\frac{V_{t+1}}{\theta}}\right]}\right)\right]\\
&\left.-E\left[\frac{e^{-\frac{V_{t+1}}{\theta}}\left(c_{t+1}-\frac{h_{t+1}^{\omega}}{\omega}\right)^{-\sigma}\left(V_{a,t+1}a_{k,t+1}+k_{k,t+1}V_{k,t+1}+d_{k,t+1}V_{d,t+1}\right)}{\theta E\left[e^{-\frac{V_{t+1}}{\theta}}\right]}\right]\right)
\end{aligned}
\tag{A.29}
$$

$$
\begin{aligned}
&-\sigma\left(c_t-\frac{h_t^{\omega}}{\omega}\right)^{-1-\sigma}\left(c_{a,t}-h_t^{-1+\omega}h_{a,t}\right)=\beta\left(1+rf+\left(-1+e^{d_t-df}\right)\psi\right)\\
&\left(E\left[e^{-\frac{V_{t+1}}{\theta}}\left(\frac{\left(c_{t+1}-\frac{h_{t+1}^{\omega}}{\omega}\right)^{-\sigma}E\left[\frac{e^{-\frac{V_{t+1}}{\theta}}\left(a_{a,t+1}V_{a,t+1}+k_{a,t+1}V_{k,t+1}+d_{a,t+1}V_{d,t+1}\right)}{\theta}\right]}{E\left[e^{-\frac{V_{t+1}}{\theta}}\right]^2}\right.\right.\right.\\
&\left.\left.-\frac{\sigma\left(c_{t+1}-\frac{h_{t+1}^{\omega}}{\omega}\right)^{-1-\sigma}\left(c_{a,t+1}-h_{t+1}^{-1+\omega}h_{a,t+1}\right)}{E\left[e^{-\frac{V_{t+1}}{\theta}}\right]}\right)\right]\\
&\left.-E\left[\frac{e^{-\frac{V_{t+1}}{\theta}}\left(c_{t+1}-\frac{h_{t+1}^{\omega}}{\omega}\right)^{-\sigma}\left(a_{a,t+1}V_{a,t+1}+k_{a,t+1}V_{k,t+1}+d_{a,t+1}V_{d,t}\right)}{\theta E\left[e^{-\frac{V_{t+1}}{\theta}}\right]}\right]\right)
\end{aligned}
\tag{A.30}
$$

$$
\begin{aligned}
&-\sigma\left(c_t-\frac{h_t^{\omega}}{\omega}\right)^{-1-\sigma}\left(c_{t,\chi}-h_t^{-1+\omega}h_{\chi,t}\right)=\beta\left(1+rf+\left(-1+e^{d_t-df}\right)\psi\right)\\
&\left(E\left[e^{-\frac{V_{t+1}}{\theta}}\left(\frac{\left(c_{t+1}-\frac{h_{t+1}^{\omega}}{\omega}\right)^{-\sigma}E\left[\frac{e^{-\frac{V_{t+1}}{\theta}}\left(V_{\chi,t+1}+a_{\chi,t+1}V_{a,t+1}+k_{\chi t+1}V_{k,t+1}+d_{\chi,t+1}V_{d,t+1}\right)}{\theta}\right]}{E\left[e^{-\frac{V_{t+1}}{\theta}}\right]^2}\right.\right.\right.\\
&\left.\left.-\frac{\sigma\left(c_{t+1}-\frac{h_{t+1}^{\omega}}{\omega}\right)^{-1-\sigma}\left(c_{\chi,t+1}-h_{t+1}^{-1+\omega}h_{\chi,t+1}\right)}{E\left[e^{-\frac{V_{t+1}}{\theta}}\right]}\right)\right]\\
&\left.-E\left[\frac{e^{-\frac{V_{t+1}}{\theta}}\left(c_{t+1}-\frac{h_{t+1}^{\omega}}{\omega}\right)^{-\sigma}\left(V_{\chi,t+1}+a_{\chi,t+1}V_{a,t+1}+k_{\chi,t+1}V_{k,t+1}+d_{\chi,t+1}V_{d,t+1}\right)}{\theta E\left[e^{-\frac{V_{t+1}}{\theta}}\right]}\right]\right)
\end{aligned}
\tag{A.31}
$$

$$
\begin{aligned}
&-\sigma(1+(-k_t+k_{t+1})\phi)\left(c_t-\frac{h_t^{\omega}}{\omega}\right)^{-1-\sigma}\left(c_{d,t}-h_t^{-1+\omega}h_{d,t}\right)+\phi\left(c_t-\frac{h_t^{\omega}}{\omega}\right)^{-\sigma}k_{d,t+1}=\beta\Bigg(E\Bigg[e^{-\frac{V_{t+1}}{\theta}}\\
&\left(\frac{\left(1+a_{t+1}h_{t+1}^{1-\alpha}k_{t+1}^{-1+\alpha}\alpha-\delta+(-k_{t+1}+k_{t+2})\phi\right)\left(c_{t+1}-\frac{h_{t+1}^{\omega}}{\omega}\right)^{-\sigma}E\left[\frac{e^{-\frac{V_{t+1}}{\theta}}\left(V_{a,t+1}a_{d,t+1}+V_{k,t+1}k_{d,t+1}+d_{d,t+1}V_{d,t+1}\right)}{\theta}\right]}{E\left[e^{-\frac{V_{t+1}}{\theta}}\right]^2}\right.\\
&+\frac{1}{E\left[e^{-\frac{V_{t+1}}{\theta}}\right]}\Bigg(-\sigma\left(1+a_{t+1}h_{t+1}^{1-\alpha}k_{t+1}^{-1+\alpha}\alpha-\delta+(-k_{t+1}+k_{t+2})\phi\right)\left(c_{t+1}-\frac{h_{t+1}^{\omega}}{\omega}\right)^{-1-\sigma}\left(c_{d,t+1}-h_{t+1}^{-1+\omega}h_{d,t+1}\right)\\
&+\left(c_{t+1}-\frac{h_{t+1}^{\omega}}{\omega}\right)^{-\sigma}\left(\alpha\left(h_{t+1}^{1-\alpha}k_{t+1}^{-1+\alpha}a_{d,t+1}+a_{t+1}\left(h_{t+1}^{-\alpha}k_{t+1}^{-1+\alpha}(1-\alpha)h_{d,t+1}+h_{t+1}^{1-\alpha}k_{t+1}^{-2+\alpha}(-1+\alpha)k_{d,t+1}\right)\right)\right.\\
&\left.+\phi\left(k_{a,t+2}a_{d,t+1}-k_{d,t+1}+k_{k,t+2}k_{d,t+1}+d_{d,t+1}k_{d,t+2}\right)\right)\Bigg)\Bigg]\\
&-E\left[\frac{e^{-\frac{V_{t+1}}{\theta}}\left(1+a_{t+1}h_{t+1}^{1-\alpha}k_{t+1}^{-1+\alpha}\alpha-\delta+(-k_{t+1}+k_{t+2})\phi\right)\left(c_{t+1}-\frac{h_{t+1}^{\omega}}{\omega}\right)^{-\sigma}\left(V_{a,t+1}a_{d,t+1}+V_{k,t+1}k_{d,t+1}+d_{d,t+1}V_{d,t+1}\right)}{\theta E\left[e^{-\frac{V_{t+1}}{\theta}}\right]}\right]\Bigg)
\end{aligned}
\tag{A.32}
$$

$$
\begin{aligned}
&-\sigma(1+(-k_t+k_{t+1})\phi)\left(c_t-\frac{h_t^{\omega}}{\omega}\right)^{-1-\sigma}\left(c_{k,t}-h_t^{-1+\omega}h_{k,t}\right)+\phi\left(c_t-\frac{h_t^{\omega}}{\omega}\right)^{-\sigma}(-1+k_{k,t+1})=\beta\Bigg(E\Bigg[e^{-\frac{V_{t+1}}{\theta}}\\
&\left(\frac{\left(1+a_{t+1}h_{t+1}^{1-\alpha}k_{t+1}^{-1+\alpha}\alpha-\delta+(-k_{t+1}+k_{t+2})\phi\right)\left(c_{t+1}-\frac{h_{t+1}^{\omega}}{\omega}\right)^{-\sigma}E\left[\frac{e^{-\frac{V_{t+1}}{\theta}}\left(V_{a,t+1}a_{k,t+1}+k_{k,t+1}V_{k,t+1}+d_{k,t+1}V_{d,t+1}\right)}{\theta}\right]}{E\left[e^{-\frac{V_{t+1}}{\theta}}\right]^2}\right.\\
&+\frac{1}{E\left[e^{-\frac{V}{\theta}}\right]}\Bigg(-\sigma\left(1+a_{t+1}h_{t+1}^{1-\alpha}k_{t+1}^{-1+\alpha}\alpha-\delta+(-k_{t+1}+k_{t+2})\phi\right)\left(c_{t+1}-\frac{h_{t+1}^{\omega}}{\omega}\right)^{-1-\sigma}\left(c_{k,t+1}-h_{t+1}^{-1+\omega}h_{k,t+1}\right)\\
&+\left(c_{t+1}-\frac{h_{t+1}^{\omega}}{\omega}\right)^{-\sigma}\left(\alpha\left(h_{t+1}^{1-\alpha}k_{t+1}^{-1+\alpha}a_{k,t+1}+a_{t+1}\left(h_{t+1}^{-\alpha}k_{t+1}^{-1+\alpha}(1-\alpha)h_{k,t+1}+h_{t+1}^{1-\alpha}k_{t+1}^{-2+\alpha}(-1+\alpha)k_{k,t+1}\right)\right)\right.\\
&\left.+\phi\left(k_{a,t+2}a_{k,t+1}-k_{k,t+1}+k_{k,t+1}k_{k,t+2}+d_{k,t+1}k_{d,t+2}\right)\right)\Bigg)\Bigg]\\
&-E\left[\frac{e^{-\frac{V_{t+1}}{\theta}}\left(1+a_{t+1}h_{t+1}^{1-\alpha}k_{t+1}^{-1+\alpha}\alpha-\delta+(-k_{t+1}+k_{t+2})\phi\right)\left(c_{t+1}-\frac{h_{t+1}^{\omega}}{\omega}\right)^{-\sigma}\left(V_{a,t+1}a_{k,t+1}+k_{k,t+1}V_{k,t+1}+d_{k,t+1}V_{d,t+1}\right)}{\theta E\left[e^{-\frac{V_{t+1}}{\theta}}\right]}\right]\Bigg)
\end{aligned}
\tag{A.33}
$$

$$
\begin{aligned}
&-\sigma(1+(-k_t+k_{t+1})\phi)\left(c_t-\frac{h_t^{\omega}}{\omega}\right)^{-1-\sigma}\left(c_{a,t}-h^{-1+\omega}h_{a,t}\right)+\phi\left(c_t-\frac{h_t^{\omega}}{\omega}\right)^{-\sigma}k_{a,t+1}=\beta\Bigg(E\Bigg[e^{-\frac{V_{t+1}}{\theta}}\\
&\Bigg(\frac{\left(1+a_{t+1}h_{t+1}^{1-\alpha}k_{t+1}^{-1+\alpha}\alpha-\delta+(-k_{t+1}+k_{t+2})\phi\right)\left(c_{t+1}-\frac{h_{t+1}^{\omega}}{\omega}\right)^{-\sigma}E\left[\frac{e^{-\frac{V_{t+1}}{\theta}}\left(a_{a,t+1}V_{a,t+1}+k_{d,t+1}V_{k,t+1}+d_{a,t+1}V_{d,t+1}\right)}{\theta}\right]}{E\left[e^{-\frac{V_{t+1}}{\theta}}\right]^2}\\
&+\frac{1}{E\left[e^{-\frac{V_{t+1}}{\theta}}\right]}\Bigg(-\sigma\left(1+a_{t+1}h_{t+1}^{1-\alpha}k_{t+1}^{-1+\alpha}\alpha-\delta+(-k_{t+1}+k_{t+2})\phi\right)\left(c_{t+1}-\frac{h_{t+1}^{\omega}}{\omega}\right)^{-1-\sigma}\left(c_{a,t+1}-h_{t+1}^{-1+\omega}h_{a,t+1}\right)\\
&+\left(c_{t+1}-\frac{h_{t+1}^{\omega}}{\omega}\right)^{-\sigma}\Big(\alpha\Big(h_{t+1}^{1-\alpha}k_{t+1}^{-1+\alpha}a_{a,t+1}+a_{t+1}\left(h_{t+1}^{-\alpha}k_{t+1}^{-1+\alpha}(1-\alpha)h_{a,t+1}+h_{t+1}^{1-\alpha}k_{t+1}^{-2+\alpha}(-1+\alpha)k_{a,t+1}\right)\Big)\\
&+\quad\phi\left(-k_{a,t+1}+a_{a,t+1}k_{a,t+2}+k_{a,t+1}k_{k,t+2}+d_{a,t+1}k_{d,t+2}\right)\Big)\Big)\Bigg)\Bigg]\\
&-E\left[\frac{e^{-\frac{V_{t+1}}{\theta}}\left(1+a_{t+1}h_{t+1}^{1-\alpha}k_{t+1}^{-1+\alpha}\alpha-\delta+(-k_{t+1}+k_{t+2})\phi\right)\left(c_{t+1}-\frac{h_{t+1}^{\omega}}{\omega}\right)^{-\sigma}\left(a_{a,t+1}V_{a,t+1}+k_{a,t+1}V_{k,t+1}+d_{a,t+1}V_{a,t+1}\right)}{\theta E\left[e^{-\frac{V_{t+1}}{\theta}}\right]}\right]\Bigg]\Bigg)
\end{aligned}
\tag{A.34}
$$

$$
\begin{aligned}
&-\sigma(1+(-k_t+k_{t+1})\phi)\left(c_t-\frac{h_t^{\omega}}{\omega}\right)^{-1-\sigma}\left(c_{\chi,t}-h_t^{-1+\omega}h_{\chi,t}\right)+\phi\left(c_t-\frac{h_t^{\omega}}{\omega}\right)^{-\sigma}k_{\chi,t+1}=\beta\Bigg(E\Bigg[e^{-\frac{V_t+1}{\theta}}\\
&\Bigg(\frac{\left(1+a_{t+1}h_{t+1}^{1-\alpha}k_{t+1}^{-1+\alpha}\alpha-\delta+(-k_{t+1}+k_{t+2})\phi\right)\left(c_{t+1}-\frac{h_{t+1}^{\omega}}{\omega}\right)^{-\sigma}E\left[\frac{e^{-\frac{V_{t+1}}{\theta}}\left(V_{\chi,t+1}+a_{t+1}V_{a,t+1}+k_{\chi t+1}V_{k,t+1}+d_{\chi,t+1}V_{d,t+1}\right)}{\theta}\right]}{E\left[e^{-\frac{V_{t+1}}{\theta}}\right]^2}\\
&+\frac{1}{E\left[e^{-\frac{V_{t+1}}{\theta}}\right]}\Bigg(-\sigma\left(1+a_{t+1}h_{t+1}^{1-\alpha}k_{t+1}^{-1+\alpha}\alpha-\delta+(-k_{t+1}+k_{t+2})\phi\right)\left(c_{t+1}-\frac{h_{t+1}^{\omega}}{\omega}\right)^{-1-\sigma}\left(c_{\chi,t+1}-h_{t+1}^{-1+\omega}h_{\chi,t+1}\right)\\
&+\left(c_{t+1}-\frac{h_{t+1}^{\omega}}{\omega}\right)^{-\sigma}\Big(\alpha\Big(h_{t+1}^{1-\alpha}k_{t+1}^{-1+\alpha}a_{\chi,t+1}+a_{t+1}\left(h_{t+1}^{-\alpha}k_{t+1}^{-1+\alpha}(1-\alpha)h_{\chi,t+1}+h_{t+1}^{1-\alpha}k_{t+1}^{-2+\alpha}(-1+\alpha)k_{\chi,t+1}\right)\Big)\\
&+\phi\left(-k_{\chi t+1}+k_{\chi,t+2}+a_{\chi,t+1}k_{a,t+2}+k_{t+1}k_{k,t+2}+d_{\chi,t+1}k_{d,t+2}\right)\Big)\Big)\Bigg)\Bigg]-E\Bigg[e^{-\frac{V_{t+1}}{\theta}}\\
&\frac{\left(1+a_{t+1}h_{t+1}^{1-\alpha}k_{t+1}^{-1+\alpha}\alpha-\delta+(-k_{t+1}+k_{t+2})\phi\right)\left(c_{t+1}-\frac{h_{t+1}^{\omega}}{\omega}\right)^{-\sigma}\left(V_{\chi,t+1}+a_{\chi,t+1}V_{a,t+1}+k_{\chi,t+1}V_{k,t+1}+d_{\chi,t+1}V_{d,t+1}\right)}{\theta E\left[e^{-\frac{V_{t+1}}{\theta}}\right]}\Bigg]\Bigg]\Bigg)
\end{aligned}
\tag{A.35}
$$

A.3 Simulation results

The results in the following table show the compensating variation in consumption τ for different values of risk aversion σ and the concern for robustness θ. Calculations for the benchmark scenario are performed with Schmitt-Grohé and Uribe's (2003) original calibration. The second scenario, assumes financial frictions understood as a high sensitivity of domestic interest rate to debt. Parameter characterizing the sensitivity of the sovereign risk to domestic debt stock (ψ) is approximately 1000 times greater than the original scenario.

Table A.1: Welfare loss estimations due to aggregate uncertainty, for different combinations of risk aversion and concern for model uncertainty; no financial frictions case (consumption equivalent variation, τ)

Risk Aversion	Model		
	Risk ($\theta = \infty$)	Risk+MU ($\theta = 8$)	Risk+MU ($\theta = 1.2$)
$\sigma = 0.1$	-0.01242487494	-0.0047773802	0.02940232963
$\sigma = 0.5$	-0.0104147361	-0.00031245048	0.04775196528
$\sigma = 1$	-0.00827278276	-0.00476982253	0.07859008227
$\sigma = 1.5$	-0.0062126702	0.01546052899	0.1288140583
$\sigma = 2$	-0.0041754876	0.0282842252	0.2021151683
$\sigma = 3$	-0.0001189552	0.07435342758	0.4793921136
$\sigma = 4$	0.003933465942	0.1769271301	1.092818428
$\sigma = 5$	0.007985148696	0.4081522957	2.332218983

Table A.2: Welfare loss estimations due to aggregate uncertainty, for different combination of risk aversion and concern for model uncertainty; with financial frictions (consumption equivalent variation, τ)

Risk Aversion	Model		
	Risk ($\theta = \infty$)	Risk+MU ($\theta = 8$)	Risk+MU ($\theta = 1.2$)
$\sigma = 0.1$	-0.01136340832	-0.00387704322	0.02937390047
$\sigma = 0.5$	-0.0078994127	0.001558175204	0.04598087782
$\sigma = 1$	-0.00479431469	0.006784700022	0.0723374851
$\sigma = 1.5$	-0.00213717794	0.01675790425	0.1144456391
$\sigma = 2$	0.000296700405	0.0278397621	0.1740893674
$\sigma = 3$	0.00480046866624	0.06535471154	0.3941810435
$\sigma = 4$	0.009047364944	0.1453796449	0.8741916885
$\sigma = 5$	0.01313991887	0.3218276507	1.856003205