The Asymptotics of Price and Strategy in the Buyer's Bid Double Auction*

Mark A. Satterthwaite
Northwestern University

Steven R. Williams
University of Melbourne

Konstantinos E. Zachariadis
Queen Mary University of London

July 26, 2019

Abstract

Solving for Bayesian-Nash equilibrium is complicated in most double auction models due to the multiplicity of traders and the asymmetry of behavior between buyers and sellers. In a model with correlated and interdependent values/costs, we identify the asymptotic limits of distributions in the first order conditions for optimal bidding/asking in the buyer's bid double auction. Substitution of these asymptotic distributions into the first order conditions can in some cases permit the solution for approximately optimal bids/asks that provide insight into what is "first order" in a trader's strategic decision-making. The effectiveness of the approximations along with several comparative statics predictions are then tested numerically.

JEL Classification: C63, D44, D82

Keywords: double auction, asymptotic analysis, rational expectations equilibrium, interdependent values

1 Introduction

The theory of double auctions examines the impact upon markets of private information and the strategic behavior that it facilitates. One aim is to investigate the conjecture that the inefficiency of bilateral trade established by Myerson and Satterthwaite (1983, Cor. 1) is significant in only the smallest of markets, i.e., price theory and its assumption of price-taking behavior leading to efficient trade are meaningful even with only a modest number of traders. These objectives make it difficult to use the common assumption to simplify the mathematics of studying markets, namely, a large number of traders, for this assumption diminishes the very issues that are the intended focus of study. The theory of double auctions has therefore faced the problem of studying finite, multi-player games of incomplete information, which have for the most part proven to be very complex.

*This is a preliminary draft. Please do not quote or circulate without the authors' permission.

This paper, in contrast, makes use of asymptotic results for large numbers of traders to provide insight into the outcome of trading and the strategic decisions of traders in even small markets. The first order condition for a trader's optimal choice of his bid/ask in a game of incomplete information depends on the distribution of bids/asks from other traders. This first order condition is stated formally; with a modest restriction on bids/asks from other traders that is consistent with prior work on equilibrium behavior, the relevant distributions of bids/asks can be characterized asymptotically. These asymptotic distributions are then substituted into the first order condition, which in certain cases permits its solution for a bid/ask. This approach allows the derivation of formulas for bids/asks in terms of the fundamentals of the model (in particular, the distributions that define the game of incomplete information). The formulas are simple in comparison to the first order conditions for equilibrium and they provide insight into what is most fundamental in trader decision-making in a way that is not at all apparent from the strict game-theoretic approach. The effectiveness of these approximations is evaluated by comparing them to numerically-computed examples of equilibrium in small markets.

Model. We consider a double auction in an informational environment from Satterthwaite, Williams, and Zachariadis (2014, hereafter, SWZ) with a linear structure that facilitates analysis. Fix m, $n \in \mathbb{N}$. For *market size* $\eta \in \mathbb{N}$, we consider ηm buyers, each of whom wishes to buy at most one item, and ηn sellers, each of whom has one item to sell. A state μ is drawn from the *uniform improper prior* on $\mathbb{R}$. We discuss our use of this improper distribution below. A *preference term* ε_i is independently drawn for each trader i from a distribution G_ε on $\mathbb{R}$; trader i's value/cost then equals $\mu + \varepsilon_i$. Utility for each trader is quasilinear in his value/cost and money, with utility normalized to zero in the case of no trade and no monetary transfer. A noise term δ_i is independently drawn for each trader i from a distribution G_δ on $\mathbb{R}$, with trader i observing the signal $\sigma_i = \mu + \varepsilon_i + \delta_i$. This defines a *correlated interdependent values (CIV) model*, with the interdependence referring to the fact that learning another trader's signal or value/cost may cause a trader to update his estimate of his own value/cost. The special case in which the noise distribution G_δ is degenerate and each trader directly observes his own value/cost is the *correlated private values (CPV)* special case, with correlation among values/costs arising through the state μ.

This informational structure is used here because it is sufficiently restrictive to allow a much deeper analysis of strategic behavior in our double auction than in more general environments, while still retaining the correlation and interdependence of values/costs that are prominent features of actual markets. The uniform improper prior can be thought of informally as "the uniform distribution across the entire real line." DeGroot (1970, p. 190) motivates it as modeling a situation in which forming a proper prior ex ante is costly or complicated, and the decision maker knows that he will receive valuable information at the interim stage on which to define his posterior beliefs, which are then well-defined. Its real value for our purposes is that implies an *invariance property* for a trader's decision problem: a trader's beliefs about the values/costs and signals of others in reference to his own signal are the same for each possible value of his signal. A trader's decision problem is therefore simply translated linearly as his signal changes and he in this sense solves the same problem at every value of his signal. This in turn greatly simplifies the analysis of strategic behavior in ways that we will point out throughout this paper.

The *buyer's bid double auction* (BBDA) operates as follows. Knowing their signals, each buyer submits a bid and each seller submits an ask. The bids/asks are ordered in a list $s_{(1)} \leq s_{(2)} \leq \ldots \leq s_{(\eta(m+n))}$. The BBDA's price equals $s_{(\eta m+1)}$ with buyers whose bids are at or above this price buying from sellers whose asks are strictly below this price.[1,2] A seller trades in the BBDA only if his ask is below the price $s_{(\eta m+1)}$; he can not influence the price at which he trades. He therefore acts as a price-taker and chooses his ask to place himself on the correct side of the realized market price, taking into account both his signal and the information that he learns from the market price in the event that he trades. In the CPV case in which he knows his cost, this reduces to submitting it as his ask, i.e., $S(c) = c$ is a seller's dominant strategy. A buyer, however, sets the price at which he trades in the event that his bid equals $s_{(\eta m+1)}$. He takes this possibility into account in choosing his bid, bidding below what his bid would otherwise be if he instead acted as a price-taker.

Results. We build upon the analysis of Bayesian-Nash equilibrium in SWZ, and in particular, its analysis of the first order conditions (FOCs) for optimal bidding/asking. In the ordered list of bids/asks $s_{(1)} \leq s_{(2)} \leq \ldots \leq s_{(\eta(m+n)-1)}$ from the other $\eta(m+n)-1$ traders, a trader focuses on $s_{(\eta m)}$, for a buyer trades if and only if his bid is above this bid/ask while a seller sells if and only if his ask is strictly below it. A buyer additionally focuses on $s_{(\eta m+1)}$ because his bid sets the price when it lands between $s_{(\eta m)}$ and $s_{(\eta m+1)}$, in which case it is the $(\eta m+1)^{\text{st}}$-smallest bid/ask overall and determines the price. Theorem 1 states the asymptotic distributions of the critical order statistics $s_{(\eta m)}$ and $s_{(\eta m+1)}$ when a trader is faced with strategies by the others satisfying a particular bound. This bound is in fact satisfied by equilibrium strategies, as studied in SWZ (secs. 5.2-5.3).

Additionally, Theorem 1 states the asymptotic distribution of BBDA's price. In a pioneering paper, Reny and Perry (2006) show that a double auction's equilibrium price in a continuum market coincides with the *rational expectations equilibrium* (REE) price. It is also proven that a noncooperative equilibrium exists in a sufficiently large but finite market that approximately implements the REE price of the continuum limit. Reny and Perry (2006) in this sense build a strategic foundation for REE. SWZ (sec. 5.5) proves convergence of the BBDA's price to the REE price as the market size η increases within the context of the informational environment studied in this paper and shows that its expected error is $\Theta\left(1/\sqrt{\eta}\right)$. Theorem 1 goes further and characterizes the asymptotic normality of the BBDA's price with mean equal to the limiting REE price. The normality of the price mirrors the approximate normality of changes in stock prices that is well-documented empirically and that underlies the Black-Scholes formula for option pricing. Within the

[1] Suppose $s_{(\eta m)} < s_{(\eta m+1)}$ and let d denote the number of bids at or above $s_{(\eta m+1)}$. There are then $\eta n - d$ asks at or above $s_{(\eta m+1)}$, and so there are $\eta n - (\eta n - d) = d$ asks below $s_{(\eta m+1)}$. The market thus clears at the price $s_{(\eta m+1)}$. If $s_{(\eta m)} = s_{(\eta m+1)}$, then a fair lottery may be needed to complete the allocation among traders on the short side of the market whose bids/asks equal $s_{(\eta m)} = s_{(\eta m+1)}$. Additional details can be found in Satterthwaite and Williams (1989, p. 480–81).

[2] For each $k \in [0,1]$, the *k-double auction* generalizes the BBDA (i.e., the 1-double auction) by selecting as the market price from the ordered list of bids/asks. We reference this market mechanism below and in section 3.

informational environment of this paper, the BBDA's price is therefore a consistent, asymptotically unbiased and normal estimator of the REE price. Numerical examples in section 3.0.1 demonstrate that the asymptotic distribution can meaningfully approximate a sample distribution of equilibrium prices even in very small markets.

We next turn to the analysis of a trader's decision problem, which we address by substituting the asymptotic distributions of the critical order statistics $s_{(\eta m)}$ and $s_{(\eta m+1)}$ into the FOCs for optimal bidding/asking. The resulting equations are the *asymptotic first order conditions* (AFOCs). We then focus on two special cases, first the CPV case and then the CIV model in which G_ε and G_δ are each normal, wherein the AFOCs can be solved for trader bids/asks.

As noted above, in the CPV special case a seller's dominant strategy is to submit his cost as his ask. The issue in the CPV case is thus a buyer's effort to influence price in his favor in which he weighs the marginal expected gain from picking up a trade by raising his bid against the marginal expected cost of raising the price he pays by doing so. Solving the buyer's AFOC, the amount that a buyer adds to his value to determine his bid has

$$-\frac{1}{\eta(m+n)-1}\frac{1}{g_\varepsilon(\xi_q^\varepsilon)} \tag{1}$$

as its first-order approximation, where the minus sign indicates underbidding. Here, $g_\varepsilon(\xi_q^\varepsilon)$ is the density of G_ε evaluated at a particular quantile ξ_q^ε of this distribution that is determined by the number m of buyers and the number n of sellers; as discussed in section 2.1.2, it is closely tied to the value of the REE price. Formula (1) is interpreted in section 4.1 and its comparative statics predictions are discussed in section 4.2.1. The effectiveness of this formula as an approximation to equilibrium underbidding in small markets and the comparative statics predictions are tested with numerical examples in section 4.2.2. We note here some elementary observations: (i) (1) is remarkably simple in comparison to the first order condition for a buyer's optimal equilibrium choice of his bid, which is an integral differential equation that cannot be solved in closed form;[3] (ii) (1) exhibits the $O(1/\eta)$ rate of convergence to truthful bidding that has been proven for double auctions using lengthy combinatorial arguments; (iii) while equilibrium bidding depends on the distribution G_ε over the entire real line, (1) depends on its density at a single point ξ_q^ε that is tied to where the price is most likely to be; (iv) while the FOC is so complex in part because of the difference in bidding/asking behavior between buyers and sellers, (1) identifies the total number $(m+n)\eta - 1$ of traders that a buyer faces as critical, with the relative size m/n of demand and supply appearing only in the quantile ξ_q^ε that indicates where the price is most likely to be. Point (iv) reflects the fact that buyers and sellers increasingly bid/ask in the same way as the market size η increases, namely, as price-takers.

There are two FOCs in the CIV case. The seller's FOC consists of a *price-taking term* that determines his optimal ask given his inference from the price at which he sells. A buyer's FOC

[3] We know of only two closed-form solutions for equilibrium in k-double auctions, each of which is in the case in which buyer values and seller costs are independently drawn from the uniform distribution on $[0, 1]$: (i) the equilibrium of the $k = 1/2$ bilateral double auction of Chatterjee and Samuelson (1983, sec. 3); (ii) the equilibrium in the multilateral BBDA of Satterthwaite and Williams (1989, eq. 5.11) and Williams (1991, ex. A.1).

adds to his price-taking term a *strategic term* that captures his incentive to influence price in his favor. The AFOCs that are determined after substitution using Theorem 1 resist a general solution largely because of the complexity of the price-taking terms. In the case in which G_ε and G_δ are each normal with mean zero and respective variances υ_ε and υ_δ, however, the AFOCs can be solved for offsets. The amount that a buyer adds to his signal to determine his bid can be resolved into an amount that arises from his strategic term plus an amount that arises from his price-taking term. The amount that arises from his strategic term generalizes (1) from the CPV case and so it is interpreted similarly. The amount that a seller adds to his signal to determine his ask and the amount that arises from a buyer's price-taking term are given by a single formula in m, n, a value of a density at a quantile tied to the limiting price, and the variances υ_ε and υ_δ. This formula provides rare insight into a trader's inference from the market price by allowing us to investigate how a trader's effort to protect himself from a winner's curse changes as the measure of noise υ_δ in the signals varies in comparison to the measure υ_ε of the breadth of individual preferences in the model. In particular, for fixed m, n and market size η, we examine limiting bid/ask behavior as: (i) noise is eliminated from the model and a private value model is approached; (ii) individual preferences are eliminated from the model and a pure common value model is approached. The effectiveness of the approximation of equilibrium in small markets is demonstrated with computed examples in section ?? and our comparative statics predictions concerning how equilibrium bidding/asking varies with the amount of noise and the breadth of preferences in the model are investigated numerically in section ??.

The paper is organized as follows. Section 2 completes the model and summarizes results from SWZ that are needed in this paper, including results concerning REE. Section 3 addresses the asymptotic distribution of the BBDA's price and the order statistics among bids/asks that are critical in a trader's decision-making. We then turn in section 4 to analyzing a trader's decision problem by substitution into the FOCs for optimal bidding/asking, with the CPV special case addressed first followed by the CIV case in which the distributions are normal. Section 5 is our conclusion and all proofs are in the Appendix.

2 The Model

The informational structure of our model and the rules of the BBDA are discussed in the Introduction. We make the following assumption on the distributions:

A1: *The distributions G_ε and G_δ are continuously differentiable with finite first moments and positive densities g_ε and g_δ on $\mathbb{R}$ that are symmetric about* 0.

The following notation is useful in our discussion of convergence as the market size η increases. Let

$$q \equiv \frac{m}{m+n}, \tag{2}$$

the relative size of demand in the market. Define

$$\xi_q^{\varepsilon+\delta} \equiv G_{\varepsilon+\delta}^{-1}(q), \tag{3}$$

the q^{th} quantile of distribution $G_{\varepsilon+\delta}^{-1}$ of the sum of a trader's preference term ε and his noise term δ. In the CPV special case, we write $\xi_q^{\varepsilon} \equiv G_{\varepsilon}^{-1}(q)$. Define

$$V(\sigma) \equiv \mathbb{E}[z|\mu = 0, \sigma], \tag{4}$$

the expected value/cost z of a trader when $\mu = 0$ and $\sigma \in \mathbb{R}$ is his signal. We will in some instances add the following assumption on G_ε and G_δ:

A2: *$V(\sigma)$ is strictly increasing in $\mathbb{R}$.*

This is a strict version of first order stochastic dominance. It is satisfied by the normal, Laplace,and Cauchy distributions that we use as test cases for numerical work. A2 of course always holds in the CPV special case, wherein $V(\sigma) = \sigma$.

2.1 Summary of Results from SWZ

We summarize in this section some notation and results from SWZ that are needed for this paper. Bayesian-Nash equilibria in the results of SWZ all share the following property:

A3: Symmetric Offset Equilibrium. *In the market of size η, each buyer uses the same function $B^\eta : \mathbb{R} \to \mathbb{R}$ to select his bid as a function of his signal and each seller uses the same function $S^\eta : \mathbb{R} \to \mathbb{R}$ to select his ask. Moreover, each strategy B^η and S^η is an offset strategy in the sense that*

$$B^\eta(\sigma) = \sigma + \lambda_B^\eta, S^\eta(\sigma) = \sigma + \lambda_S^\eta$$

for all values of the signal $\sigma \in \mathbb{R}$, where the offsets λ_B^η and λ_S^η are constants.

It should be noted that deviations from equilibrium are not restricted in any sense by A3: it is a property of equilibrium, not a restriction on what is required for equilibrium. As noted in the Introduction, a trader's decision problem is translated linearly as his signal changes and he in this sense solves the same problem in selecting his bid/ask at every value of his signal. An offset strategy specifies the same solution to this problem at each value of the signal. It thus extends to strategic behavior the invariance property mentioned above.[4]

[4]The focus on offset strategies is also motivated by numerical work in SWZ (sec. 4.1.2) in the CPV special case, which suggests that the only symmetric equilibrium in increasing, C^1 strategies is an offset equilibrium, and it is uniquely determined.

2.1.1 First Order Conditions, Equilibrium and Price-Taking Behavior

We begin with the first order conditions (FOCs) that offsets λ_B^η, λ_S^η for buyers and sellers satisfy if they define a Bayesian-Nash equilibrium in the market of size η:

$$\sigma_B + \lambda_B^\eta = \underbrace{\mathbb{E}[v|\sigma_B, x^\eta = \sigma_B + \lambda_B^\eta]}_{\text{price-taking term}} - \underbrace{\frac{\Pr[x^\eta < \sigma_B + \lambda_B^\eta < y^\eta|\sigma_B]}{f_x^B(\sigma_B + \lambda_B^\eta|\sigma_B)}}_{\text{strategic term}} \tag{5}$$

$$\sigma_S + \lambda_S^\eta = \underbrace{\mathbb{E}\left[c|\sigma_S, x^\eta = \sigma_S + \lambda_S^\eta\right]}_{\text{price-taking term}}. \tag{6}$$

Equation (5) is the FOC from the perspective of a focal buyer for the selection of his offset λ_B^η and (6) is the FOC from the perspective of a focal seller who chooses λ_S^η, each calculated assuming that every nonfocal trader uses the offset λ_B^η, λ_S^η for his side of the market. Here and in the remainder of the paper, x^η and y^η denote respectively the ηm^{th} and $(\eta m+1)^{\text{st}}$ order statistics of other traders' bids and asks from the perspective of the focal trader that are critical in determining his outcome of trading.[5] The labeling of the terms in (5) and (6) reflect their roles in the FOCs: the strategic term of a buyer originates in his consideration of the possibility of moving the price in his favor, while his price-taking term would determine his optimal bid if he ignored this possibility; a seller's FOC has only a price-taking term, as he has no ability to influence price in his favor. Each price-taking term is simply the trader's value/cost in the CPV special case.

Reflecting invariance, offsets λ_B^η, λ_S^η solve (5) and (6) for all σ_B, σ_S if and only if they solve these equations in the special case of $\sigma_B = \sigma_S = 0$, which simplifies the solution of these equations. With restrictions on G_ε and G_δ in addition to A1, SWZ (Thm. 1) asserts the existence of offset solutions λ_B^η, λ_S^η to the FOCs (5) and (6) that satisfy $\lambda_B^\eta - \lambda_S^\eta < 0$ so long as the number m of buyers strictly exceeds 1. When such solutions define equilibrium,[6] we refer to the equilibrium values of the labeled terms (5) and (6) as the *equilibrium strategic* term of a buyer and the *equilibrium price-taking* terms of a buyer and of a seller, with "equilibrium" emphasizing its strategic nature.

2.1.2 Rational Expectations Equilibrium

The rational expectations (REE) price is the target for convergence of BBDA's equilibrium price as the market size η increases. As the REE price reveals information known individually by no trader but collectively by all traders in the limiting continuum market, this convergence demonstrates that the BBDA's equilibrium price can aggregate information dispersed across traders in a meaningful way in a finite market. Before addressing convergence below, we first review the results of SWZ

[5]We thus use the same notation for these order statistics even though a focal buyer and a focal seller face different samples of $\eta(m+n)-1$ bids/asks.

[6]Sufficiency of the first order approach is addressed in SWZ (Sec. 4.2): (i) it is straightforward to determine numerically whether or not a pair λ_B^η, λ_S^η of solutions to (5) and (6) determine an equilibrium, either by graphing expected utility as the bid/ask varies or marginal expected utility; (ii) it is difficult in the double auction framework to identify conditions directly on G_ε and G_δ that ensure sufficiency of the first order approach; (iii) there are examples of m, n, $\eta = 1$, G_ε and G_δ in which the solution to (5) and (6) fails to define an equilibrium, but sufficiency appears to hold in such cases once the market size η is sufficiency increased (SWZ (sec. 4.2.1)).

(sec. 5.1) concerning REE.

The *limit market* in state μ consists of m times a unit mass of buyers and n times a unit mass of sellers, with values/costs and signals generated using the distributions G_ε, G_δ from our finite model. The *REE function* $p^{\text{REE}} : \mathbb{R} \to \mathbb{R}$ determines the REE price $p^{\text{REE}}(\mu)$ in the limit market for state μ. It is defined by two properties. First, it is invertible and thus reveals the state. Let Λ denote the function that recovers the state μ from the REE price, $\Lambda\left(p^{\text{REE}}(\mu)\right) = \mu$. Second, $p^{\text{REE}}(\mu)$ clears the limit market in the state μ when each trader learns his signal σ, observes $p^{\text{REE}}(\mu)$, and calculates his expected value/cost $\mathbb{E}\left[z|\Lambda\left(p^{\text{REE}}(\mu)\right), \sigma\right]$. If he is a buyer, he buys one unit if and only if $\mathbb{E}\left[z|\Lambda\left(p^{\text{REE}}(\mu)\right), \sigma\right] \geq p^{\text{REE}}(\mu)$, and if he is a seller, he sells his unit if and only if $\mathbb{E}\left[z|\Lambda\left(p^{\text{REE}}(\mu)\right), \sigma\right] \leq p^{\text{REE}}(\mu)$.

Applying an argument of Reny and Perry (2006, sec. 3), SWZ (Thm. 3) states that if G_ε, G_δ satisfy A1 and A3, then the unique REE price in state μ is

$$p^{\text{REE}}(\mu) \equiv \mu + V\left(\xi_q^{\varepsilon+\delta}\right), \tag{7}$$

and the one-to-one mapping from the REE price to the state μ is therefore $\Lambda\left(p^{\text{REE}}\right) \equiv p^{\text{REE}} - V\left(\xi_q^{\varepsilon+\delta}\right)$. When the BBDA operates in the limit market, all traders use the equilibrium offset

$$\lambda^{\infty} \equiv V\left(\xi_q^{\varepsilon+\delta}\right) - \xi_q^{\varepsilon+\delta} \tag{8}$$

with the equilibrium price equaling $\mu + V\left(\xi_q^{\varepsilon+\delta}\right) = p^{\text{REE}}(\mu)$. The BBDA thus implements the REE price in the limit market. This result is easily specialized to the CPV case: (i) $V\left(\xi_q^{\varepsilon+\delta}\right)$ in (7) is replaced by ξ_q^{ε}, the q^{th} population quantile of G_ε, as $V(z) = z$ in this case; (ii) a trader's equilibrium strategy in the limit market is to report his value/cost.

2.1.3 Convergence

For fixed m, n, G_ε and G_δ, consider a sequence of symmetric offset equilibria $\left(\lambda_B^\eta, \lambda_S^\eta\right)_{\eta\in\mathbb{N}}$ in markets indexed by the market size η. SWZ (Thm. 4) proves the convergence of the buyer's equilibrium strategic term in this sequence to zero at the rate $O(1/\eta)$, reflecting a buyer's diminishing ability to influence price in his favor. Additionally, SWZ (sec. 5.3) conjectures that the sequences of equilibrium offsets $\left(\lambda_B^\eta\right)_{\eta\in\mathbb{N}}$ and $\left(\lambda_S^\eta\right)_{\eta\in\mathbb{N}}$ converge to the equilibrium offset λ^∞ in (8) of the limit market at the rate $O(1/\eta)$, which it supports with numerical examples.[7] Let $p_{\text{eq}}^\eta(\mu)$ denote the random variable of the BBDA's equilibrium price in the market of size η in state μ. Finally, for each value of the state μ, SWZ (Thm. 6) addresses the convergence of $p_{\text{eq}}^\eta(\mu)$ in the market of size η to the REE price $p^{\text{REE}}(\mu)$ for that state. It is shown that the expected absolute error

$$\mathbb{E}\left[\left|p_{\text{eq}}^\eta(\mu) - p^{\text{REE}}(\mu)\right| \middle| \mu\right]$$

[7]The reliance here upon a conjecture along with supportive numerical work reflects the difficulty of addressing a trader's inference from the market price in a finite market, i.e., the price-taking terms of traders.

of the BBDA's price as an estimate of the REE price is $\Theta\left(1/\sqrt{\eta}\right)$.

3 Asymptotic Distributions

As noted above, SWZ establishes convergence in expectation of the BBDA's equilibrium price to the REE price p^{REE}. We address in this section the properties of the BBDA's price as a estimator of the REE price and then demonstrate numerically its usefulness in this role even in very small markets. We also address the asymptotic distributions of the ηm^{th} and $(\eta m+1)^{\text{st}}$ order statistics in a sample of $\eta(m+n)-1$ bids/asks that are critical in a trader's decision problem. Our efforts to simplify and explain the solution to a trader's decision in section 4 starts with these distributions.

For fixed m and n, consider a sequence of markets indexed by the market size η. Let

$$(\boldsymbol{B}^{\eta},\boldsymbol{S}^{\eta})_{\eta\in\mathbb{N}} \equiv \left(B_i^{\eta},S_j^{\eta}\right)_{\eta\in\mathbb{N},1\le i\le \eta m,1\le j\le \eta n}$$

denote a sequence of strategy profiles in a sequence of markets. We use the following assumption on the sequence of strategies:

A4: The sequence of strategies $(\boldsymbol{B}^{\eta},\boldsymbol{S}^{\eta})_{\eta\in\mathbb{N}}$ has the following two properties:

1. each strategy B_i^{η},S_j^{η} in the sequence is an increasing, C^1 function;
2. there exists a constant $K(G_\varepsilon,G_\delta,m,n)$ such that

$$\left|\left(B_i^{\eta}(\sigma)-\sigma\right)-\lambda^{\infty}\right|, \left|\left(S_j^{\eta}(\sigma)-\sigma\right)-\lambda^{\infty}\right| < \frac{K(G_\varepsilon,G_\delta,m,n)}{\eta} \tag{9}$$

   for all $\eta\in\mathbb{N}$, $1\le i\le \eta m$, $1\le j\le \eta n$, and $\sigma\in\mathbb{R}$, where λ^{∞} is the equilibrium offset of the limit market defined in (8).

Property 1. ensures that densities for the critical order statistics in a trader's decision problem exist and are continuous. As discussed in section 2.1.3, property 2. is motivated by the analysis in SWZ, which offers theoretical and numerical evidence that supports A4 holding for symmetric offset equilibria. The defining inequality (9) of A4 is written to highlight the distinction between (i) the difference between a trader's bid/ask and his signal and (ii) the limiting value λ^{∞} of this difference. This assumption, however, does not require that the difference between a trader's bid/ask and his signal be constant (as in an offset strategy), that it be the same across all buyers and across all sellers (as required in a symmetric equilibrium), or even the restriction of equilibrium. We move away here from these restrictions because they are needed for the following characterization of the BBDA's price as an estimator of the REE price and the critical order statistics for a trader's choice of his optimal bid/ask: it is (9) that is key, not constant offsets, symmetry or equilibrium, which implies a robustness in both the BBDA's ability to identify the REE price and a trader's decision problem.

Theorem 1 *For fixed m and n, consider a sequence of markets indexed by the market size η. Let $(\boldsymbol{B}^{\eta}, \boldsymbol{S}^{\eta})_{\eta\in\mathbb{N}}$ be a sequence of strategy profiles that satisfies A4.*

1. *Asymptotic Distribution of Price. Let $p^{\eta}_{eq}(\mu)$ denote the sequence of the random variables of the BBDA's price that is determined by this strategy sequence in the state μ. For each state $\mu \in \mathbb{R}$ and for its corresponding REE price $p^{\mathrm{REE}}(\mu) = \mu + V(\xi_q^{\epsilon+\delta})$, we have*

$$p^{\eta}_{eq}(\mu) \sim \mathcal{AN}\left(p^{\mathrm{REE}}(\mu), \frac{mn}{\eta(m+n)^3}\frac{1}{g^2_{\varepsilon+\delta}(\xi_q^{\epsilon+\delta})}\right). \tag{10}$$

2. *Asymptotic Distribution of Critical Order Statistics. From the perspective of either a focal buyer or a focal seller faced with the strategies of others in the profile $(\boldsymbol{B}^{\eta}, \boldsymbol{S}^{\eta})_{\eta\in\mathbb{N}}$, the random variables $x^{\eta}(\mu)$ and $y^{\eta}(\mu)$ of the ηm^{th} and $(\eta m+1)^{st}$ order statistics of other traders' bids and asks given the state μ satisfy*

$$x^{\eta}(\mu), y^{\eta}(\mu) \sim \mathcal{AN}\left(p^{\mathrm{REE}}(\mu), \frac{mn}{(\eta(m+n)-1)(m+n)^2}\frac{1}{g^2_{\epsilon+\delta}(\xi_q^{\epsilon+\delta})}\right). \tag{11}$$

3. *Statements (10) and (11) holds in the CPV special case by replacing $g_{\epsilon+\delta}(\xi_q^{\epsilon+\delta})$ with $g_{\epsilon}(\xi_q^{\epsilon})$ and $V(\xi_q^{\epsilon+\delta})$ with $V(\xi_q^{\epsilon})$.*

4. *As a consequence of (10) and (11), $p^{\eta}(\mu)$, $x^{\eta}(\mu)$, $y^{\eta}(\mu)$ are consistent, asymptotically unbiased and normal estimators of the REE price in each state μ.*

The difference between the asymptotic distribution of the equilibrium price $p^{\eta}_{\mathrm{eq}}(\mu)$ in (10) and the asymptotic distribution of the order statistics $x^{\eta}(\mu)$ and $y^{\eta}(\mu)$ in (11) lies in the variance, with the factor $1/(m+n)\cdot\eta$ in (10) replaced with $1/(m+n-1)\cdot\eta$ in (11). This reflects the fact that $p^{\eta}_{\mathrm{eq}}(\mu)$ is determined by a sample of $(m+n)\cdot\eta$ bids/asks while $x^{\eta}(\mu)$ and $y^{\eta}(\mu)$ are determined by samples of $\eta(m+n)-1$ bids/asks. The proofs of (10) and (11) generalize standard results (Serfling (1980, sec. 2.3.3)) on the asymptotic distribution of a sample quantile to a case where the sample is not identically distributed, which reflects the asymmetry here between buyer and seller behavior along with the fact that A4 allows asymmetry among the strategies on each side of the market.

It is straightfoward to show that the characterization (10) of the asymptotic distribution of the BBDA's price also holds for other formulas for selecting a price from the interval $[s_{(\eta m)}, s_{(\eta m+1)}]$ of market-clearing prices for the market of size η, and not just the BBDA's rule of selecting $s_{(\eta m+1)}$ as the price. In particular, it holds for every k-double auction (where, for $k \in [0,1]$, $p = ks_{(\eta m+1)} + (1-k)s_{(\eta m)}$) along with randomized rules for price selection in this interval. The key is assumption (9). We have chosen to state this theorem solely for the BBDA because this procedure is the subject of the remainder of this paper: the rule for selecting price affects the choice of strategies, and it is in the special case of the BBDA that we have made progress in studying strategic behavior.

η	Buyer's Offset λ_B^η	Seller's Offset λ_S^η
2	-1.3404	0.4124
4	-0.7036	0.2172
8	-0.3417	0.1091
16	-0.1684	0.0549

Table 1: For $G_\varepsilon,\ G_\delta$ standard normal, $m = n = 1$, and market sizes $\eta = 2, 4, 8, 16$, the equilibrium offsets of buyers and of sellers are given in columns 2 and 3.

3.0.1 Numerical Example: The Relevance of Theorem 1 to Small Markets

We evaluate in this section the effectiveness of the asymptotic distribution (10) of the BBDA's equilibrium price as an approximation. We begin with an example in which it is meaningful as an approximation even in very small markets. Consider $m = n = 1$, G_ε, G_δ standard normal, and market sizes $\eta = 2, 4, 8, 16$.[8] Table 1 lists the offset equilibria computed in this case for each size η of market. SWZ (sec. 5.3.1, fn. 22) shows that in the case of $m = n$ the equilibrium offset λ^∞ of the limit market equals 0. Because $(B_i^\eta(\sigma) - \sigma) - \lambda^\infty$ and $\left(S_j^\eta(\sigma) - \sigma\right) - \lambda^\infty$ therefore reduce here to the equilibrium offsets λ_B^η and λ_S^η, respectively, assumption A4 is shown to hold here by the halving of the offset values in this table as the market size η doubles.

Figure 1 and Table 2 then address the asymptotic distribution (10). Monte Carlo simulations are used to calculate the sample density function of the BBDA's equilibrium price $p_{\mathrm{eq}}^\eta(\mu)$ for $\eta = 2, 4, 8, 16$ based upon the offset equilibrium in Table 1. Because $m = n$, we have $q = 1/2$ and the asymptotic median is

$$p^{\mathrm{REE}}(\mu) = \mu + V\left(\xi_q^{\varepsilon+\delta}\right) = \mu.$$

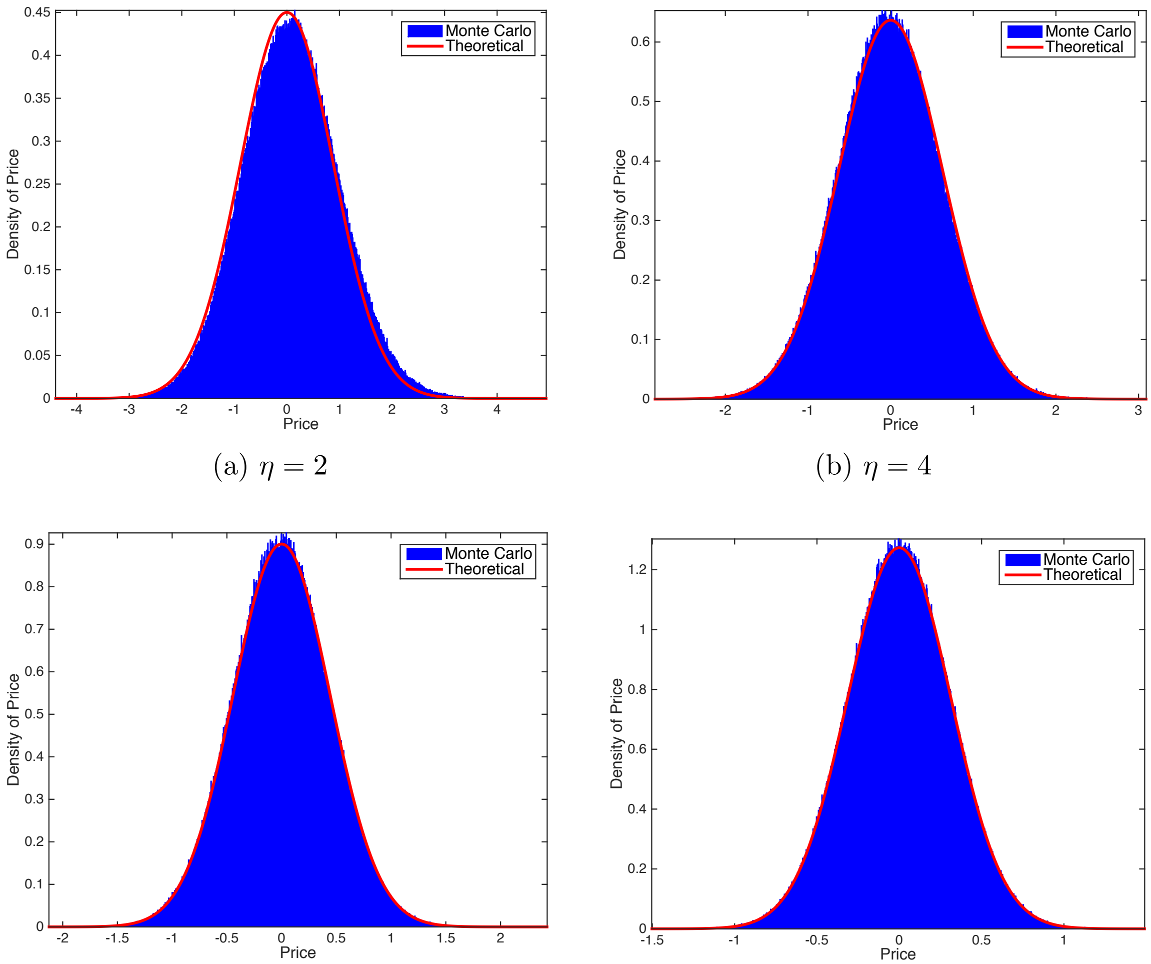
Here, we have substituted

$$V(\xi_q^{\varepsilon+\delta}) = V(0) \equiv \mathbb{E}[z|\mu = 0, \sigma = 0] = 0$$

using formula (4) for $V(\cdot)$. The invariance property of our model together with the use of offset strategies in equilibrium allows us to set $\mu = 0$ for our simulations without loss of generality. Figure 1 plots the sample densities against their asymptotic limit for the different market sizes η, from which it is clear that asymptotic normality meaningfully represents the sample density even in these very small markets.

Further evidence is provided in Table 2, which reports the mean, variance, skewness and excess kurtosis of the sample density of $p_{\mathrm{eq}}^\eta(\mu = 0)$ along with the asymptotic variance as given in Theorem 1. Because the asymptotic distribution is normal, the distance between the sample distribution and the asymptotic distribution is measured by how fast the magnitudes of skewness and excess kurtosis converge to zero. We see that: (i) the sample mean converges at an exponential rate to its limit of

[8] Results of similar calculations in the cases of $G_\varepsilon = G_\delta$ standard Laplace and standard Cauchy can be found in the Online Appendix to this paper.



(a) $\eta = 2$ (b) $\eta = 4$

(c) $\eta = 8$ (d) $\eta = 16$

Figure 1: The figure depicts the sample density of the BBDA's equilibrium price $p^{\eta}_{\text{eq}}(\mu)$ calculated using Monte Carlo simulations in the case of $G_\varepsilon,\ G_\delta$ standard normal, $m = n = 1$, and market sizes $\eta = 2, 4, 8, 16$, graphed against its asymptotic density as given in Theorem 1.

zero; (ii) the sample variance is very close to its asymptotic value and the difference diminishes at a rate that is at least $O(1/\eta)$;[9] (iii) skewness and kurtosis approach zero at rates that are at least as fast as $O(1/\eta)$, which indicates how close the sample densities are to normal.

When Does the Approximation Fail? We next ask, "When is the the normal distribution in (10) a *poor* approximation for the sample distribution of the equilibrium price $p^{\eta}_{\text{eq}}(\mu)$?" There are two cases in which we know that it may fail. The first case is a well-known problem in statistical sampling. For $q = m/(m+n)$, (10) asserts the asymptotic normality of an over-approximation $p^{\eta}_{\text{eq}}(\mu) = s_{(\eta m+1)}$ to the q^{th} quantile among $\eta\,(m+n)$ bids/asks. Consider the case of $q = 1/2$ addressed above in which the quantile approximated is the sample median among bids/asks. For

[9]The rates in (i) and (ii) are in accord with standard results in statistics, namely Serfling (1980, Thm. 2.3.2) and Serfling (1980, Thm. 2.3.3-A).

η	Mean	Variance	Asymptotic Variance	Skewness	Excess Kurtosis
2	0.0847	0.8412	0.7854	0.0996	0.0454
4	-0.0139	0.3906	0.3927	0.0319	0.0400
8	-0.0058	0.1936	0.1963	0.0112	0.0114
16	-0.0013	0.0973	0.0982	0.0008	0.0096

Table 2: For G_δ standard normal and G_ε $m = n = 1$, and market sizes $\eta = 2, 4, 8, 16$, the sample mean, variance, (theoretical) asymptotic variance, skewness and excess kurtosis of the equilibrium price $p^\eta_{\text{eq}}(\mu = 0)$ are presented along with the asymptotic variance.

sufficiently small samples, multimodality in the distribution being sampled can cause the distribution of the sample median to be multimodal, and hence "un-normal" in appearance. SWZ (sec. 4.2.1) contains an example of distributions G_ε and G_δ in which the offset solutions to the FOCs determine a bimodal distribution of bids/asks. This problem, however, quickly disappears as the sample size (indexed here by the market size η) increases and should therefore be a problem only in very small markets.[10]

Approaching the Common Value Special Case. The second case is motivated by a well-known problem in the theory of financial markets, namely, nonexistence of equilibrium in which trade occurs in a common value model. This case is particularly important because it explains the increasing inaccuracy of our approximation of trader offsets in sections 4.1 and 4.3 as the common value case in approached. The special case of $G_\varepsilon \sim \mathcal{N}(0, \upsilon_\varepsilon)$ and $G_\delta \sim \mathcal{N}(0, \upsilon_\delta)$ in studied in these sections; we consider it here in the case of $m = 2$, $n = 1$ and $\eta = 2$. In the limiting case when the variance υ_ε of the preference term equals zero, each trader i's value/cost is the state μ and he observes the noisy signal $\sigma_i = \mu + \delta_i$. Fixing the variance of the noise term equal to $\upsilon_\delta = 1$ (i.e., G_δ standard normal), Table 3 compiles the equilibrium offsets for a sequence of values of υ_ε as it decreases to 0, with υ_ε in the first column, the offset λ^η_B of a buyer in column 2, the offeset λ^η_S of a seller in column 3, and the difference $\lambda^\eta_B - \lambda^\eta_S$ of their offsets in column 4. Consistent with intuition, it appears that the buyer's offset converges to $-\infty$ and the seller's offset to ∞ as υ_ε decreases to zero, with the equilibrium thus converging to a no-trade equilibrium at the limiting common value model. Figure 2 illustrates this convergence by graphing the difference $\lambda^\eta_B - \lambda^\eta_S$ of the offsets as a function of υ_ε, which appears to approach $-\infty$ along a hyperbola as $\upsilon_\varepsilon \to 0$.

The implication of this for the accuracy of the asymptotic distribution (10) of price as an approximation for the sample distribution is as follows. Assumption A4 in Theorem 1 posits a constant $K(G_\varepsilon, G_\delta, m, n)$ so that the bound (9) holds. If for fixed market size η the values

$$\left|(B^\eta_i(\sigma) - \sigma) - \lambda^\infty\right| = \left|\lambda^\eta_B - \lambda^\infty\right|, \left|(S^\eta_i(\sigma) - \sigma) - \lambda^\infty\right| = \left|\lambda^\eta_S - \lambda^\infty\right| \tag{12}$$

[10] The more notable problem identified in SWZ (sec. 4.2.1), however, is that multimodality in the distribution of bids/asks can lead to insufficiency of the first order approach, i.e., a symmetric solution to the FOCs may fail to define an equilibrium. This problem, however, also appears to quickly resolve itself as the market size increases.

υ_ε	Buyer's Offset λ_B^η	Seller's Offset λ_S^η	$\lambda_B^\eta - \lambda_S^\eta$
1	-1.2189	0.1332	-1.3521
0.5	-1.3379	0.3485	-1.6864
0.25	-1.6323	0.7760	-2.4083
0.125	-2.1556	1.5226	-3.6782
0.0625	-2.9979	2.6090	-5.6069
0.0312	-4.1762	3.9268	-8.1030
0.0156	-5.7898	5.6143	-11.4041
0.0078	-8.0939	7.9694	-16.0633
0.0039	-11.3794	11.2912	-22.6706

Table 3: For G_δ standard normal and$G_\varepsilon \sim \mathcal{N}(0, \upsilon_\varepsilon)$, $m = 2$, $n = 1$, and market size $\eta = 2$, the equilibrium offsets of buyers and of sellers are calculated as the variance υ_ε of the preference term in the model grows small and the common value model is approached.

are large in magnitude, then a large constant $K(G_\varepsilon, G_\delta, m, n)$ is required so that (9) holds, which means that a large market size η is needed to make these expressions small and the approximation by the asymptotic distributon accurate. Similarly, holding the market size η constant and allowing υ_ε to go to zero, the terms in (12) grow large and the approximation becomes less and less meaningful. Figures 3 and 4 illustrate this problem. Figure 3 depicts the sample density of the BBDA's price together with its asymptotic limit in four of the cases in Table 3. The asymptotic distribution clearly grows worse and worse as an approximation as υ_ε decreases. Figure 4 illustrates that the issue is that the market size must increase as υ_ε becomes small in order for the approximation to be meaningful. It depicts in the case of $\upsilon_\varepsilon = 0.0625$ and $\eta = 2$, 4, 8, and 16 the sample density in comparison to its asymptotic limit. In contrast to the standard normal case in Figure 1, we see that the approximation does not become accurate until η increases to 16.

Finally, we note that the equilibrium of the BBDA for the limiting continuum market of section 2.1.2 exists in the common value case and there is no discontinuity as the common value case is approached within the limiting continuum case (i.e., when υ_ε is decreased to zero *after* the limit $\eta \to \infty$ is applied first). In the common value case, we have

$$V(\sigma) \equiv \mathbb{E}[z|\mu = 0, \sigma] = 0$$

because $z = \mu$ in the state μ. From (7), we then have

$$p^{\text{REE}} \equiv \mu + V\left(\xi_q^{\varepsilon+\delta}\right) = \mu.$$

Applying (8), each trader's offset is

$$\lambda^\infty \equiv V\left(\xi_q^{\varepsilon+\delta}\right) - \xi_q^{\varepsilon+\delta} = -\xi_q^{\varepsilon+\delta}.$$

Each trader therefore bids/asks $\sigma - \xi_q^{\varepsilon+\delta}$. The BBDA's price is the qth quantile of the distribution

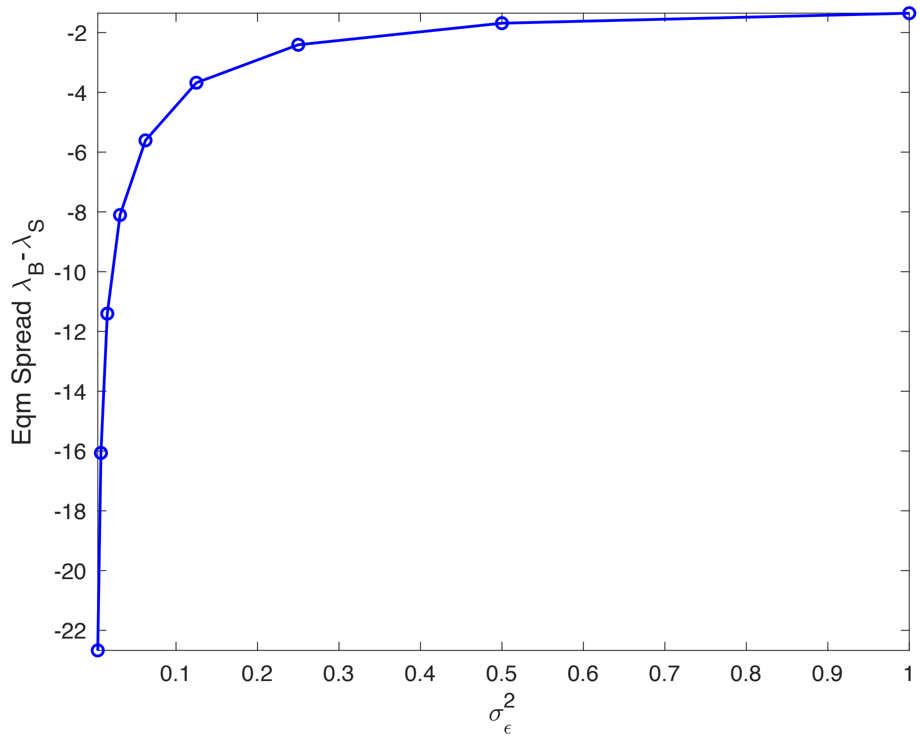


Figure 2: Given $M = 2$, $N = 1$, $\eta = 2$, $G_\varepsilon \sim \mathcal{N}(0, \upsilon_\varepsilon)$ and $G_\delta \sim \mathcal{N}(0, \upsilon_\delta = 1)$, the difference $\lambda_B^\eta - \lambda_S^\eta$ of the equilibrium offsets is graphed as a function of the variance υ_ε of the preference term.

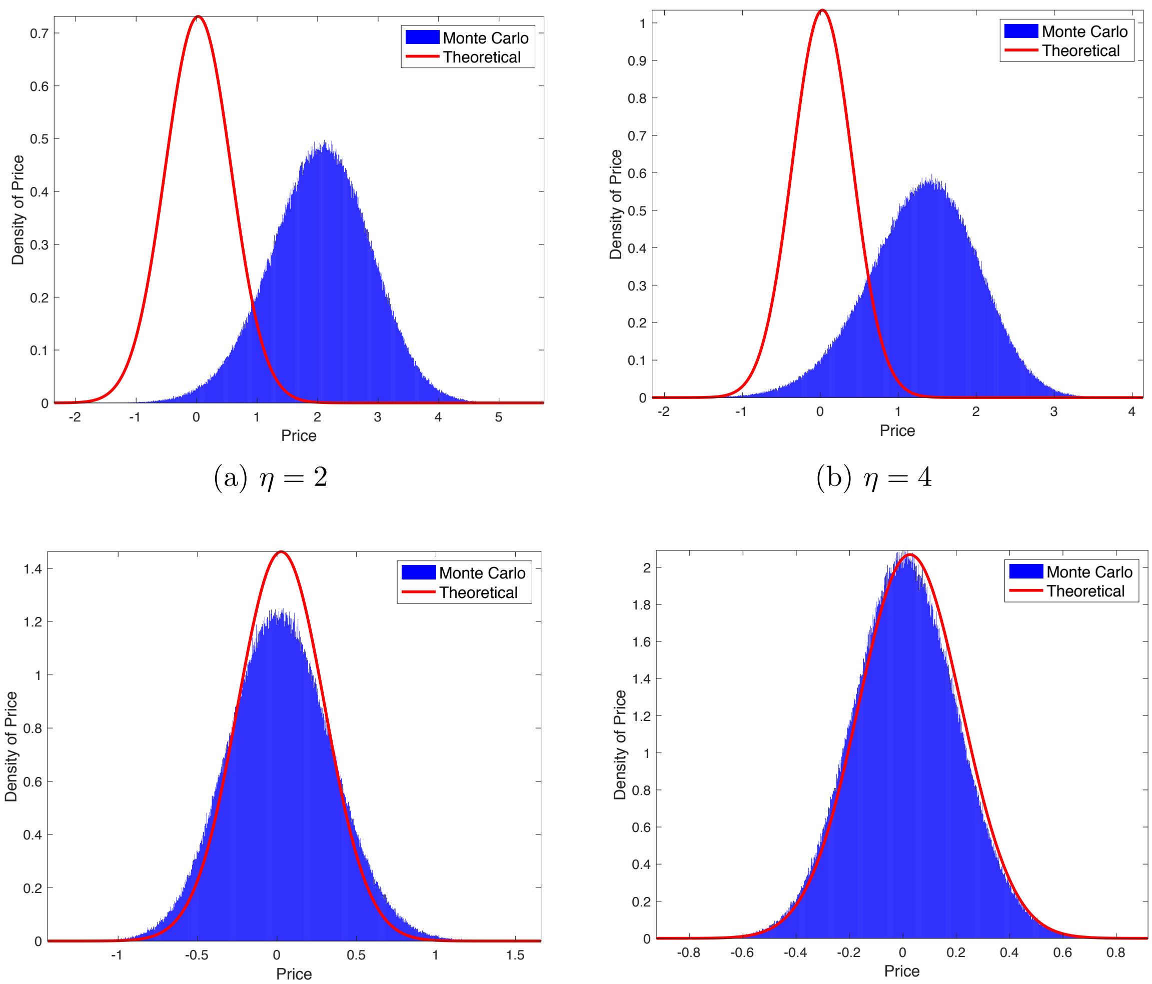
of bids/asks, i.e., $\mu = p^{\mathrm{REE}}$, and trade occurs between buyers whose signals are above $\mu + \xi_q^{\varepsilon+\delta}$ and sellers whose signals are below it.[11]

4 An Asymptotic Analysis of a Trader's Decision Problem

Theorem 1 characterizes the asymptotic distributions of the order statistics $x^\eta(\mu)$ and $y^\eta(\mu)$ among bids/asks given the state μ that are critical in a trader's decision problem in the BBDA. We now substitute these asymptotic distributions into the FOCs for optimal bidding/asking to determine the *asymptotic FOCs* (AFOCs). In the special instances of (i) the CPV case and (ii) the CIV case in which G_ε and G_δ are both normal, the AFOCs can be solved to determine the *asymptotic offsets*. Moreover, in the particular case (ii) that we are able to analyze in the CIV model, we separate the buyer's asymptotic offset into asymptotic *price-taking* and *strategic* terms, with the seller's asymptotic offset consisting solely of a price-taking term. While we do not establish a formal sense in which these asymptotic expressions approximate their equilibrium counterparts, numerical examples demonstrate their value as approximations even in small markets. Most importantly,

[11] While this seems to contradict the theorem of Milgrom and Stokey (1982, Theorem 1) on "no information-based trading", that result rests not on inference from the act of trading, but instead upon complete contracts ex ante, which are not a feature of our model.

Milgrom, P., and N. Stokey, "Information, Trade and Common Knowledge," *Journal of Economic Theory* vol. 26, no. 1, February 1982, pp. 17-27.



(a) $\eta = 2$ (b) $\eta = 4$

(c) $\eta = 8$ (d) $\eta = 16$

Figure 3: The figure depicts the sample density of the BBDA's equilibrium price $P^{\eta}_{\text{eq}}(\mu)$ calculated using Monte Carlo simulations in the case of G_δ standard normal and $\varepsilon \sim \mathcal{N}(0, 0.0625)$, $m = 2$, $n = 1$, and market sizes $\eta = 2, 4, 8, 16$, graphed against its asymptotic density as given in Theorem 1.

the asymptotic expressions are interpretable and provide insight into both strategic behavior and the very complicated inference from the market price that a trader addresses in the CIV model. These asymptotic expressions suggest comparative statics predictions concerning equilibrium bidding/asking behavior that are then investigated numerically.

4.1 Summary of Results

We begin with the special CPV case in which sellers submit their costs as their asks. In this case, we will show that the buyer's AFOC has the unique solution

$$\tilde{\lambda}^{\eta}_{B} \equiv v - b = -\frac{1}{\eta(m+n)-1}\frac{1}{g_{\varepsilon}(\xi^{\varepsilon}_{q})} + O\left(\frac{1}{\eta^2}\right), \tag{13}$$

which is approximated from a first-order perspective by the constant

$$\lambda^{\eta}_{B,\text{approx}} = -\frac{1}{\eta(m+n)-1}\frac{1}{g_{\varepsilon}(\xi^{\varepsilon}_{q})}. \tag{14}$$

Because it applies the asymptotic distributions of order statistics in Theorem 1, the only assumption concerning the behavior of the non-focal traders is that their strategies satisfy A4. In this sense, the buyer's optimal bidding is uniquely determined from a first order perspective against a range of behavior by the other traders that includes asymmetric and/or non-equilibrium strategies.

The uniqueness of the solution (13) is consistent with the support provided in SWZ (sec. 4.1.2) for the conjecture that a symmetric equilibrium in the BBDA is uniquely determined in the CPV special case, when it exists. We also note that (13) and its approximation (14) are $O(1/\eta)$, which is consistent with the rate derived in SWZ (Cor. 2) for the convergence of a buyer's strategic term to zero. Finally, we note that (14) identifies what is first-order in a buyer's decision problem in this case, namely, (i) the total number $(m+n)\eta - 1$ of traders that he faces and (ii) the value of the density $g_{\varepsilon}(\xi^{\varepsilon}_{q})$ at the single point

$$\xi^{\varepsilon}_{q} = G^{-1}_{\varepsilon}\left(\frac{m}{m+n}\right),$$

which is the quantile of interest in the CPV case. Part (i) reflects the fact that buyers and sellers increasingly bid/ask in the same way as the market size η increases; the distinction between how buyers bid and sellers ask becomes increasingly irrelevant as η increases, and all that matters asymptotically is how many traders the focal buyer faces and not their roles. Part (ii) reflects the fact that the focus of the buyer's decision problem in every state μ is near its limiting equilibrium price, namely, $p^{REE}(\mu) = \mu + \xi^{\varepsilon}_{q}$, and the uncertainty that he faces is thus summarized from a first order perspective with the single term $g_{\varepsilon}(\xi^{\varepsilon}_{q})$.

The CIV model is richer because it adds to a buyer's effort to influence price the inference problem that each trader faces concerning what he learns from the market price about his value/cost. While the AFOCs can be derived in the CIV model simply by substituting the asymptotic distributions of the order statistics in Theorem 1 into the FOCs for optimal bidding/asking, we impose normality of G_{ε} and G_{δ} in order to complete the second step of then solving explicitly for the asymptotic offsets. Let $G_{\varepsilon} \sim \mathcal{N}(0, \upsilon_{\varepsilon})$ and $G_{\delta} \sim \mathcal{N}(0, \upsilon_{\delta})$, with the notation υ_{ε} and υ_{δ} for the respective variances important in the discussion that follows. The buyer and seller asymptotic offsets are respectively

$$\tilde{\lambda}^{\eta}_{B} = -\left(\frac{\upsilon_{\delta}}{\widetilde{\upsilon}_{x|\mu}+\upsilon_{\varepsilon}}\right) V\left(\xi^{\varepsilon+\delta}_{q}\right) - \left(\frac{\widetilde{\upsilon}_{x|\mu}+\upsilon_{\varepsilon}+\upsilon_{\delta}}{\widetilde{\upsilon}_{x|\mu}+\upsilon_{\varepsilon}}\right)\frac{1}{\left(\eta(m+n)-1\right) g_{\varepsilon+\delta}\left(\xi^{\varepsilon+\delta}_{q}\right)} + O\left(\frac{1}{\eta^{2}}\right) \tag{15}$$

and

$$\tilde{\lambda}^{\eta}_{S} = -\left(\frac{\upsilon_{\delta}}{\widetilde{\upsilon}_{x|\mu}+\upsilon_{\varepsilon}}\right) V\left(\xi^{\varepsilon+\delta}_{q}\right), \tag{16}$$

where

$$\widetilde{v}_{x|\mu} = \frac{mn}{(m+n)^2} \frac{1}{\eta(m+n)-1} \frac{1}{g^2_{\varepsilon+\delta}\left(\xi^{\varepsilon+\delta}_q\right)}$$

is the asymptotic density of the order statistic $x^\eta(\mu)$ as stated in (11).

We focus upon the approximate offsets

$$\lambda^\eta_{B,\text{approx}} = -\left(\frac{\upsilon_\varepsilon \upsilon_\delta}{\left(\widetilde{v}_{x|\mu}+\upsilon_\varepsilon\right)\left(\upsilon_\varepsilon+\upsilon_\delta\right)}\xi^{\varepsilon+\delta}_q + \frac{\widetilde{v}_{x|\mu}+\upsilon_\varepsilon+\upsilon_\delta}{\widetilde{v}_{x|\mu}+\upsilon_\varepsilon}\frac{1}{\left((m+n)\eta-1\right)g_{\varepsilon+\delta}\left(\xi^{\varepsilon+\delta}_q\right)}\right) \tag{17}$$

and

$$\tilde{\lambda}^\eta_S = -\frac{\upsilon_\varepsilon \upsilon_\delta}{\left(\widetilde{v}_{x|\mu}+\upsilon_\varepsilon\right)\left(\upsilon_\varepsilon+\upsilon_\delta\right)}\xi^{\varepsilon+\delta}_q \tag{18}$$

where we use normality to substitute[12]

$$V\left(\xi^{\varepsilon+\delta}_q\right) = \frac{\upsilon_\varepsilon}{\upsilon_\varepsilon+\upsilon_\delta}\xi^{\varepsilon+\delta}_q.$$

The term $\tilde{\lambda}^\eta_S$ in (18) represents the seller's effort to protect himself from a winner's curse with his ask. It is identical as a function to the asymptotic buyer's price-taking term in (17). The buyer's approximate offset $\lambda_{B,\text{approx}}$ sums this price-taking term with the asymptotic strategic term

$$-\left(\frac{\widetilde{v}_{x|\mu}+\upsilon_\varepsilon+\upsilon_\delta}{\widetilde{v}_{x|\mu}+\upsilon_\varepsilon}\right)\frac{1}{\left((m+n)\eta-1\right)g_{\varepsilon+\delta}\left(\xi^{\varepsilon+\delta}_q\right)}, \tag{19}$$

which generalizes (14) to the CIV case by replacing $g_\varepsilon\left(\xi^\varepsilon_q\right)$ with $g_{\varepsilon+\delta}\left(\xi^{\varepsilon+\delta}_q\right)$ and by assigning a weight to this expression.

The interpretation of the buyer's strategic term (19) is the same here as in the CPV case above. We thus turn to the dependence of the price-taking terms and the offsets in total on the variances υ_ε and υ_δ of the preference and noise distributions. In the normal case considered here, the values of the quantile $\xi^{\varepsilon+\delta}_q$ and the density $g_{\varepsilon+\delta}\left(\xi^{\varepsilon+\delta}_q\right)$ are determined by the sum $\upsilon_\varepsilon+\upsilon_\delta$ of these variances; consequently, the value of $\widetilde{v}_{x|\mu}$ in the above formulas remains fixed as the variances change so long as their sum remains fixed. Keeping $\upsilon_\varepsilon+\upsilon_\delta = K$ for some constant $K>0$, we thus explore how the approximate offsets change as υ_δ varies across $[0, K]$, where υ_δ measures how much noise there is in a trader's signal concerning his value/cost. Note that: (i) as $\upsilon_\delta \to 0$ and $\upsilon_\varepsilon \to K$, the CIV model converges to a CPV special case as noise is eliminated from the signals; (ii) as $\upsilon_\delta \to K$ and $\upsilon_\varepsilon \to 0$, the CIV model converges to a common value case in which all values/costs are the same and equal to the state μ.

Consider first convergence to the CPV special case. As $\upsilon_\delta \to 0$, and $\upsilon_\varepsilon \to K$, the seller's approximate offset $\lambda^\eta_{S,\text{approx}}$ converges to zero: as noise is eliminated from the signals, a seller's signal converges to his cost, which he submits as his ask in the CPV special case. The buyer's approximate

[12] There are no second order terms in $\tilde{\lambda}^\eta_S$ to drop, as they originate in $\tilde{\lambda}^\eta_B$ solely from the buyer's ability to influence the price that he pays in his favor.

price-taking term similarly goes to zero, while the weight in parentheses in (19) converges to one. His approximate offset $\lambda^{\eta}_{B,\text{approx}}$ thus converges to its value (14) in the CPV case.

Convergence to the common value model is more interesting because it is discontinous in the limit. As $\upsilon_\delta \to K$, $\upsilon_\varepsilon \to 0$, and the CIV model converges to a common value model, the seller's approximate offset $\tilde{\lambda}^{\eta}_{S}$ converges to 0 while the buyer's approximate offset $\lambda^{\eta}_{B,\text{approx}}$ converges to the finite value

$$-\left(\frac{\widetilde{\upsilon}_{x|\mu}+K}{\widetilde{\upsilon}_{x|\mu}}\right)\frac{1}{\left((m+n)\eta-1\right)g_{\varepsilon+\delta}\left(\xi_q^{\varepsilon+\delta}\right)}.$$

This disagrees with the intuition that the offset of buyers should go to $-\infty$ while the sellers offset goes to $+\infty$ as the common value is approached, thus implementing in the limit a no-trade outcome, which is supported by the equilibria that are reported in Table 3 and in Figure 2. Note in particular that the approximate price-taking terms of both buyers and sellers converge to zero as the common value model is approached, which is especially unintuitive because it suggests a diminished effort by traders to protect themselves from a winner's curse as the necessity of doing so grows more acute.

Why do the approximations $\lambda^{\eta}_{B,\text{approx}}$ and $\tilde{\lambda}^{\eta}_{S}$ perform so poorly as the common value model is approached? The approximations rest upon the accuracy of the asymptotic distribution (11) in Theorem 1 as an approximation of the equilibrium distributions of the order statistics $x^{\eta}(\mu)$ and $y^{\eta}(\mu)$. As suggested by the results presented in Figure 3 for the BBDA's equilibrium price $p^{\eta}(\mu)$, the accuracy of this approximation diminishes as the common value model is approached with the market size η held constant. The exercise we carry out here is for fixed market size η; because we fail to increase η as $\upsilon_\varepsilon \to 0$ as needed to maintain the accuracy of the approximation, the asymptotic distribution (11) becomes worse and worse as an approximation as the common value model is approached, which explains why the approximations $\lambda^{\eta}_{B,\text{approx}}$ and $\tilde{\lambda}^{\eta}_{S}$ similarly become less and less meaningful.

4.2 Asymptotic Analysis in the CPV Case

We begin with a formal statement of our approximation result.

Theorem 2 (Approximate Solution) *For fixed m and n, consider a sequence of markets indexed by the market size η. Let $(\boldsymbol{B}^{\eta},\boldsymbol{S}^{\eta})_{\eta\in\mathbb{N}}$ be a sequence of strategy profiles that satisfies A4. Faced with the strategies in this sequence, suppose a focal buyer considers strategies $\widetilde{B}(\cdot;\eta)$ in which his underbidding is $O\left(1/\eta^{\epsilon}\right)$ for some $\epsilon>0$. Then the unique strategy that solves the focal buyer's AFOC is the offset $\tilde{\lambda}(\eta)$ given in (13).*

The odd but rather weak assumption that the focal buyer considers strategies whose underbidding is $O\left(1/\eta^{\epsilon}\right)$ for some $\epsilon>0$ is used in the proof to start the derivation of (13) by allowing a Taylor polynomial expansion of $\widetilde{B}(\cdot;\eta)$.

4.2.1 Comparative Statics

Formula (14) facilitates the comparative statics exercise of exploring the dependence of equilibrium upon the distribution G_ε, which is new to the double auctions literature. The approximate offset $\lambda^\eta_{B,\text{approx}}$ depends upon g_ε only through its value at the quantile ξ^ε_q. As stated in Theorem 1, this reflects the fact that in each state μ, the critical order statistics $x^\eta(\mu)$, $y^\eta(\mu)$ are with high probability near $p^{REE}(\mu) = \mu + \xi^\varepsilon_q$ for large market size η. Moreover, $\lambda^\eta_{B,\text{approx}}$ decreases as $g_\varepsilon(\xi^\varepsilon_q)$ increases, which suggests that equilibrium buyer underbidding should also vary with $g_\varepsilon(\xi^\varepsilon_q)$ in this way: if bids/asks are less concentrated at $\mu + \xi^\varepsilon_q$, then the focal buyer is more likely to be able to influence price in his favor without losing a profitable trade and consequently underbids by more. This prediction is not at all apparent from inspection of the buyer's FOC (**??**).

As noted earlier, $\lambda^\eta_{B,\text{approx}}$ depends upon the relative sizes of m and n through their determination of q and $g_\varepsilon(\xi^\varepsilon_q)$; otherwise, the dependence on m and n is limited to the sum $m+n$. Equilibrium bidding depends separately on each variable m and n because the distribution of a buyer's bid is different from the distribution of a seller's ask. This asymmetry, however, becomes inconsequential asymptotically as buyers' bids converge to their true values and all nonfocal traders bid/ask increasingly in the same way. The asymmetry of behavior thus matters less and less as $\eta \to \infty$ and all that does matter is the sum $m + n$ in its role in determining the total number $(m+n)\eta - 1$ of nonfocal traders.[13] The value of q continues to matter to a buyer as the market increases in size because it determines where the price is selected within the distribution of bids/asks. These observations suggests a second comparative statics exercise. Consider for fixed k_1, $k_2 \in \mathbb{N}$ markets with (i) ηk_1 buyers, ηk_2 sellers and (ii) ηk_2 buyers, ηk_1 sellers. Because of the assumption that g_ε is symmetric about 0, it follows that $g_\varepsilon\left(\xi^\varepsilon_q\right) = g_\varepsilon\left(\xi^\varepsilon_{1-q}\right)$ where $q = k_1/(k_1+k_2)$ and ξ^ε_q, ξ^ε_{1-q} are respectively the quantiles of interest in markets (i) and (ii). Consequently, the value of $\lambda^\eta_{B,\text{approx}}$ is exactly the same across the two markets, and so equilibrium bidding behavior by buyers should be approximately the same across them.

4.2.2 Numerical Example

Numerical calculations presented below suggest that $\lambda^\eta_{B,\text{approx}}$ as given by (14) approximates the equilibrium offset of buyers quite well even in small markets. The two comparative statics predictions that are discussed above are also illustrated by the calculations that follow.

We consider the four mixtures of normals depicted in Figure 5. In (a) we have the standard normal $\mathcal{N}(0,1)$, in (b) an equal mixture of two normals $\mathcal{MN}(\{0.5,0,1\}, \{0.5,0,4\})$ centered at zero but with different variances, in (c) an equal mixture of two normals $\mathcal{MN}(\{0.5,-1,1\}, \{0.5,1,1\})$ centered at -1 and 1 with equal variances, and in (d) an equal mixture of two normals $\mathcal{MN}(\{0.5,-1.5,1\}, \{0.5,1.5,1\})$ centered at -1.5 and 1.5 with equal variances, which produces

[13] This contrasts with the strategies $B(v) = \eta m v/(\eta m + 1)$, $S(c) = c$, which define an equilibrium in the BBDA for *all* η, m *and* n when buyer values and seller costs are independently drawn from the uniform distribution on $[0,1]$ (Satterthwaite and Williams (1989, eq. 5.11), Williams (1991, ex. A.1)). Underbidding in this case thus depends only on the number ηm of buyers and not on the number ηn of sellers. The emphasis above on the total number of traders instead of their roles as buyers or sellers is a feature of the particular informational structure of this paper.

a bimodal distribution.

Table 4 concerns the case of $m = n = 1$ and $\eta = 2, 4, 8, 16$. Its four panels A–D correspond to

Panel A: $\mathcal{N}(0,1)$, $\xi_q^\varepsilon = 0$, $g_\varepsilon(\xi_q^\varepsilon) = 0.3989$.

η	λ_B^η	$\lambda_{B,\text{approx}}^\eta$	$\left\lvert\lambda_{B,\text{approx}}^\eta - \lambda_B^\eta\right\rvert$	$\frac{\left\lvert\lambda_{B,\text{approx}}^\eta - \lambda_B^\eta\right\rvert}{\lambda_B^\eta}$
2	-0.6896	-0.8355	0.1459	0.2116
4	-0.3398	-0.3581	0.0183	0.0539
8	-0.1639	-0.1671	0.0031	0.0195
16	-0.0805	-0.0809	0.0004	0.0050

Panel B: $\mathcal{MN}\left(\{0.5, 0, 1\}, \{0.5, 0, 4\}\right)$, $\xi_q^\varepsilon = 0$, $g_\varepsilon(\xi_q^\varepsilon) = 0.2992$.

η	λ_B^η	$\lambda_{B,\text{approx}}^\eta$	$\left\lvert\lambda_{B,\text{approx}}^\eta - \lambda_B^\eta\right\rvert$	$\frac{\left\lvert\lambda_{B,\text{approx}}^\eta - \lambda_B^\eta\right\rvert}{\lambda_B^\eta}$
2	-0.9304	-1.1141	0.1837	0.1974
4	-0.4617	-0.4775	0.0158	0.0342
8	-0.2215	-0.2228	0.0065	0.0293
16	-0.1077	-0.1078	0.0001	0.0009

Panel C: $\mathcal{MN}\left(\{0.5, -1, 1\}, \{0.5, 1, 1\}\right)$, $\xi_q^\varepsilon = 0$, $g_\varepsilon(\xi_q^\varepsilon) = 0.2420$.

η	λ_B^η	$\lambda_{B,\text{approx}}^\eta$	$\left\lvert\lambda_{B,\text{approx}}^\eta - \lambda_B^\eta\right\rvert$	$\frac{\left\lvert\lambda_{B,\text{approx}}^\eta - \lambda_B^\eta\right\rvert}{\lambda_B^\eta}$
2	-1.0468	-1.3776	0.3308	0.3160
4	-0.5305	-0.5904	0.0599	0.1129
8	-0.2610	-0.2755	0.0145	0.0556
16	-0.1296	-0.1333	0.0037	0.0285

Panel D: $\mathcal{MN}\left(\{0.5, -1.5, 1\}, \{0.5, 1.5, 1\}\right)$, $\xi_q^\varepsilon = 0$, $g_\varepsilon(\xi_q^\varepsilon) = 0.1295$.

η	λ_B^η	$\lambda_{B,\text{approx}}^\eta$	$\left\lvert\lambda_{B,\text{approx}}^\eta - \lambda_B^\eta\right\rvert$	$\frac{\left\lvert\lambda_{B,\text{approx}}^\eta - \lambda_B^\eta\right\rvert}{\lambda_B^\eta}$
2	-1.4650	-2.5737	1.1087	0.7568
4	-0.7626	-1.1030	0.3404	0.4464
8	-0.3948	-0.5147	0.1199	0.3037
16	-0.2084	-0.2491	0.0407	0.1953

Table 4: For different market sizes η, the equilibrium offset λ_B^η is compared to its approximation $\lambda_{B,\text{approx}}^\eta$ in the case of $m = 1$ buyer and $n = 1$ seller and the four distributions depicted in Figure 5.

the four distributions (a)–(d) in Figure 5. Column 1 in each panel lists the market size η, column 2 lists the equilibrium offset λ_B^η, column 3 lists the approximate offset $\lambda_{B,\text{approx}}^\eta$, column 4 lists the absolute error in the approximation of equilibrium, and column 5 lists the absolute error as a fraction of the equilibrium offset.

Beginning with Panel A in Table 4 we see that the relative error in the approximation diminishes quickly so that the approximate solution performs well even for modest market sizes. This is true also in Panels B and C. The most challenging case is the bimodal distribution in Panel D. The poor performance of the approximate formula is to be expected because a larger sample size is required in the case of a bimodal distribution to achieve the same accuracy in convergence (in the sense of the central limit theorem) in comparison to unimodal distributions. Nonetheless, the approximate formula is already quite accurate for 16 traders on each side of the market even in this case. We have also computed these values for the four distributions in the asymmetric case of $m = 1, n = 2$ and obtained qualitatively similar results.

Because $m = n = 1$ in Table 4, we have $q = 1/2$ for each of the four distributions and $\xi_q = 0$. The density $g_\varepsilon(\xi_q^\varepsilon)$ decreases as one moves from distribution (a) to (b) to (c) to (d). As noted above, the approximate solution $\lambda^\eta_{B,\text{approx}}$ therefore increases for each size η of market as the distributions change in this way; moreover, the equilibrium offset also increases as the distribution changes, even in the case of $\eta = 2$. Our first comparative statics prediction suggested by the asymptotically derived, approximate formula (14) thus holds even in these very small markets.

Table 5 addresses our second comparative statics prediction in the case of the standard normal

η	$\lambda^\eta_{1,2}$	$\lambda^\eta_{2,1}$	$\left\lvert\lambda^\eta_{1,2} - \lambda^\eta_{2,1}\right\rvert$	$\frac{\lvert\lambda^\eta_{1,2}-\lambda^\eta_{2,1}\rvert}{\lvert\lambda^\eta_{1,2}\rvert}$
2	-0.5027	-0.5085	0.0058	0.0115
4	-0.2433	-0.2441	0.0008	0.0033
8	-0.1184	-0.1185	0.0001	0.0008
16	-0.0583	-0.0583	0	0

Table 5: For different market sizes η and G_ε standard normal, the equilibrium offset $\lambda^\eta_{1,2}$ for the case of $m = 1$ buyer, $n = 2$ sellers is compared to the equilibrium offset $\lambda^\eta_{2,1}$ for the case of $m = 2$ buyers, $n = 1$ seller.

distribution. We consider the cases of $m = 1$, $n = 2$ and $m = 2$, $n = 1$ with $\eta = 2, 4, 8$, and 16. It is observed above that buyer underbidding should be approximately the same in these two sequences for η sufficiently large. Column 1 of the table again lists η, column 2 lists the equilibrium offset $\lambda^\eta_{1,2}$ for the sequence of markets with $m = 1$ and $n = 2$, and column 3 lists the equilibrium offset $\lambda^\eta_{2,1}$ for the sequence of markets with $m = 2$ and $n = 1$. Notice that the two lists of offsets are very close to one another even in the smallest of markets, and they are in fact identical to our level of computational accuracy for $\eta = 16$. Column 4 lists the absolute difference $\left\lvert\lambda^\eta_{1,2} - \lambda^\eta_{2,1}\right\rvert$ of these equilibrium offsets. Column 5 lists this absolute difference as a fraction of the absolute offset $\left\lvert\lambda^\eta_{1,2}\right\rvert$, which is done to provide some sense of scale for the absolute error. Even in the case of $\eta = 2$, this relative error is barely larger than 1%.

4.3 Asymptotic Analysis in the CIV Case with G_ε, G_δ Normal

Our approximation result in normal special case of the CIV model is as follows.

Theorem 3 (Approximate Solution) *Assume that $G_\varepsilon \sim \mathcal{N}(0, \upsilon_\varepsilon)$ and $G_\delta \sim \mathcal{N}(0, \upsilon_\delta)$. For fixed m and n, consider a sequence of markets indexed by the market size η. Let $(\boldsymbol{B}^\eta, \boldsymbol{S}^\eta)_{\eta\in\mathbb{N}}$ be a sequence of strategy profiles that satisfies A4. Faced with the strategies of the others in this* *sequence, suppose a focal buyer considers strategies $\widetilde{B}(\cdot;\eta)$ in which his underbidding is $O\left(1/\eta^{\epsilon}\right)$ for some $\epsilon > 0$. The unique solutions to the AFOCs (**??**) and (**??**) are $\tilde{\lambda}_B^{\eta}$ in (15) and $\tilde{\lambda}_S^{\eta}$ in (16).*

As in the CPV special case, the assumption that the focal buyer considers strategies where underbidding is $O\left(1/\eta^{\epsilon}\right)$ for some $\epsilon > 0$ is used in the proof to start the derivation of (13) by allowing a Taylor polynomial expansion of $\widetilde{B}(\cdot;\eta)$.

4.3.1 Numerical Example

In Tables 6 and 7 we present numerical results for the case of $G_{\varepsilon} = G_{\delta}$ standard normal (i.e., and $v_{\varepsilon} = v_{\delta} = 1$). We consider market sizes $\eta \in \{2, 4, 8, 16, 32, \infty\}$ and three different supply/demand cases, $m = n = 1$, $m = 1$, $n = 2$ and $m = 2, n = 1$. Columns 2 and 3 in Table 6 present the equilibrium offsets for buyers and sellers calculated by solving the FOCs (5)-(6), columns 4 and 5 present the approximate offsets for buyers and sellers as given by (17) and (18), and columns 6 and 7 present the absolute difference between the equilibrium solution and the approximate solution. Columns 2 and 3 in Table 7 present the absolute difference between the equilibrium solutions and their corresponding limit λ^{∞} defined in (8, columns 4 and 5 present the strategic term of a buyer computed exactly by using (5)-(6) and approximately by using (19), and column 6 presents the absolute difference between equilibrium and approximate strategic terms.[14]

The approximate and equilibrium offsets have the same magnitudes, mostly the same sign and converge to their common limit at the same rate. In Figure **??** we depict their absolute difference as a function of $\eta \in \{2, 4, 8, 16, 32\}$ (i.e., the last two columns of Table 6). Their difference falls at a rate of $O(1/\eta)$, the same as the rate at which they tend to their common limit. This is clearly a worse approximation than in the CPV case. The extra error in the approximation is due to the fact that in the asymptotic approach, the strategic behavior of others is ignored by the focal buyer/seller and hence his price-taking term does not address it. To see this more concretely, observe that the approximation for the strategic term of a buyer is extremely good, on par with the CPV case. It is therefore the error in the price-taking term that worsens the approximation. Even if the approximation as a whole is not as good as in the CPV case, it is still useful as a formula for easily calculating the offsets and most importantly in understanding what is first order in a trader's calculation of his bid/ask. For example the asymptotic analysis suggests that offsets should tend to their limit at a rate $O(1/\eta)$, which is verified for the equilibrium offsets as can be seen from Columns 1 and 2 in Table 7.

5 Conclusion

We analyze the equilibrium price and a trader's decision problem in the buyer's bid double auction from an asymptotic perspective. The asymptotic distribution of the equilibrium price is characterized. It reveals that this price is a consistent, asymptotically unbiased and normal estimator of the

[14] **SRW:** There is a footnote at the end of the section 4.1 that discusses some possible numerical work for this section. At present, it's boring. We need to figure out some interesting computational exercises, especially those that shed light on the price-taking terms (the distinguishing feature of the CIV model).

η	λ_S	λ_B	$\widetilde{\lambda}_S$	$\widetilde{\lambda}_B$	$\lvert\lambda_S - \widetilde{\lambda}_S\rvert$	$\lvert\lambda_B - \widetilde{\lambda}_B\rvert$
$m = 1$						
$n = 1$						
2	0.4124	-1.3403	0	-1.7588	0.4124	0.4185
4	0.2172	-0.7036	0	-0.856	0.2172	0.1524
8	0.1091	-0.3417	0	-0.4317	0.1091	0.09
16	0.0549	-0.1684	0	-0.2182	0.0549	0.0498
32	0.0312	-0.0874	0	-0.1099	0.0312	0.0225
∞	0	0	0	0	0	0
$m = 1$						
$n = 2$						
2	0.4912	-0.8372	0.1821	-1.0609	0.3091	0.2237
4	0.3948	-0.2657	0.2333	-0.3911	0.1615	0.1254
8	0.3487	0.0219	0.2657	-0.0509	0.083	0.0728
16	0.3262	0.1643	0.2842	0.1243	0.042	0.04
32	0.311	0.2455	0.2942	0.2137	0.0168	0.0318
∞	0.3046	0.3046	0.3046	0.3046	0	0
$m = 2$						
$n = 1$						
2	0.1332	-1.2189	-0.1821	-1.4252	0.3153	0.2063
4	-0.0787	-0.743	-0.2333	-0.8577	0.1546	0.1147
8	-0.1893	-0.5167	-0.2657	-0.5824	0.0764	0.0657
16	-0.2467	-0.4086	-0.2842	-0.4442	0.0375	0.0356
32	-0.2861	-0.3517	-0.2942	-0.3746	0.0081	0.0229
∞	-0.3046	-0.3046	-0.3046	-0.3046	0	0

Table 6: Comparison of the approximately computed bid/ask with the equilibrium bid/ask for different market sizes and demand/supply ratios (ε, δ standard normal).

η	$\lvert\lambda_S - \left(V\left(\xi_q^{\varepsilon+\delta}\right) - \xi_q^{\varepsilon+\delta}\right)\rvert$	$\lvert\lambda_B - \left(V\left(\xi_q^{\varepsilon+\delta}\right) - \xi_q^{\varepsilon+\delta}\right)\rvert$	$\frac{\Pr[x<\lambda_B<y\mid 0]}{f^B_{x\mid\sigma}(\lambda_B\mid 0)}$	$\frac{\widetilde{\Pr}[x<\widetilde{\lambda}_B<y\mid 0]}{\widetilde{f}^B_{x\mid\sigma}(\widetilde{\lambda}_B\mid 0)}$	$\left\lvert\frac{\Pr[x<\lambda_B<y\mid 0]}{f^B_{x\mid\sigma}(\lambda_B\mid 0)} - \frac{\widetilde{\Pr}[x<\widetilde{\lambda}_B<y\mid 0]}{\widetilde{f}^B_{x\mid\sigma}(\widetilde{\lambda}_B\mid 0)}\right\rvert$
$m=1$, $n=1$					
2	0.4124	1.3403	0.9279	1.1816	0.2537
4	0.2172	0.7036	0.4864	0.5064	0.02
8	0.1091	0.3417	0.2326	0.2363	0.0037
16	0.0549	0.1684	0.1135	0.1144	0.0009
32	0.0312	0.0874	0.0562	0.0563	0.0001
∞	0	0	0	0	0
$m=1$, $n=2$					
2	0.1866	1.1418	0.7107	0.7779	0.0672
4	0.0902	0.5703	0.3456	0.3536	0.008
8	0.0441	0.2827	0.1676	0.1691	0.0015
16	0.0216	0.1403	0.0824	0.0828	0.0004
32	0.0064	0.0591	0.0406	0.0409	0.0003
∞	0	0	0	0	0
$m=2$, $n=1$					
2	0.4378	0.9143	0.7233	0.7779	0.0546
4	0.2259	0.4384	0.3476	0.3536	0.006
8	0.1153	0.2121	0.1679	0.1691	0.0012
16	0.0579	0.104	0.0824	0.0828	0.0004
32	0.0185	0.0471	0.0407	0.0409	0.0002
∞	0	0	0	0	0

Table 7: Comparison of the approximately computed bid/ask with the equilibrium bid/ask for different market sizes and demand/supply ratios (ε, δ standard normal).

rational expectations price. The rational expectations price is thus approximately implemented in a finite market by a strategically determined, market-clearing price, and numerical evidence suggests that this can be true even in very small markets. The asymptotic first order condition is determined by identifying the asymptotic values of probabilities in a trader's first order condition. In the correlated private values model, the solution to this equation is a simple formula in the fundamentals of our model that determines a unique offset strategy of a buyer. Numerical investigation suggests that this strategy closely approximates the equilibrium bidding strategy even in small markets. This simple formula identifies the first order considerations in a buyer's selection of his bid and thereby allows a comparative statics analysis of equilibrium. A similar analysis is carried out in the correlated, interdependent values case in which the distributions are normal, and all comparative statics predictions from the correlated private values case are also shown numerically to hold in the correlated interdependent values case.

References

Chatterjee, K., and W. Samuelson (1983): "Bargaining under Incomplete Information," *Operations Research*, 31, 835–851.

DeGroot, M. H. (1970): *Optimal Statistical Decisions*. John Wiley and Sons, Hoboken, New Jersey.

Myerson, R., and M. A. Satterthwaite (1983): "Efficient Mechanisms for Bilateral Trading," *Journal of Economic Theory*, 29, 265–281.

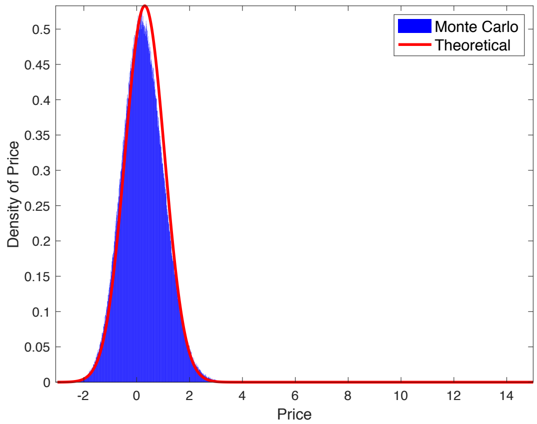
Reny, P. J., and M. Perry (2006): "Toward a Strategic Foundation for Rational Expectations Equilibrium," *Econometrica*, 74(5), 1231–1269.

Satterthwaite, M. A., and S. R. Williams (1989): "The Rate of Convergence to Efficiency in the Buyer's Bid Double Auction as the Market Becomes Large," *Review of Economic Studies*, 56(4), 477–498.

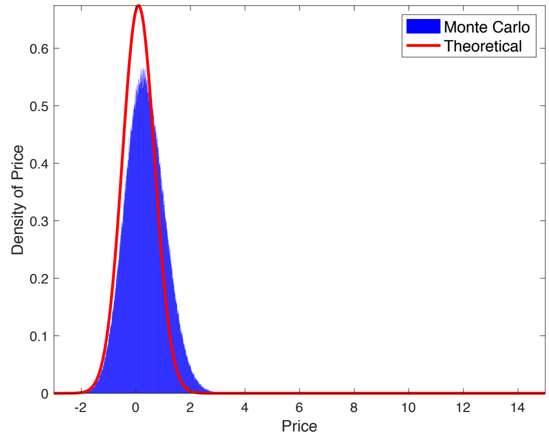
Satterthwaite, M. A., S. R. Williams, and K. E. Zachariadis (2014): "Price Discovery," *mimeo*.

Serfling, R. J. (1980): *Approximation Theorems of Mathematical Statistics*. John Wiley and Sons, New York.

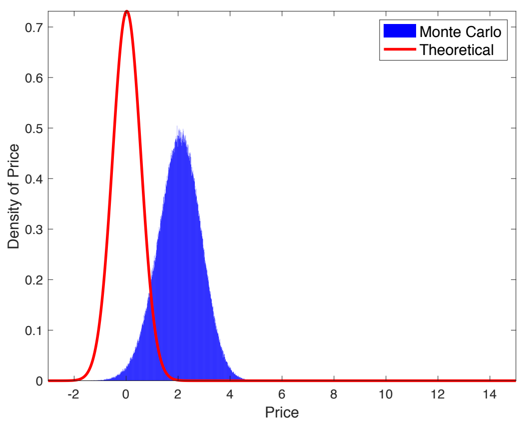
Williams, S. R. (1991): "Existence and Convergence of Equilibria in the Buyer's Bid Double Auction," *Review of Economic Studies*, 58(2), 351–374.



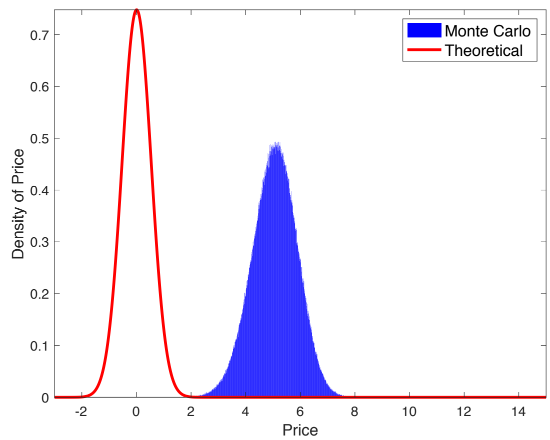
(a) $v_\varepsilon = 1$



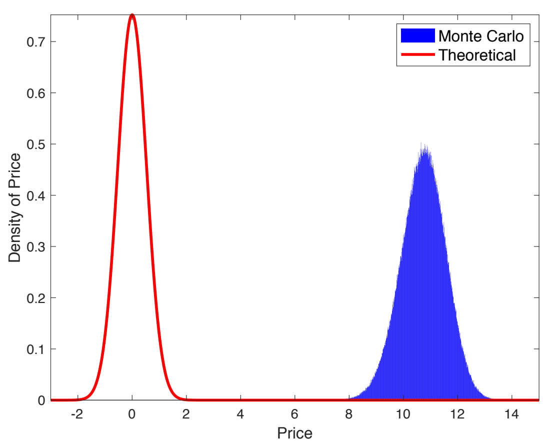
(b) $v_\varepsilon = 1/4$



(c) $v_\varepsilon = 1/16$

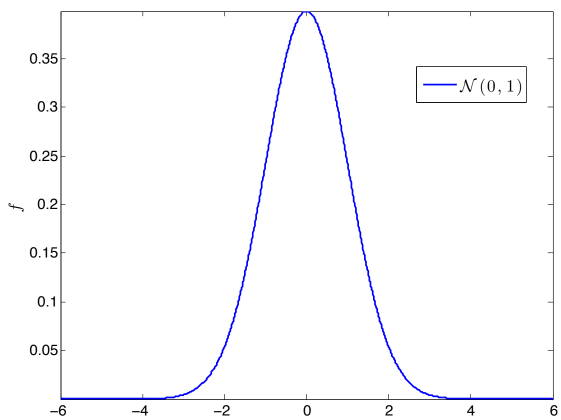


(d) $v_\varepsilon = 1/64$

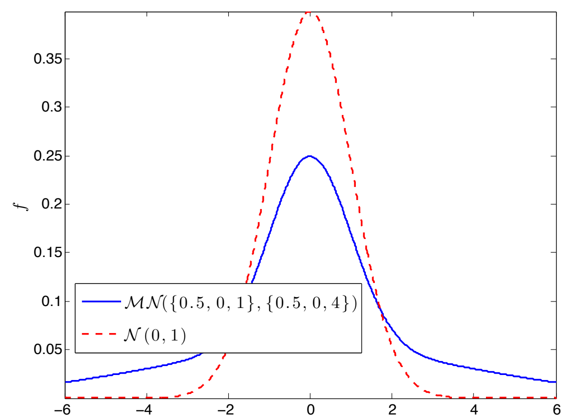


(e) $v_\varepsilon = 1/256$

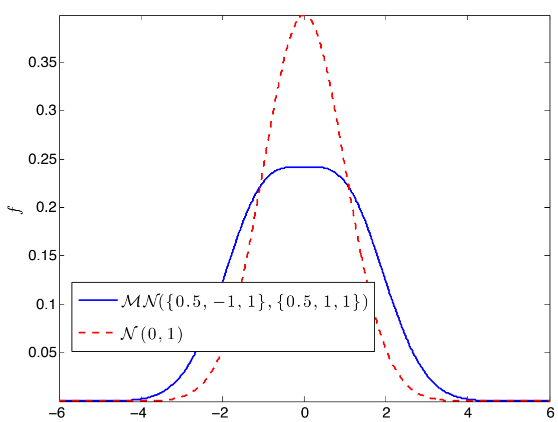
Figure 4: The figure depicts the sample density of the BBDA's equilibrium price $p^\eta_{\text{eq}}(\mu)$ calculated using Monte Carlo simulations in the case of G_δ standard normal and $\varepsilon \sim \mathcal{N}(0, v_\varepsilon)$, $m = 2$, $n = 1, \eta = 2$, and variance of the preference term $v_\varepsilon = 1, 1/4, 1/16, 1/64, 1/256$, graphed against its asymptotic density as given in Theorem 1.



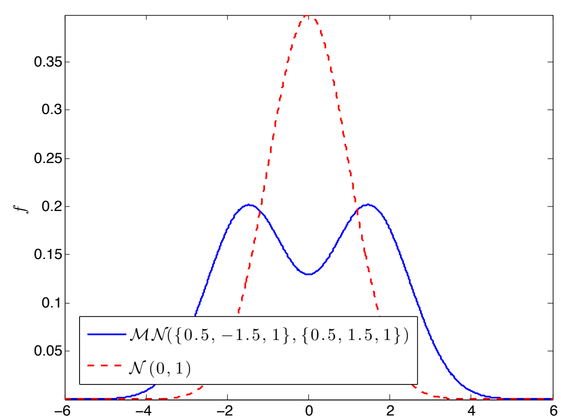
(a) $\mathcal{N}(0,1)$



(b) $\mathcal{MN}\left(\{0.5,0,1\},\{0.5,0,4\}\right)$



(c) $\mathcal{MN}\left(\{0.5,-1,1\},\{0.5,1,1\}\right)$



(d) $\mathcal{MN}\left(\{0.5,-1.5,1\},\{0.5,1.5,1\}\right)$

Figure 5: The densities g_ε of the mixture of normals that we use for our numerical illustration in comparison to the density of the standard normal.