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Uneven recessions and optimal firm subsidies[☆]

Caio Machado

Instituto de Economía, Pontificia Universidad Católica de Chile, Chile

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ABSTRACT

How should policymakers distribute firm subsidies when supply shocks hit some firms and spill over to others through demand externalities? I propose a model to answer that question. The optimal policy depends on the severity of the supply shock and the degree of substitution across goods. If shocks are not too large or widespread, it is optimal to subsidize only firms not hit by supply shocks. For larger and more widespread shocks, the results depend on the elasticity of substitution: If complementarities across the varieties produced by firms are high, firms facing supply shocks should be prioritized, and the opposite is true if complementarities are low.

1. Introduction

Many crises start with shocks that hit some firms much more than others (uneven shocks). To name a few examples: The Russian invasion of Ukraine more deeply affected sectors that relied on Russian energy sources; COVID-19 particularly disrupted the operations of businesses that require intensive social contact; Climate shocks are likely to impact more strongly firms that require agricultural inputs; During large devaluations in developing economies, firms that rely on imported inputs tend to be the most vulnerable ones. As emphasized by Guerrieri et al. (2022), even if such supply disruptions are contained to a subset of firms in the economy, they may indirectly affect other sectors due to demand linkages, potentially leading to widespread business closures even in sectors that were not directly hit by the shock.

As a response to those concerns, governments often provide subsidies to support firms' operations. Some of those programs do not have specific targets, such as the Paycheck Protection Program in the US, which offered funds to a broad set of firms that could be used to fund their operations.[1] Others are more tailored to firms directly hit by a shock, such as the Restaurant Revitalization Fund in the US or recent energy subsidies offered to German firms highly dependent on (Russian) gas.[2] A goal of such subsidy programs is to prevent businesses from stopping operating, possibly containing negative spillovers to the rest of the economy.

In this paper, I ask what is the most efficient way to subsidize firms after uneven shocks, when negative supply shocks in some sectors may reduce demand in other sectors. Should subsidy programs prioritize firms that were directly hit by a supply shock? Or should they focus on firms that were only indirectly affected due to the negative demand spillovers of such shocks? I study these issues in a simple model that is able to shed light on the economic forces that determine the answers to those questions.

Model. The model consists of a simple static monopolistic competition environment with fixed operating costs. Each firm produces a different variety. Varieties are complements among themselves, according to a standard CES aggregator. There are two types of firms, A and B. A-firms can produce positive quantities (operate) only if they pay a fixed operating cost. B-firms face no fixed operating costs and hence always operate. An operating firm can produce up to some exogenous limit (its productivity) at zero marginal cost.

Incentives to operate are increasing in the number of firms that operate, as goods are complements among themselves. That is, there are positive demand externalities: If a restaurant operates, it increases the demand for parking services close to it, and vice versa. Initially, I assume that all firms are identical and productive enough so that the unique equilibrium consists of all firms operating.

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E-mail address: caio.machado@uc.cl.

[1] Those funds were called "forgivable loans", since they consisted of a loan that did not have to be paid if a firm used it to pay operating expenses. So, in practice, they worked like transfers received conditional on a firm operating.

[2] As argued by German authorities when releasing their energy subsidies program: *"The programme is precisely targeted. It serves to cut the costs of the increased gas and electricity prices for particularly affected energy-intensive and trade-intensive companies"* (BMWK, 2022).

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Uneven shocks and business closure cascades. I then hit this economy with a supply shock that reduces the productivity of a proportion of firms in the economy, leaving the productivity of the remaining ones unaffected. A shock is characterized by its size, which is the productivity loss it induces, and its reach, which is the proportion of firms that are hit by it (how widespread it is). If the shock is large and widespread enough, there may be business closure cascades: There is an equilibrium in which both A-firms hit and not hit by the shock do not operate, and this may be the unique equilibrium for very large and widespread shocks.

Intuitively, firms' goods are complements, and a firm that faces no supply shock may find it optimal to cease operating simply because it anticipates other firms in the economy will not produce as much, reducing the demand for its good. Supply shocks in some sectors reduce the demand in other sectors, potentially leading to widespread business closure. Coming back to the restaurant and parking lot example, a decrease in the supply or the quality of gas makes restaurants less productive but should have little effect on the operations of a parking lot. However, the parking lot may find it useful to send its staff home and shut down if its demand is highly dependent on the restaurant's customers.

Optimal subsidies in an uneven recession. In many circumstances, all A-firms cease to operate in equilibrium because they do not internalize the positive demand externalities their production causes on others, but a social planner would like them all to operate even after the shock. Whenever there is an equilibrium in which all firms stop operating after the shock, but a social planner would like them all to operate, I say that the economy is subject to an *inefficient business closure cascade*. Assuming a policymaker has enough resources at its disposal, it can implement the first best in which all firms operate by subsidizing their operating costs. But suppose it would like to do so while spending as little as possible on subsidies. Who should be prioritized in the subsidy scheme? The firms directly hit by the supply shock (the restaurant, in my previous example) or those indirectly affected through demand linkages (the parking lot)?

To answer those questions in a model where multiple equilibria can arise, I rely on a unique implementation approach, as in Bernstein and Winter (2012) and Halac et al. (2020). In particular, I build on the insight that in supermodular games, the optimal subsidy problem can be thought of as an optimal permutation problem. I then say that a subsidy scheme is optimal in an inefficient business closure cascade if it implements the first best as the unique equilibrium outcome at the lowest possible cost for the policymaker. The problem boils down to choosing how to rank firms. Once a ranking has been chosen, the optimal policy implies giving each firm the lowest subsidy that makes it willing to enter if it believes only firms ranked above it will do so. This implies that, all else constant, the higher a firm's ranking, the more generous the subsidy it receives.

The optimal policy can always be thought of as a sequential problem, in which the policymaker starts offering a subsidy generous enough to guarantee the entry of a given firm, say firm 1, regardless of its belief about the decisions of other firms. Then, the policymaker moves to the next firm in the ranking, say firm 2, and offers a subsidy that is generous enough to guarantee its entry even if it believes that only firm 1 will enter and so on. That is, the policymaker sequentially ensures the entry of firms. This is what is often referred to as divide-and-conquer schemes and is common in the contracting literature with supermodular games (Bernstein and Winter, 2012). The question of how to rank firms then ultimately boils down to whether the policymaker should first guarantee the entry of firms hit or not hit by the shock (or a little bit of both).

I show that the optimal ranking and subsidy scheme depend on two features: (i) the severity of the supply shock (how large and widespread it is) and (ii) the amount of substitution across goods, which ultimately determines the strength of demand externalities and strategic complementarities.

Smaller and less widespread shocks. For less severe shocks, but still severe enough to possibly trigger a business closure cascade, I show that it is optimal to rank firms not directly hit by the supply shock first and only pay subsidies to those firms. In other words, the optimal policy only subsidizes firms that have a demand problem, and not those with a supply problem. Intuitively, when shocks are not so severe, guaranteeing that firms not hit by the supply shock will operate is enough to convince firms hit by the shock to keep operating due to complementarities. Since firms not hit by the shock tend to be more productive, convincing them to operate is cheaper. As long the entry of the most productive ones is enough to guarantee the entry of other firms, subsidizing only firms not hit by the shock is the optimal policy.

Larger and more widespread shocks. For very severe shocks, such that all firms need to get some subsidy to operate, the optimal policy depends on the degree of complementarities across goods. If complementarities are strong (precisely, if the elasticity of substitution across goods is smaller than two), it is optimal to rank firms hit by the shock first. In this case, firms directly hit by the shock get larger subsidies than firms not hit by the shock. But if complementarities are not so strong (meaning that the elasticity of substitution across goods is above two), firms not hit by the shock should be ranked first. In that case, they do not necessarily get larger subsidies than those not hit, since more productive firms require lower subsidies to operate for a given belief about the action of others, but they get larger rents in equilibrium after the subsidy.

The intuition for that result is the following. Firms not hit by the shock are more productive, which implies they have two distinctive features relative to firms hit by the shock. First, their entry generates more demand for others, as they produce more and goods are complements. Second, they are more sensitive to the actions of other firms: Since they can produce more, their revenue reacts more to changes in their demand schedule. The former suggests that high-productivity firms should be placed up in the ranking, since guaranteeing their entry from the start allows the policymaker to pay much lower subsidies to other firms, as they generate larger demand externalities. But the latter suggests that they should be placed at the end of the ranking: As they react more to the actions of others, the policymaker could potentially benefit from guaranteeing the entry of firms with revenues less sensitive to demand (those hit by the shock), so that those that are more sensitive to demand (those not hit by the shock) can be paid much lower subsidies. Hence, there are forces pushing in opposite directions, and the ideal target to be prioritized is not obvious.

Why does the elasticity of substitution across goods determine which effect dominates? When goods are very strong complements, guaranteeing the entry of some, but not all, firms has little effect on the subsidy a firm requires to enter. With strong complementarities, firms require the entry of almost all firms to have guaranteed a high enough level of demand for their good for them to be willing to enter, for a given partial subsidy. Hence, firms placed up in the ranking will require almost all of their costs to be subsidized, regardless of who is placed before them. The benefit of placing a high-productivity firm instead of a low-productivity one up in the ranking is small: Convincing a firm to operate when the entry of just a few firms has been guaranteed is very expensive, regardless of how productive those few firms are. Hence, the policymaker prioritizes placing the firms most sensitive to the action of others down in the ranking, where entry of almost all firms is guaranteed. As complementarities weaken (goods become closer substitutes), the benefit of guaranteeing the entry of some very productive firms early on increases, as firms become more sensitive to the entry of a small number of firms because goods are not such strong complements.

Intermediate cases. For intermediate shocks, the optimal ranking involves first guaranteeing the entry of firms not hit by the supply shock whenever the elasticity of substitution is larger than two. If that is not the case, then the optimal ranking can be non-monotonic: The policymaker may, for instance, want to guarantee the entry of a few firms hit by the shock, then move on to the ones not hit and go back to those hit by the shock. In short, for intermediate shocks, the optimal subsidy scheme looks like a convex combination of the more extreme cases previously discussed.

Implementation with shocks of uncertain size. When complementarities are not so strong (elasticity of substitution larger than two), the optimal policy can be easily implemented even if the policymaker is uncertain about the size of the shock. It starts offering subsidies sequentially to firms not hit by the shock, to trigger their entry, and only moves on to subsidizing the firms directly hit by the shock if that is not enough to end the business closure cascade.

Related literature. This paper relates to the literature that emphasizes how demand externalities combined with some non-convexity in the production function, such as fixed costs, can lead to regime switches in economic activity, building on the seminal contributions of Kiyotaki (1988) and Cooper and John (1988) (see Guimaraes et al., 2016 and Schaal and Taschereau-Dumouchel, 2018 for some recent examples). It is particularly related to Guimaraes and Machado (2018), who study the optimal timing of stimulus policies (subsidies) in an economy subject to such regime switches. Differently from that paper, here shocks affect firms differently, and the focus is on how to distribute subsidies among different firms at a given point in time.

The paper is also related to Guerrieri et al. (2022), who show how business closure cascades can arise in equilibrium in an economy subject to the demand externalities assumed in this paper, and to Bigio et al. (2020), who study the trade-off between credit policies and transfers in a setting related to Guerrieri et al. (2022). Here, the focus is on how to optimally distribute subsidies intended to fund firms' operating expenses. Cesa-Bianchi and Ferrero (2021) provide evidence on the empirical relevance of the externalities emphasized in this line of work.

On the theoretical front, this paper builds on the machinery introduced by Bernstein and Winter (2012) and further developed by Halac et al. (2020) to study the funding problem of an entrepreneur seeking to raise capital for an investment. However, while my optimal policy shares the same divide-and-conquer property that arises in those papers, I cannot directly apply their results to compute the optimal ranking, as the structure of payoffs in my setting does not fit into their environments. This explains why my results can differ substantially from theirs depending on the characteristics of the shock hitting the economy. For instance, while in Halac et al. (2020) funding application it is optimal to prioritize the agents with a higher impact on others (for reasonable distributional assumptions), here, instead, when complementarities are strong and the shock is severe enough, it is optimal to prioritize the agents that have less impact on others. This paper also relates to other papers studying how to optimally target different types of interventions in settings with strategic complementarities, such as Segal (2003), Sakovics and Steiner (2012), Akerlof and Holden (2019) and Galeotti et al. (2020), to name a few.[3]

Layout. The remainder of this paper is organized as follows. Section 2 presents the basic model and discusses the equilibrium after an uneven shock hits the economy, as well as the social optimum. Section 3 studies how to optimally implement the first best, characterizing the optimal subsidy scheme. Section 4 discusses some extensions of the main model and Section 5 concludes. All proofs are in the Appendix.

[3] In an open-economy model similar to Melitz (2003) and Pflüger and Südekum (2013) analyze firms' subsidies aimed at inducing entry, as in this paper, but they do not allow subsidies to differ across firms, which is the focus of my work.

2. A model with inefficient business closure

This section presents a simple model that illustrates how inefficient business closure can happen both for firms that were and were not directly hit by a supply shock, and that is tractable enough to allow for an analytical characterization of optimal subsidies.

The economy consists of several independent local economies (islands). A representative island is composed of two types of firms, A-firms and B-firms, and a household. A-firms need to pay fixed costs to produce positive quantities, while B-firms do not. One can think of B-firms as large firms with small fixed costs relative to their total operations, and A-firms as smaller firms that face a non-trivial operating decision, as fixed costs represent a relevant fraction of their expected sales. $N_A \geq 2$ and $N_B \geq 2$ are the number of A- and B-firms in a given island, respectively, and $N = N_A + N_B$. We index A-firms by $i \in \{1, 2, \ldots, N_A\}$ and B-firms by $i \in \{N_A + 1, N_A + 2, \ldots, N\}$. Firms are owned by the household, who receives all the profits on its island.

The production of each firm is used by a competitive producer to produce Y units of the composite good according to

$$Y = \left(\sum_{i=1}^{N} y_i^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}}, \tag{1}$$

where y_i denotes the amount of good i used, and $\theta > 1$ is the elasticity of substitution. I denote by p_i the price of variety i and normalize the price of the composite good P to one. The household utility is $U = C$, where C is her consumption of the composite good.

The production technology of firms is as follows. If firm i is of type A, it can produce up to Z_i units only if it pays a fixed cost $\psi > 0$, in units of the composite good, otherwise it produces zero. If firm i is of type B, it can produce up to Z_i units without paying any fixed cost. Producing above Z_i is prohibitively costly. This technology then abstracts from the intensive margin in firms' production decisions, since my focus is on studying the optimal distribution of subsidies intended to keep firms operating, and also serves to keep the optimal subsidy problem more tractable.[4]

Since the composite good producer is competitive and $P = 1$, $p_i = \frac{\partial Y}{\partial y_i}$ and hence we have the usual demand schedule:

$$p_i = y_i^{-\frac{1}{\theta}} Y^{\frac{1}{\theta}}. \tag{2}$$

I refer to Y as the aggregate demand, as it shifts firms' demand upwards. Hence, when other firms on an island produce more, the demand schedule of firm i shifts to the right, that is, there are positive demand externalities, as in Guimaraes and Machado (2018) and Guerrieri et al. (2022). Intuitively, the goods in this economy are complements: The more people go to restaurants, the more parking services they demand. Firms maximize profits and are subject to (1), (2) and $y_i \leq Z_i$, if they operate, and to $y_i = 0$, if they do not operate.

2.1. Discussion of assumptions

Remark 1. Technically, the number of firms can be arbitrarily large, but I invite the reader not to think of N as being the total number of firms in a given country. When I study optimal subsidies, I allow the policymaker to offer different subsidies to each firm, which is unreasonable if one thinks of N as the total number of firms in the economy. However, if one thinks of N as the total number of firms closely connected to each other in a smaller geographical area (island), then my policy exercise makes more practical sense.

[4] In Appendix B, I present and solve a model where firms also use labor to produce and the main results are similar.

For instance, a typical island could be composed of a restaurant, a parking lot, a gas station, a drugstore, a supermarket, a department store and a music venue, with the economy as a whole being the collection of many of those islands, that, for simplicity, are assumed not to interact with each other. The optimal policy exercise will pin down the optimal subsidy to each of those types of firms, after some but not all are hit by a supply shock.

In reality, firms' decisions are likely to depend on what happens in the economy as a whole, but it is also likely that firms care more about firms closely connected to them. For instance, when deciding whether to open an ice cream shop during a weekend, the ice cream seller cares directly about whether a movie theater next to it will be operating during that same weekend but does not care as much about whether movie theaters in a distant neighborhood (island) will operate. In order to have a clear analytical characterization of the optimal subsidy, my analysis abstracts from those less direct spillovers.

Remark 2. My focus is on studying how to optimally distribute subsidies among a set of firms that face positive demand externalities among themselves. Of course, in many crises, some sectors may actually benefit from negative supply shocks in other sectors. For instance, online retailers may have seen their demand increase during the COVID-19 crisis. If I were to add firms that benefit from a negative supply shock in other sectors, the optimal subsidy for those would be equal to zero (as they have even higher incentives to keep operating after the shock), and hence adding those firms to the model adds little insight.

Remark 3. I assume that the only dimension of heterogeneity among A-firms is their productivity Z_i, and that it is observed by the regulator. This is motivated by the fact that in a typical crisis triggered by a productivity shock, regulators often have a good idea of which firms were directly hit by that shock. For instance, in the aftermath of the COVID-19 crisis, it was clear that sectors intensive in social contact were those with more disruptions in their operations. After a shortage of Russian gas hit Europe, it is not hard to know which sectors rely the most on gas in their production process.

2.2. Equilibrium

The equilibrium is solved backward. In a second step, taking as given firms' operating decisions (paying or not the fixed cost), firms decide how much to produce. This step is trivial: One can easily verify that a firm's revenue is always increasing in the amount y_i produced, and hence operating firms produce up to their limit Z_i. Hence, hereafter I characterize the Nash Equilibrium on the first step, in which A-firms decide whether to pay the fixed operating cost.

We can write the revenue of an operating firm i as

$$R_i = p_i y_i = Z_i^{\frac{\theta-1}{\theta}} Y^{\frac{1}{\theta}},$$

and aggregate demand as

$$Y = \left(\sum_{i=1}^{N_A} e_i Z_i^{\frac{\theta-1}{\theta}} + \sum_{i=N_A+1}^{N} Z_i^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}},$$

where $e_i \in \{0,1\}$ denotes firm i's entry decision, with $e_i = 1$ meaning that firm i pays the fixed operating cost. Firms choose e_i to maximize their expected profits $\mathbb{E}\left[e_i\left(R_i - \psi\right)\right]$, taking the entry decision of others as given.

I first discuss the equilibrium in an even economy in which all firms face the same productivity. Then, I introduce uneven shocks, assuming a fraction of the firms face a reduction in their productivity Z_i and discuss how that may lead to inefficient business closure cascades.

2.2.1. Equilibrium before the shock

Suppose that initially $Z_i = Z$ for all i. I denote by $q \in \left\{0, 1, \ldots, N_A\right\}$ the number of A-firms that operate. The next proposition characterizes the equilibrium set in this case.

Proposition 1. *Suppose an even economy: $Z_i = Z$ for all i. Define $\underset{\sim}{Z} = \psi N^{-\frac{1}{\theta-1}}$ and $\widetilde{Z} = \psi\left(1+N_B\right)^{-\frac{1}{\theta-1}}$. Then:*

1. *If $Z > \widetilde{Z}$, there is a unique equilibrium, in which $q = N_A$;*
2. *If $Z < \underset{\sim}{Z}$, there is a unique equilibrium, in which $q = 0$;*
3. *If $Z \in \left[\underset{\sim}{Z}, \widetilde{Z}\right]$, both $q = N_A$ and $q = 0$ are equilibrium outcomes.*

Intuitively, if firms are very productive, it is optimal for them to pay the fixed cost even if they expect no other A-firm to operate (and hence expect low demand Y). On the other extreme, in which productivity is low, A-firms are not willing to operate even if they expect all others to operate and a high demand. In the intermediate range, firms' operating decisions depend on their expectations: If they expect others to operate, demand is high, and then it is optimal to operate. If they are pessimistic about demand, then firms cease operations, and demand is indeed low. This type of multiplicity is standard in models where non-convexities (e.g., fixed costs) create enough feedback from demand to firms' production decisions, as in Kiyotaki (1988), Guimaraes and Machado (2018) and Schaal and Taschereau-Dumouchel (2018). I say the economy is in *good times* if $q = N_A$ is the unique equilibrium ($Z > \widetilde{Z}$).

2.2.2. Equilibrium after the shock

Now, I assume a fraction $r \in (0,1)$ of both A- and B-firms are hit by a supply shock that reduces their productivity by a factor $s \in (0,1)$, that is, $Z_i = (1-s)\,Z$ for all firms hit by the shock. I refer to such a shock as an uneven shock (s,r), where s and r are, respectively, the realized *size* and *reach* of the shock. Note that, due to discreteness in the number of firms, the occurrence of an uneven shock implicitly assumes that r, N_A and N_B are such that rN_A and rN_B are integers.

The next proposition shows that if such a shock reaches a sufficiently large number of firms and its size is sufficiently large, it may lead to what I call a *business closure cascade*: All A-firms shut down in equilibrium, regardless of whether they were (directly) hit or not by the shock. In what follows, I say that an uneven shock $\left(s_2, r_2\right)$ is larger than an uneven shock $\left(s_1, r_1\right)$ if $s_2 \geq s_1$ and $r_2 \geq r_1$.

Proposition 2. *Consider an economy that is in good times and suppose it is hit by an uneven shock (s,r). Then, there are shocks and parameters such that an equilibrium in which all A-firms exit ($q = 0$) exists, and that may be the unique equilibrium. In particular, $q = 0$ is an equilibrium outcome if*

$$Z\left[1 + (1-r)\,N_B + rN_B\,(1-s)^{\frac{\theta-1}{\theta}}\right]^{\frac{1}{\theta-1}} \leq \psi, \tag{3}$$

and it is the unique equilibrium outcome if

$$Z\left[(1-r)\,N + rN\,(1-s)^{\frac{\theta-1}{\theta}}\right]^{\frac{1}{\theta-1}} < \psi. \tag{4}$$

The left-hand side of (3) is the revenue of an operating A-firm not directly hit by the shock when it is the only A-firm that enters. The left-hand side of (4) is the revenue of that firm when all firms in the economy operate. Note the left-hand side of (3) and (4) are strictly decreasing in s and r, and hence, if any of those conditions hold for an uneven shock (s,r), it also holds for any larger uneven shock.

Intuitively, after an uneven shocks hits, firms not hit by the shock expect to face a lower demand schedule, regardless of their belief about others' entry decisions, as they anticipate some firms in the economy will produce less. If the shock is large enough, then they may refrain from operating as well.

2.3. The first best after an uneven shock

I now consider the problem of a social planner that can decide whether each firm in the economy operates, following an uneven shock

(s, r). The social planner maximizes the total utility of the representative consumer, and hence its welfare function is

$$W = Y - q\psi.$$

The next proposition discusses how the decentralized and centralized equilibrium can differ.

Proposition 3. *Consider the economy hit by an uneven shock (s, r). Then:*

1. *Whenever $q = N_A$ (all operate) is an equilibrium outcome, it is the social optimum;*
2. *If $q = 0$ (all A-firms exit) is the unique equilibrium outcome, it can still be socially optimal for all firms to operate.*[5]

Firms do not internalize that when they operate, they increase the demand schedule of other firms in the economy. It is possible that they all would be better off signing a contract that forces them all to operate, but individually it is not in their interest to honor such a commitment. The planner takes the demand externalities into account and sometimes finds it collectively optimal to induce entry of all firms, even in cases where individually all firms find it optimal not to enter.

Whenever $q = N_A$ is the social optimum but $q = 0$ is an equilibrium outcome, I say that the economy is subject to an *inefficient business closure cascade*. Throughout the paper, I focus on the interesting case of economies subject to inefficient business closure cascades.

3. Optimal subsidies in an uneven recession

Consider an economy that is subject to an inefficient business closure cascade after the uneven shock hits. There is a policymaker who can offer a subsidy $\tau_i \geq 0$ for each A-firm, which the firm can use to pay its operating costs. That is, firm i receives a transfer of τ_i units of the composite good whenever it operates. One can think that the policymaker funds this policy through a lump-sum tax on the household. I denote by $\boldsymbol{\tau} = \left\{\tau_1, \tau_2, \dots, \tau_{N_A}\right\}$ the set of subsidies, and refer to the policymaker spending or cost as $\sum_{i=1}^{N_A} \tau_i$. In what follows, $\boldsymbol{u}$ represents the unit vector $(1, \dots, 1)$.

I follow the unique-implementation approach in Bernstein and Winter (2012) and Halac et al. (2020) and assume that the goal of the policymaker is to implement the first best as the unique equilibrium outcome at the lowest possible cost. More precisely, a subsidy policy $\boldsymbol{\tau} = \left\{\tau_1, \dots, \tau_{N_A}\right\}$ is optimal if: (i) $\boldsymbol{\tau} + \varepsilon \boldsymbol{u}$ implements the first best $(q = N_A)$ as the unique equilibrium outcome, for any $\varepsilon > 0$; (ii) $\boldsymbol{\tau}$ minimizes spending in the set of policies that satisfy (i). Condition (i) is called the unique implementation constraint.

A technical remark is in order. Note that I do not require that an optimal $\boldsymbol{\tau}$ implements the first best as the unique equilibrium, but instead that $\boldsymbol{\tau} + \varepsilon \boldsymbol{u}$ does so, for any $\varepsilon > 0$. The reason is that the set of subsidies that implement the first best in the unique equilibrium sense is open, with spending achieving its infimum in this set at the frontier of it (the minimum is undefined). Hence, the role of ε in the unique implementation constraint is to close the constraint set of the policymaker.

An alternative and isomorphic way of dealing with the openness issue is to assume firms always operate whenever they are indifferent between doing so or not, as in Halac et al. (2020). In that case, one must define an equilibrium in the entry game of firms as a Nash equilibrium that further satisfies the tie-breaking convention that firms that are indifferent choose to operate. In that case, a subsidy policy $\boldsymbol{\tau}$ is optimal if: (i) it implements the first best as the unique equilibrium (as defined above) outcome; (ii) it minimizes spending in the set of policies that satisfy (i). Finally, a third alternative would be to work with a discretized set of possible subsidies.

Note that the definition of the optimal subsidy is consistent with the policymaker not being willing to bear any non-fundamental risk arising from multiple equilibria. It wants to spend as much as needed to guarantee the first best is the unique equilibrium, but, of course, it does not want to spend more than necessary to achieve that goal. If taxation was distortionary, so that the policymaker had to create other inefficiencies to raise funds, then perhaps it would be willing to accept some non-fundamental risk (multiple equilibria) in the entry game of firms, if it assigned a sufficiently high probability for the best equilibrium being played whenever there is multiplicity.

Although the game played by firms in my setting is a relatively simple coordination game, characterizing the optimal subsidy, in principle, involves evaluating the equilibrium set for any arbitrary subsidy scheme $\boldsymbol{\tau}$, which introduces arbitrary heterogeneity among firms. Fortunately, I can use the same approach as in Bernstein and Winter (2012) and Halac et al. (2020) of converting the problem into a permutation problem, as shown in the next lemma. This result and its proof follow closely Lemma 1 in Halac et al. (2020).[6]

Lemma 1. *Consider an economy subject to inefficient cascades. A subsidy scheme $\tilde{\boldsymbol{\tau}}$ satisfies the unique implementation constraint if, and only if, there is a bijective function $b : \{1, \dots, N_A\} \to \{1, \dots, N_A\}$ such that for every A-firm i, choosing $e_i = 1$ is optimal when $\tau_i = \tilde{\tau}_i$ and under the belief that only A-firms i' with $b(i') \leq b(i)$ operate.*

In other words: Once we have a ranking $b(i)$ that indicates each firm's position, the policymaker must make it sequentially dominant for firms to enter, in order to satisfy the unique implementation constraint, and any subsidy that satisfies this constraint can be constructed through this process for some ranking $b(i)$. Thinking of $b(i)$ as the position in the ranking of firm i, whenever $b\left(i'\right) < b(i)$, I say that firm i' is ranked above firm i, and that a firm is placed up (down) in the ranking if $b(i)$ is low (high).

To illustrate, suppose the policymaker decides to rank firms using $b(i) = i$. Then, firm 1 must get a subsidy that makes it willing to operate even if it believes it will be the only A-firm operating. Firm 2 must get a subsidy that makes it willing to operate under the belief that only itself and firm 1 will operate, and so on. Hence, for a given ranking $b(i)$, the policymaker's decision is trivial: It pays the lowest subsidy that makes firms willing to enter under the belief that firms placed lower in the ranking will not operate and those placed higher will.

Formally, fix a ranking $b(i)$ and let h_i be the number of firms not hit by the shock ranked above firm i—and hence the number of firms hit by the shock ranked above firm i is $b(i) - h_i - 1$. The cheapest subsidy that makes firm i operate, under the belief that only those A-firms placed higher in the ranking will also operate, is the one that makes firm i just indifferent whenever a positive subsidy is needed:

$$\tau_i = \max\left\{\psi - Z_i^{\frac{\theta-1}{\theta}}\left[Z_i^{\frac{\theta-1}{\theta}} + \left(h_i + (1-r)N_B\right) Z^{\frac{\theta-1}{\theta}} + \left(b(i) - h_i - 1 + rN_B\right)\left(Z(1-s)\right)^{\frac{\theta-1}{\theta}}\right]^{\frac{1}{\theta-1}}, 0\right\}. \tag{5}$$

The first term in the maximum operator is just the difference between firm i's cost and its revenue, under the proposed belief. The problem then boils down to finding an optimal ranking $b(i)$, considering the subsidy scheme is determined by (5) for each possible ranking (a permutation problem).

Eq. (5) implies that for a given position in the ranking, the subsidy is decreasing in firm i's productivity, as one should expect. Moreover,

[5] When (4) holds, $q = 0$ is the unique equilibrium outcome. For the planner to find it optimal for all firms to operate, we need conditions (A.19) and (A.20) in Appendix A.9 to hold.

[6] Their proof ultimately does not rely on the particular structure of their model, but only on the fact that their binary-action game features strategic complementarities, as is the case here.

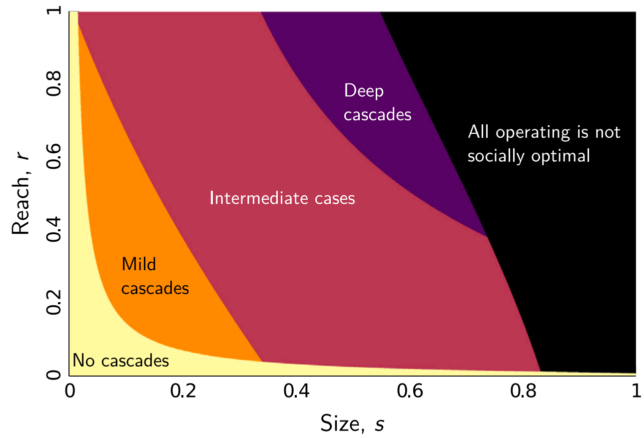

Fig. 1. Classification of shocks, a numerical example. *Notes.* The following parameters were used: $\theta = 2$, $Z = 1$, $\psi = 40.7$, $N_A = 10$, $N_B = 40$.

not only one's ranking but also the composition of those ranked above matters: the subsidy is decreasing in h_i, as firms not hit by the shock create more aggregate demand (convincing a firm to enter after the entry of a firm not hit by the shock is easier than after the entry of a firm hit by the shock, as the former generates more demand). Hence, how other firms are distributed across the ranking affects the subsidy received by each firm.

Lemma 1 also implies that the solution of the problem will always have a dynamic interpretation: The policymaker chooses a ranking and sequentially guarantees the entry of firms, following the order dictated by the ranking. The question of this paper can then be thought of as that of which firms to approach first.

Another implication of Lemma 1 is that even if firms are identical, the subsidies offered to them will differ. The firm placed first in the ranking, can be thought of as the first firm whose entry the policymaker wants to guarantee, and hence it will get a large subsidy so that it operates even under the most pessimistic belief possible. Suppose now that the second firm in the ranking is identical to the first. Now, the second-ranked firm will get a lower subsidy than the first-ranked one, as the policymaker now wants to guarantee it enters after having already established the entry of one firm.

Before solving the problem (finding the optimal ranking), it is useful to define some additional terminology. In light of Lemma 1, I say that $\boldsymbol{\tau} = \left\{\tau_1, \ldots, \tau_{N_A}\right\}$ is the subsidy scheme associated to a ranking $b(i)$ if τ_i is given by (5) for all i. I say that a ranking $b(i)$ is optimal if its associated subsidy scheme is optimal.

3.1. A taxonomy of shocks

With the aid of Lemma 1, I am now ready to characterize the optimal subsidy scheme assuming shocks of different intensities. As mentioned before, I focus on the interesting case of a shock that generates inefficient business closure cascades.[7] I use the following taxonomy to classify such inefficient closure cascades:

- **Mild cascades:** An inefficient cascade is a *mild cascade* if it is optimal for firms hit by the supply shock to enter if they believe all firms not hit will enter;
- **Deep cascades:** An inefficient cascade is a *deep cascade* if all A-firms strictly prefer not to enter even if they believe all others will enter;
- **Intermediate cases:** All inefficient cascades that do not classify as mild or deep.

Note that even if all firms find it optimal not to enter under the belief that all others will do so, the social planner may still be willing to induce entry of all firms, since it internalizes the positive demand externalities among them. That is, deep cascades are situations where the planner's solution is such that no firm is *individually* willing to enter, but they are all better off by collectively agreeing to do so.

To illustrate how inefficient closure cascades relate to the reach and size of the shock, Fig. 1 shows a numerical example of how shocks are classified according to those two dimensions.[8]

3.2. Mild business closure cascades

I first start analyzing shocks that are not so large (lower s) or so widespread (lower r)—but that, of course, are still severe enough to generate a business closure cascade.

Proposition 4. *In a mild cascade, only firms not hit by the supply shock are subsidized under the optimal policy.*

An immediate corollary of that result is that in a mild cascade, it is optimal to rank firms not hit by the shock first and firms hit by the shock last, since this implies that firms not hit by the shock get zero subsidies.[9]

Intuitively, for a given position in the ranking, firms not hit by the supply shock require lower subsidies to be willing to operate (under the belief that only firms placed higher than them in the ranking will enter). Hence, if it is possible to avoid a business closure cascade by only subsidizing such firms, doing so is the cheapest way to achieve the first best. Therefore, firms not hit by the supply shock are ranked first.

To illustrate, Fig. 2 shows the cheapest subsidy that makes entering sequentially dominant under two alternative rankings: one in which

[7] If a cascade is efficient—in the sense that a social planner would like all firms to stop operating—then it is obviously optimal to give no subsidies. If the social planner would like only one group of firms not to exit, then the optimal subsidy consists of giving subsidies only for firms in that group, in the cheapest way that makes it sequentially dominant for firms in this group to enter.

[8] See Appendix A.9 for the algebraic definition of those regions.

[9] Technically, this is not necessarily the only ranking that leads to an optimal subsidy, as one could always rank a low-productivity firm before a high-productivity one as long as, under the associated subsidy, the low-productivity one gets no subsidies and it is optimal not to subsidize all high-productivity firms. Hence, using such a ranking in those circumstances instead of ranking high-productivity firms first would have no impact on the associated subsidy scheme.

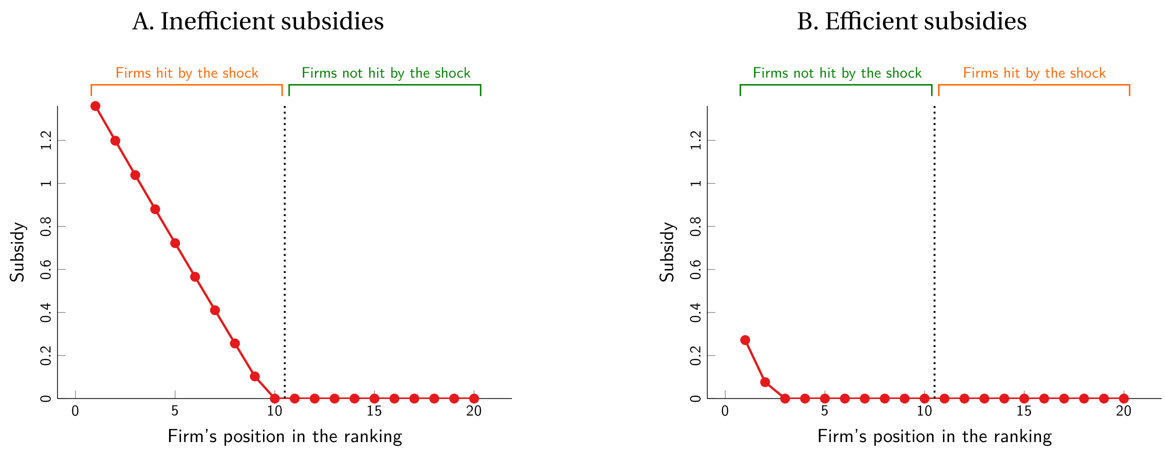

Fig. 2. Examples of subsidies in a mild cascade. *Notes.* The following parameters were used: $\theta = 2.5$, $Z = 1$, $\psi = 11.8$, $N_A = 20$, $N_B = 40$, $s = 0.15$, $r = 0.5$. The left panel shows the subsidy scheme associated to a ranking that places firms hit by the shock first. The right panel shows the optimal subsidy, which is the subsidy scheme associated to a ranking that places firms not hit by the shock first. The subsidy is measured in units of the composite good.

firms hit by the shock are ranked first and one in which those firms are ranked last. As one can see, the former is more expensive for two reasons. First, the policymaker needs to pay subsidies for a larger number of firms, as low-productivity firms create less demand for others in the economy, and therefore subsidizing them creates less *indirect subsidies*—defined as the increase in incentives to enter that firms gain from knowing others will operate. Second, it must pay higher subsidies for the firms that are subsidized: Low-productivity firms have lower incentives to operate and hence require larger subsidies for a given belief about demand.

To get an analytical expression for the optimal subsidy in a mild cascade, without loss of generality, assume firms $1, \ldots, (1-r) N_A$ were not hit by the shock and firms $(1-r) N_A + 1, \ldots, N_A$ were hit. Then, in a mild cascade, the ranking $b(i) = i$ is optimal, which is a corollary of Proposition 4. Under that optimal ranking, firms hit by the shock will receive zero subsidies, and using (5), the subsidy for firms not hit will be given by

$$\tau_i^* = \max \left\{ \psi - Z \left[i + (1-r) N_B + r N_B (1-s)^{\frac{\theta-1}{\theta}} \right]^{\frac{1}{\theta-1}}, 0 \right\}.$$

That is, the subsidy for firms not hit is decreasing in their position in the ranking (i), possibly reaching zero for some of the last high-productivity firms.

3.3. Deep business closure cascades

I now analyze the case of more severe/widespread shocks.

Proposition 5. *Consider a deep cascade. Then, under the optimal policy:*

1. *If $\theta > 2$, firms not hit by the supply shock are ranked first;*
2. *If $\theta < 2$, firms hit by the supply shock are ranked first;*
3. *If $\theta = 2$, any ranking is optimal.*

In a deep cascade, all A-firms need to get subsidies to implement the first best. Hence, to obtain the (optimal) subsidy associated to the optimal ranking of Proposition 5, one only needs to compute the subsidy that makes each firm indifferent between entering or not when only A-firms ranked above it enter. In Appendix A.10, I present the expressions for the optimal subsidy, and Fig. 3 shows examples of optimal subsidy schemes for different values of θ.

A corollary of Proposition 5 is that for $\theta < 2$, firms hit by the supply shock get larger subsidies than firms not hit. This is because they require lower subsidies than firms not hit, for a given position in the ranking, and also because they are ranked above firms not hit.

For $\theta > 2$, it is not obvious which firms get larger subsidies. Firms not hit are ranked first, which makes them receive subsidies generous enough for them to enter under pessimistic expectations. However, since they are more productive, they end up needing less subsidies than firms hit by the shock, for a given expectation about the decision of others. Still, in equilibrium under the optimal policy, firms not hit by the shock make more profits in a deep cascade when $\theta > 2$. To see that, denote by $\tilde{Y}_i$ the aggregate demand when firm i and A-firms ranked above it enter, and the remaining A-firms do not. Denote by $\overline{Y}$ the output when all firms enter. Then, under the subsidy scheme $\tau = \left\{ \tau_1, \ldots, \tau_{N_A} \right\}$ associated to the considered ranking, the profit of firm i can be written as

$$Z_i^{\frac{\theta-1}{\theta}} \overline{Y}^{\frac{1}{\theta}} - \psi + \tau_i = Z_i^{\frac{\theta-1}{\theta}} \left(\overline{Y}^{\frac{1}{\theta}} - \tilde{Y}_i^{\frac{1}{\theta}} \right),$$

where in the equality above I used that fact that the subsidy associated to the considered ranking makes firms indifferent between entering or not under the belief that $Y = \tilde{Y}_i$, and hence $\tau_i = \psi - Z_i^{\frac{\theta-1}{\theta}} \tilde{Y}_i^{\frac{1}{\theta}}$. Therefore, when $\theta > 2$, under the optimal subsidy, firms not hit by the shock have larger profits in equilibrium both because they have larger productivity Z_i and because they have a lower $\tilde{Y}_i$, as they are ranked first.

Why does the optimal ranking depend on the elasticity of substitution θ? Placing a high-productivity firm (one not hit by the shock) up in the ranking, instead of a low-productivity one, has a clear benefit: High-productivity firms generate more aggregate demand externalities when computing the subsidy for those placed lower in the ranking, as their entry generates a larger increase in Y. Hence, placing a high-productivity firm up in the ranking instead of a low-productivity one allows the policymaker to pay lower direct subsidies to the remaining ones—it creates more indirect subsidies. This suggests that high-productivity firms should always be ranked before low-productivity ones.

However, the argument so far misses that there is an opportunity cost of placing high-productivity firms up in the ranking. More productive firms have revenues that are more sensitive to aggregate demand, as $\frac{dR_i}{dY} = \frac{1}{\theta} Z_i^{\frac{\theta-1}{\theta}} Y^{\frac{1-\theta}{\theta}}$ is increasing in Z_i. That is, since they produce more, their revenue reacts more to changes in their demand schedule. Placing a productive firm up in the ranking means placing a firm that is less sensitive to aggregate demand (a low-productivity one) lower in the ranking, which compromises the effectiveness of indirect subsidies. This opportunity cost would disappear if both low- or high-productivity firms placed at the end of the ranking required no subsidy. But in a deep cascade this never happens, all firms must get some subsidy to operate (remember that in a deep cascade firms are not willing to enter, absent subsidies, even if they believe all will operate).

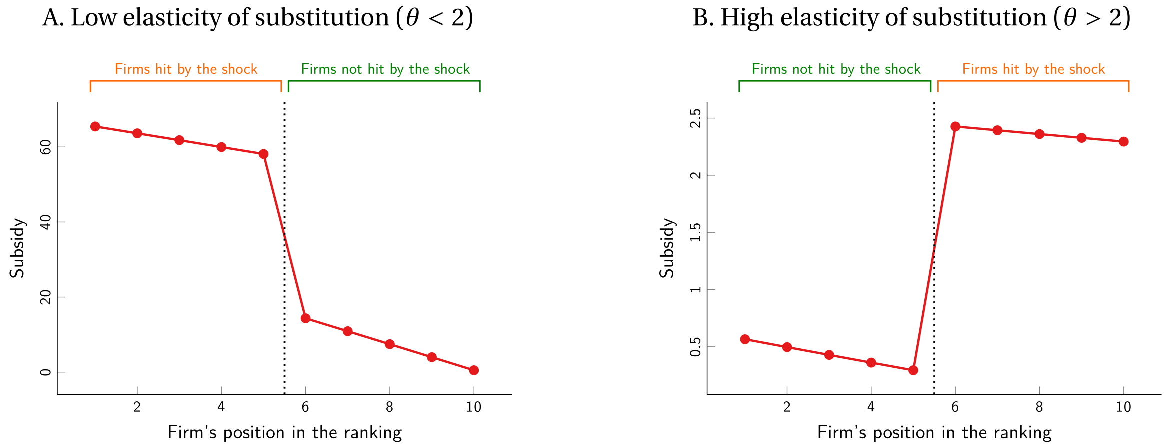

Fig. 3. Examples of optimal subsidies in a deep cascade. *Notes.* The following parameters were used: $\theta = 1.8$, $Z = 1$, $\psi = 170$, $N_A = 10$, $N_B = 60$, $s = 0.5$, $r = 0.5$ (left panel) and $\theta = 3$, $Z = 1$, $\psi = 7.8$, $N_A = 10$, $N_B = 60$, $s = 0.4$, $r = 0.5$ (right panel). The subsidy is measured in units of the composite good.

The optimal subsidy policy depends on whether the benefit of placing high-productivity firms higher in the ranking outweighs the opportunity cost of doing so. The question that remains is why the opportunity cost outweighs the benefit for low values of θ.

Consider $N_A > 2$ and a value of θ close to one. In such case, the complementarity across goods is very strong. Placing a high-productivity firm in the first position of the ranking instead of a low-productivity one will have little impact on the subsidy required for the firm placed second in the ranking, as this one will require almost all of its costs to be subsidized to enter under the belief that only one firm will enter. This is because goods are strong complements, and the value of one's variety is highly dependent on the decision of *all* firms. In fact, in the extreme case as $\theta \to 1$, Y becomes a Cobb–Douglas aggregator of all varieties and the subsidy of all but the last firm in the ranking will converge to the fixed cost ψ, regardless of the identity of the firms placed before them. In other words, with strong complementarities, when just a few firms enter, the entry of an additional firm has little effect on aggregate demand Y, be it a high- or low-productivity one. The benefit of placing a high-productivity firm up in the ranking is small, and the regulator is better off placing high-productivity firms in the last positions of the ranking—where subsidies are computed assuming all firms will enter—since those are more sensitive to the now high aggregate demand Y that will be used to compute the subsidy for the last ranked firm. As θ increases, goods are less complementary, and the differences in aggregate demand created by both types of firms for positions up in the ranking become relevant.

3.3.1. *Why $\theta = 2$ is the relevant threshold for the optimal ranking*

Having discussed why the optimal subsidy in a deep cascade depends on whether the elasticity of substitution is large or small, I now try to give the reader some insight on why the relevant cutoff for θ equals two in the main model. As will be shown, ultimately, the optimal ranking in a deep cascade depends on whether firms' revenue is a concave or convex function of the number of firms that operate, which in my model boils down to whether $\theta > 2$ or $\theta < 2$, respectively. Unfortunately, the discussion below needs to get a bit more technical, and the reader less interested in details may want to skip this section.

It is useful to start sketching the proof for the optimal subsidy in a deep cascade. Consider a deep cascade with $N_A = 3$ and $r = 2/3$, so that after the shock there is one high-productivity firm and two low-productivity ones. To ease notation, let $\tilde{s} = (1-s)^{\frac{\theta-1}{\theta}}$ and assume $Z = 1$ throughout this section. If q_L and q_H low- and high-productivity firms, respectively, operate, one can write the revenue of an operating firm i as

$$\tilde{R}\left(Z_i, X\right) = Z_i^{\frac{\theta-1}{\theta}} X\left(q_L, q_H\right),$$

where

$$X\left(q_L, q_H\right) = \left[q_H + (1-r)N_B + \left(q_L + rN_B\right)\tilde{s}\right]^{\frac{1}{\theta-1}}.$$

$X\left(q_L, q_H\right)$ is equal to $Y^{\frac{1}{\theta}}$ and hence represents the demand shifter in (2). Note that

$$\frac{\partial X\left(q_L, q_H\right)}{\partial q_L} = \tilde{s}\frac{\partial X\left(q_L, q_H\right)}{\partial q_H} \tag{6}$$

and

$$\frac{\partial X\left(q_L, q_H\right)}{\partial q_H} = \frac{1}{\theta-1} X\left(q_L, q_H\right)^{2-\theta}. \tag{7}$$

Hence, when $\theta = 2$, $X\left(q_L, q_H\right)$ is linear in both arguments. If $\theta > 2$, it is strictly concave in both arguments and if $\theta < 2$, it is strictly convex. Importantly, the derivatives above only depend on X and on exogenous parameters, and whether θ is above or below two determines whether those derivatives increase or decrease with X, which, as will be shown, is what drives the result in Proposition 5.

Assume that initially the policymaker decided to rank the low-productivity firms first (in positions 1 and 2) and is considering whether it could reduce its spending by ranking the high-productivity one first instead. If it does so, the total change in spending under the subsidy scheme associated to the new ranking is

$$\begin{aligned}\Delta = {} & \underbrace{\tilde{s}X(1,0) - X(0,1)}_{\substack{<0 \\ (\uparrow \text{ demand for 1st-ranked} \\ \text{and } \uparrow \text{ sensitivity to demand})}} + \underbrace{\tilde{s}\left[X(2,0) - X(1,1)\right]}_{\substack{<0 \\ (\uparrow \text{ demand for 2nd-ranked})}} \\ & + \underbrace{X(2,1)(1-\tilde{s})}_{\substack{>0 \\ (\text{same demand for 3rd-ranked,} \\ \text{but } \downarrow \text{ sensitivity to demand})}} .\end{aligned} \tag{8}$$

Note that the effect of this deviation on the subsidy paid to the first- and second-ranked firms go in an opposite direction than the effect for the third-ranked one, and so far it is unclear which effects dominate. After adding and subtracting $X(1,1)$ in (8), and using (6), Δ can be rewritten as

$$\begin{aligned}\Delta &= X(2,1) - X(1,1) + X(1,1) - X(0,1) \\ &\quad - \tilde{s}\left[X(1,1) - X(1,0) + X(2,1) - X(2,0)\right] \\ &= \int_0^1 \frac{\partial X(1+v,1)}{\partial q_L}dv + \int_0^1 \frac{\partial X(v,1)}{\partial q_L}dv \\ &\quad - \tilde{s}\left[\int_0^1 \frac{\partial X(1,v)}{\partial q_H}dv + \int_0^1 \frac{\partial X(2,v)}{\partial q_H}dv\right] \\ &= \tilde{s}\left\{\int_0^1 \left[\frac{\partial X(1+v,1)}{\partial q_H} - \frac{\partial X(2,v)}{\partial q_H}\right]dv\right.\end{aligned}$$

$$+\int_0^1 \left[\frac{\partial X(v,1)}{\partial q_H} - \frac{\partial X(1,v)}{\partial q_H}\right] dv \Bigg\}.$$

By (7), if $\theta = 2$, the demand shifter $X(q_L, q_H)$ is linear, and therefore the derivatives above are equal to the same constant, and hence $\Delta = 0$. If $\theta > 2$, $\frac{\partial X(q_L,q_H)}{dq_H}$ is a decreasing function of $X(q_L, q_H)$, and hence, the terms in brackets are negative—since $X(1+v,1) > X(2,v)$ and $X(v,1) > X(1,v)$, for all $v \in (0,1)$. If $\theta < 2$, we have the opposite result. Hence, such deviation is only profitable if $\theta > 2$, as expected, since in that case the optimal ranking requires placing the firm not hit by the shock in the first position.

Intuitively, suppose the policymaker has fixed positions $1, 2, \ldots, J$ in the ranking and is deciding whether to place a firm hit or not hit by the shock in some interior position $J+1$. Placing a high-productivity firm in that slot, instead a of a firm hit by the shock, has costs and benefits, as previously discussed. The cost is that doing so implies giving up placing a firm that is more sensitive to demand down the ranking. That cost should then be larger the greater the gap between firms not hit and hit by the shock in terms of the sensitivity of revenue to the demand shifter X. Given $Z = 1$, that gap boils down to

$$\frac{\partial \tilde{R}(1,X)}{dX} - \frac{\partial \tilde{R}(1-s,X)}{dX} = (1-\tilde{s}) > 0.$$

The benefit is that a high-productivity firm creates more demand (it has a stronger effect on X) for those down in the ranking, since, by (6) and (7),

$$\frac{\partial X(q_L,q_H)}{\partial q_H} - \frac{\partial X(q_L,q_H)}{\partial q_L} = \frac{1}{\theta-1} X(q_L,q_H)^{2-\theta}(1-\tilde{s}) > 0. \tag{9}$$

If $\theta > 2$, this benefit is decreasing in X and thus tends to be relatively large for the first positions in the ranking—when the subsidy is computed assuming low values for q_L and q_H—which tilts the policymaker towards placing the high-productivity firms in the first positions of the ranking. If $\theta < 2$, the opposite intuition applies.

To sum up, what drives the optimal ranking is whether the benefit in (9) is increasing or decreasing in the demand shifter X, or alternatively, whether a firm's demand (or revenue) is a concave or convex function of the number of other firms that operate, which in my model boils to whether the elasticity of substitution is above or below two.

3.4. Intermediate cases

I now study the inefficient closure cascades that cannot be classified as deep or mild cascades. In fact, for $\theta > 2$ the result is the same across all cases:

Proposition 6. *If $\theta > 2$, the firms not hit by the supply shock are always ranked first among those receiving positive subsidies, under the optimal policy. In particular, it is optimal to rank all firms not hit by the shock first.*

This result is perhaps unsurprising, given the previous results: with $\theta > 2$, for both mild and deep cascades, ranking firms not hit by the shock up in the ranking is optimal, and hence one should expect similar results for the case in between.

For $\theta < 2$, the optimal ranking may be non-monotonic in the sense that firms hit and not hit may alternate positions. Given that the results for deep and mild cascades go in opposite directions when $\theta < 2$, the intermediate cases are a mixture of the two, yielding the non-monotonicity. Fig. 4 shows a numerical example in which the optimal ranking is non-monotonic. Since there is no theoretical result that pins down an optimal ranking in an intermediate case with $\theta < 2$, I numerically find the optimal rank in that case by checking all possible alternatives. More precisely, for each possible ranking $b(i)$, I numerically compute the subsidy associated to it using (5), and then find the one that minimizes total spending (which is doable numerically as long as N_A is not too large).

Table 1
Summary of results.

	Higher substitution ($\theta > 2$)	Lower substitution ($\theta < 2$)
Deep cascades	Firms **not hit** by the shock: • Are ranked first • Get larger rents	Firms **hit** by the shock: • Are ranked first • Get larger subsidies
Mild cascades	Only subsidize firms **not hit**	Only subsidize firms **not hit**
Intermediate cases	Firms **not hit** by the shock: • Are ranked first • Get larger rents	Optimal ranking can be non-monotonic

As a final remark on optimal subsidies, the optimal subsidy in Fig. 4, and in other figures in the paper, implies one has to give a different subsidy to each firm, as to explore indirect subsidies efficiently. One may wonder how the optimal subsidy compares to simpler subsidy schemes that either pay the same subsidy to all firms, or pay the same subsidies to all firms within the same group (the groups being those A-firms hit and not hit by the shock).

To shed light on this, Fig. 5, using the same parameters of Fig. 4, compares the average subsidy paid to all A-firms, as well as this average within the groups "hit" and "not hit", when: (i) The policymaker is unconstrained and can set a different subsidy for each firm, which is the case discussed throughout the paper, and hence represents the subsidy scheme of Fig. 4; (ii) The policymaker is constrained to give the same subsidy to all A-firms of the same group; (iii) The policymaker is constrained to give the same subsidy to all A-firms. In all cases, as assumed throughout the text, the policymaker chooses the subsidy scheme that yields the lowest spending among all schemes that implement the first best as a unique equilibrium outcome (but now it may face additional restrictions).[10]

As one should expect, average spending increases as the policymaker becomes more constrained. Interestingly, in that example, the unconstrained policymaker pays slightly higher subsidies, on average, for firms hit by the shock than the constrained ones—as Fig. 4 shows, two firms hit by the shock are ranked first under the optimal policy. It does so because it allows it to pay much lower subsidies for the firms not hit by the shock.

Having discussed the optimal subsidy in all types of inefficient cascades, I summarize the main findings of the paper in Table 1.

3.5. Dynamic implementation when size of shock is unknown

Suppose the policymaker is uncertain about whether the economy is in a deep cascade, in a mild cascade, or in an intermediate case. All that it observes is that after a shock, a group of firms (A-firms) stopped operating. The policymaker believes that firms coordinate on the worst equilibrium. For $\theta > 2$, such policymaker can easily implement the optimal policy eventually by observing firms' reactions to the subsidy. In light of the previous results, it must follow these steps:

1. Offer a subsidy to a firm not directly hit by the shock, just enough to induce entry.
2. If all firms enter after the last subsidy is announced, stop; otherwise move to step 3.

[10] To numerically compute the optimal policy in each case, I use the fact that Lemma 1 still holds when the policymaker faces those additional restrictions. For each possible ranking, I then compute the lowest subsidy that makes the best ranked low- and high-productivity A-firm willing to enter when they believe only those ranked above them will also do so, which I label τ_L and τ_H, respectively . Under (ii), I then assign the subsidy τ_L for all A-firms hit by the shock and τ_H for all the firms not hit. Under (i), I assign a subsidy equal to $\max\{\tau_L, \tau_H\}$ to all A-firms. I then compare spending across all possible rankings.

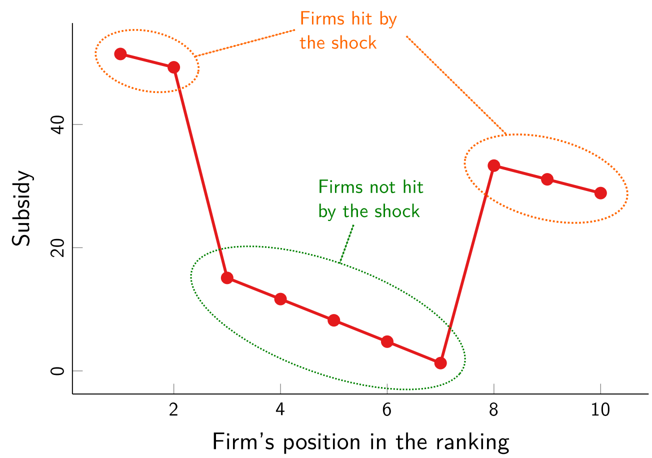

Fig. 4. Example of optimal ranking for $\theta < 2$ and intermediate case. *Notes.* The following parameters were used: $\theta = 1.8$, $Z = 1$, $\psi = 170$, $N_A = 10$, $N_B = 60$, $s = 0.4$, $r = 0.5$. The subsidy is measured in units of the composite good.

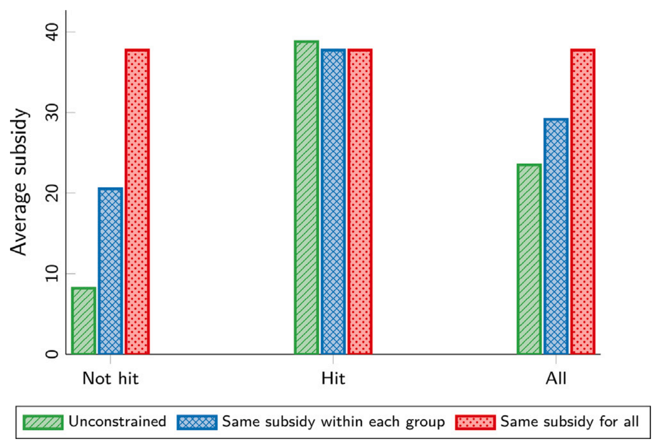

Fig. 5. Average spending under the optimal policy for different restrictions on the policymaker. *Notes.* The left and middle groups of bars represent the average subsidy among firms not hit and hit by the shock, respectively. The rightmost group of bars represents the average subsidy for all firms. Parameters are the same used in Fig. 4. The subsidy is measured in units of the composite good.

3. Offer a subsidy just enough to induce entry of a firm not hit by the shock, if there is still one not receiving subsidies. If all firms not hit by the shock are already receiving subsidies, offer subsidies for a firm hit by the shock. Go back to step 2.

In short, the policymaker should start offering subsidies for firms not hit by the shock, and just move to firms hit by the shock when that is not enough to induce entry. For $\theta < 2$, such scheme is not available. In that case, to implement the optimal policy one needs to have more information about the size of the shock.

4. Extensions

I now discuss some extensions of the main model.

4.1. Many levels of productivity

In the model presented, after the shock there are only two possible levels of productivity, low and high, depending on whether a firm was hit or not by the shock. Now consider a situation where each firm has a possibly different productivity Z_i and suppose that there is an equilibrium in which all A-firms exit, but a policymaker would like to implement $q = N_A$ as a unique equilibrium outcome, spending as little as possible.

If the economy is in a deep cascade, in the sense that no A-firm is willing to enter even if it believes $q = N_A$, then the result in Proposition 5 continues to hold in the following sense: (i) For $\theta > 2$, firms are ranked according to their productivity under the optimal policy, that is, firms with higher productivity are ranked above firms with lower productivity; (ii) For $\theta < 2$, firms with lower productivity are ranked above firms with higher productivity. This result holds because the proof of Proposition 5 does not use the fact that there are only two productivity levels, only the fact that all firms require positive subsidies to operate.[11]

For an economy that is not in a deep cascade, I can show the following result, which requires additional proof:

Proposition 7. *Suppose there is a $\bar{k} \in \{1, \ldots, N_A - 1\}$ such that: (i) $Z_i = Z^*$ if $i \leq \bar{k}$, and $Z_i < Z^*$ if $N_A \geq i > \bar{k}$; (ii) Firms $i > \bar{k}$ are willing to enter once they anticipate all firms $i \leq \bar{k}$ will enter. Then, to implement $q = N_A$ as a unique equilibrium outcome at the lowest possible cost, one should pay no subsidy for firms $i > \bar{k}$.*

This result is analogous to Proposition 4 if we interpret the firms belonging to the low-productivity group ($i > \bar{k}$) as those hit by the

[11] For instance, in the proof for $\theta > 2$, I assume, by contradiction, that there is a firm with lower productivity ranked above a firm with higher productivity under the optimal policy. Then, I swap the position of those two firms and show that one can implement $q = N_A$ in the unique equilibrium sense at a lower cost after the swap. The opposite is done for $\theta < 2$. Note that the arguments in the proof of Lemma 1 also work in the case considered here.

shock and the ones belonging to the high-productivity group ($i \leq \bar{k}$) as the ones not hit by the shock. However, now firms hit by the shock are allowed to be hit with different intensities, as their productivity can differ.

4.2. Intensive margin

For simplicity, and given the focus on the paper on correcting inefficiencies in firms' operating decisions (extensive margin), the main model uses a production technology that implies that the only relevant decision of firms is whether or not to operate. That delivers tractability when discussing the optimal subsidies, as the only source of inefficiency in the model regards firms' entry decisions. However, for robustness, I show that the main results still hold in a less parsimonious model where firms use labor (intensive margin) and have a more standard production function with decreasing returns to scale. The model with labor and the results regarding the optimal subsidy scheme are presented in detail in Appendix B.

4.3. Heterogeneous costs

For simplicity, it was assumed throughout the text that all firms face the same fixed cost ψ. I now briefly discuss the direction in which my results could go if one relaxed this assumption, leaving a more detailed analysis for future work.

All the results regarding the optimal ranking in a deep cascade (Proposition 5) continue to hold if firms have different fixed costs (an inspection of the proof of Proposition 5 shows that any heterogeneity in fixed costs would cancel out when computing the relevant variables for that proof). Of course, the subsidy received by each firm would change: the higher the fixed cost, the higher the subsidy, for a given position in the ranking. More precisely, if we take an economy in an inefficient deep cascade and marginally increase the fixed cost of a given firm, its subsidy would increase one to one with the increase in the cost under the optimal policy.

Outside of a deep cascade, the results could depend in a more meaningful way on the heterogeneity in fixed costs. For instance, if firms hit by the shock face sufficiently lower fixed costs than those not hit—which is perhaps an unnatural assumption in the examples of uneven recessions discussed in the introduction—it could be the case that incentivizing the entry of firms hit by the shock is cheaper, by the simple reason that those have lower operating costs to start with. When that is so, I conjecture that for shocks with sufficiently small size and reach, but that still trigger a cascade, the policymaker will only want to guarantee the entry of some firms hit by the shock, since this is cheaper and may be enough to guarantee the entry of the remaining ones. In that case, what would be driving the result is not the fact that some firms are directly hit by a productivity shock and others are not, but instead, the heterogeneity in fixed costs.

If firms hit by the supply shock face higher fixed costs than the ones not hit—which is perhaps more natural than the opposite in the examples used to motivate this paper—then this additional dimension of heterogeneity would further provide incentives for policymakers not to subsidize those hit by the shock in a mild cascade, as guaranteeing the entry of those would be even more expensive, reinforcing the results of the main model.

5. Concluding remarks

This paper studies the optimal distribution of firm subsidies when supply shocks hit some firms and spill over to other firms due to demand externalities. The broader message is that the optimal main target of those policies depends crucially on: (i) The size and the reach of the supply shock; (ii) The degree of substitution across firms' goods. For smaller shocks, subsidies should focus on firms that were only indirectly affected by an uneven supply shock, that is, those that have a demand problem but not a supply-side problem. For larger shocks, the optimal policy depends on the strength of complementarities, and a characterization is provided in terms of the elasticity of substitution across goods.

I conclude with some reflections on how my results can shed light on a recent policy debate. Some opponents of the Paycheck Protection Program (PPP) in the US often criticized that policy for distributing subsidies to firms that were not hit as hard by the COVID-19 shock, at the expense of providing lower assistance to those in greater need.[12] The analysis presented in this paper shows that it is not obvious that the firms most hit by a supply shock should be the main recipient of funds when there are demand spillovers among producers. As demonstrated, it is likely that the optimal policy implies giving some subsidies to firms not directly hit by a supply shock.[13] In some cases, in fact, the optimal policy implies giving subsidies *only* to firms that face no supply shock, but that were adversely affected through demand spillovers (this happens in a mild cascade).

However, it is important to clarify that the analysis in this paper does not support the idea of subsidizing firms that benefit from a given shock. Rather, it lends support to the view that prioritizing firms that suffer through demand spillovers, and not directly from supply disruptions, can be desirable.

Declaration of competing interest

The author declares that he has no relevant or material financial interests that relate to the research described in this paper.

Appendix A. Technical appendix

To avoid repetition, I introduce some notation and terminology that will be referenced in different proofs. In what follows, $R\left(\tilde{q}_L, \tilde{q}_H, Z_i\right)$ refers to the revenue, after the shock, of a firm i, with productivity $Z_i \in \{(1-s)Z, Z\}$, when it enters and a quantity $\tilde{q}_L$ and $\tilde{q}_H$ of *other* firms with low and high productivity, respectively, enter. That is,

$$R\left(\tilde{q}_L, \tilde{q}_H, Z_i\right) = Z_i^{\frac{\theta-1}{\theta}} \left\{ Z_i^{\frac{\theta-1}{\theta}} + \left[\tilde{q}_H + (1-r) N_B\right] Z^{\frac{\theta-1}{\theta}} + \left(\tilde{q}_L + r N_B\right) [Z(1-s)]^{\frac{\theta-1}{\theta}} \right\}^{\frac{1}{\theta-1}}. \tag{A.1}$$

Note that $R\left(\tilde{q}_L, \tilde{q}_H, Z_i\right)$ is strictly increasing in all arguments.

A.1. Proof of Proposition 1

We can write the aggregate demand Y as a function of q:

$Y(q) = Z\left(q + N_B\right)^{\frac{\theta}{\theta-1}}$.

For $q = N_A$ to be an equilibrium, we need A-firms to be willing to enter if all others enter, which requires

$Z^{\frac{\theta-1}{\theta}} Y\left(N_A\right)^{\frac{1}{\theta}} = Z N^{\frac{1}{\theta-1}} \geq \psi$.

Hence, if $Z \geq \psi N^{-\frac{1}{\theta-1}}$, $q = N_A$ is an equilibrium outcome. Similarly, $Z^{\frac{\theta-1}{\theta}} Y(1)^{\frac{1}{\theta}} = Z\left(1 + N_B\right)^{\frac{1}{\theta-1}} \leq \psi$ implies that firms are willing not to enter if no other A-firm enters, so whenever $Z \leq \psi\left(1 + N_B\right)^{-\frac{1}{\theta-1}}$, $q = 0$ is an equilibrium outcome. The stated results then follow. □

[12] Other more targeted programs launched after the PPP, such as the Restaurant Revitalization Fund, can be seen a response to those criticisms.

[13] This happens even in the cases least favorable to firms not hit by the shock analyzed. For instance, even in a deep cascade with $\theta < 2$, firms not hit by the shock get positive subsidies, although they are not prioritized (they are ranked last). Also, in the intermediate case analyzed in Fig. 4, firms not hit by the shock get positive subsidies as well.

A.2. Proof of Proposition 2

After the shock, we can write aggregate demand Y as a function of the number of low- and high-productivity A-firms that enter, denoted by q_L and q_H, respectively:

$$Y\left(q_L, q_H\right) = Z\left[q_H + (1-r)N_B + \left(q_L + rN_B\right)(1-s)^{\frac{\theta-1}{\theta}}\right]^{\frac{\theta}{\theta-1}}. \tag{A.2}$$

Note that whenever low-productivity firms are willing to enter, it is strictly optimal for high-productivity firms to enter. Conversely, whenever high-productivity firms are willing not to enter, it is strictly optimal for low-productivity firms not to enter. Hence, an equilibrium with $q = 0$ exists if, and only if,

$$Z^{\frac{\theta-1}{\theta}} Y(0,1)^{\frac{1}{\theta}} = Z\left[1 + (1-r)N_B + rN_B(1-s)^{\frac{\theta-1}{\theta}}\right]^{\frac{1}{\theta-1}} \leq \psi.$$

A sufficient condition for $q = 0$ to be the unique equilibrium after an uneven shock (s, r) is that high-productivity A-firms are not willing to enter even when they think all other firms will operate:

$$Z^{\frac{\theta-1}{\theta}} Y\left(rN_A, (1-r)N_A\right)^{\frac{1}{\theta}} = Z\left[(1-r)N + rN(1-s)^{\frac{\theta-1}{\theta}}\right]^{\frac{1}{\theta-1}} < \psi.$$

To show that those conditions can be satisfied for some parameters and shocks, consider $r = 1/2$. In this case, condition (3) can be written as

$$Z\left[1 + \frac{N_B}{2} + \frac{N_B}{2}(1-s)^{\frac{\theta-1}{\theta}}\right]^{\frac{1}{\theta-1}} \leq \psi.$$

As $s \to 1$, this condition becomes $Z \leq \psi\left(1 + N_B/2\right)^{-\frac{1}{\theta-1}} \equiv \overline{Z}$. Note that $\overline{Z}$ is above the bound for the economy to be in good times ($\widetilde{Z}$), and hence for $Z \in \left(\widetilde{Z}, \overline{Z}\right)$, we can construct an uneven shock that hits half of the firms in good times and delivers $q = 0$ as an equilibrium outcome, by assuming a size s large enough. To have $q = 0$ as the unique equilibrium outcome, we should check whether (4) holds, which with $r = 1/2$ boils down to

$$Z\left[\frac{N}{2} + \frac{N}{2}(1-s)^{\frac{\theta-1}{\theta}}\right]^{\frac{1}{\theta-1}} < \psi,$$

and as $s \to 1$ it becomes $Z < \psi(N/2)^{-\frac{1}{\theta-1}} \equiv \underline{Z}$. Now it is not obvious that $\underline{Z} > \widetilde{Z}$, and for that to be true we need $N_B > N_A - 2$, that is, the number of B-firms relative to A-firms must be sufficiently large. If that is the case, then for s sufficiently large and $Z \in \left(\widetilde{Z}, \underline{Z}\right)$, we get $q = 0$ as the unique equilibrium outcome after a shock that hits half of the firms. □

A.3. Proof of Proposition 3

Consider an equilibrium with $q = N_A$ and another scenario with $q < N_A$. Aggregate demand Y is larger when $q = N_A$ than in the latter scenario, and since $q = N_A$ is an equilibrium outcome, all firms must have profits larger or equal to zero in that case. Hence, all firms that operate in both scenarios have strictly larger profits in the equilibrium with $q = N_A$ than when $q < N_A$, since their revenue is $Z_i^{\frac{\theta-1}{\theta}} Y^{\frac{1}{\theta}}$ (the remaining ones are at least weakly better off with $q = N_A$). Now note that social welfare is simply the sum of firms' profits: $\sum_{i=1}^{N} e_i Z_i^{\frac{\theta-1}{\theta}} Y^{\frac{1}{\theta}} - q\psi = Y - q\psi$, concluding the proof of the first statement.

To prove the second statement, it suffices to construct an example where $q = 0$ is the unique equilibrium outcome, but the planner prefers $q = N_A$. Let $N_A = N_B = 2$, $r = 1/2$ and assume that firms 1 and 3 were hit by the shock. Welfare can be written as a function of the entry decisions $e_i \in \{0,1\}$ of firms 1 and 2:

$$W\left(e_1, e_2\right) = Y\left(e_1, e_2\right) - \left(e_1 + e_2\right)\psi,$$

where in this proof Y is written as a function of e_1 and e_2:

$$Y\left(e_1, e_2\right) = Z\left[\left(e_1 + 1\right)(1-s)^{\frac{\theta-1}{\theta}} + e_2 + 1\right]^{\frac{\theta}{\theta-1}}.$$

The planner strictly prefers all firms to enter if: (i) $W(1,1) > W(0,1)$; (ii) $W(1,1) > W(1,0)$; (iii) $W(1,1) > W(0,0)$. A sufficient condition for $q = 0$ to be the unique equilibrium outcome is that firm 2 does not want to enter when firm 1 enters, that is,

$$Z^{\frac{\theta-1}{\theta}} Y(1,1)^{\frac{1}{\theta}} < \psi. \tag{A.3}$$

Further assuming $\theta = 1.5$, $\psi = 8.7$, $s = 0.9$, $Z = 1$ and plugging those in the expressions above, one can verify that both (A.3) and (i), (ii), (iii) above are satisfied. Moreover, under such parameters we have that, before the shock, when an A-firm enters and the other does not, the revenue $Z^{\frac{\theta-1}{\theta}} Y^{\frac{1}{\theta}}$ is larger than the cost ψ, so the assumption that the economy was in good times before the shock holds. □

A.4. Proof of Lemma 1

The arguments in this proof are similar to those in Lemma 1 in Halac et al. (2020). Consider an economy subject to inefficient cascades and suppose a subsidy scheme $\boldsymbol{\tau}$ that satisfies the unique implementation constraint. Then, it must be that, under such subsidy, there is at least one A-firm that is willing to operate even if it believes all other A-firms will not do so. If this was not the case, then there would be an equilibrium in which no A-firm enters for a subsidy scheme $\boldsymbol{\tau}+\varepsilon\boldsymbol{u}$ with ε sufficiently small, contradicting the unique implementation constraint. Letting k_1 be the index of such firm, we then have

$$R\left(0, 0, Z_{k_1}\right) \geq \psi + \tau_{k_1}.$$

Now, there must be an A-firm k_2 that finds it optimal to enter when no other A-firm but firm k_1 enters. If that was not the case, then for $\varepsilon > 0$ sufficiently small and under the subsidy scheme $\boldsymbol{\tau} + \varepsilon\boldsymbol{u}$, there would be an equilibrium in which all A-firms but firm k_1 do not enter, contradicting the unique implementation constraint. Hence, $R\left(1, 0, Z_{k_2}\right) \geq \psi + \tau_{k_2}$ if $Z_{k_1} = (1-s)Z$ and $R\left(0, 1, Z_{k_2}\right) \geq \psi + \tau_{k_2}$ if $Z_{k_1} = Z$. For $N_A > 2$, a similar reasoning implies that there is an A-firm k_3 that finds it optimal to enter when it believes all other A-firms but firms k_1 and k_2 will not enter. This process continues until we reach the last A-firm k_{N_A}, which proves the "if" part of the lemma.

To prove the "only if" part, take a subsidy scheme $\boldsymbol{\tau}$ and suppose that, under such subsidy scheme, there is a bijective function $b : \left\{1, \ldots, N_A\right\} \to \left\{1, \ldots, N_A\right\}$ such that for each $i \in \left\{1, \ldots, N_A\right\}$ firm i finds it optimal to enter when it believes only A-firms i' with $b\left(i'\right) \leq b(i)$ will enter. Then, for any $\varepsilon > 0$, under the subsidy scheme $\boldsymbol{\tau} + \varepsilon\boldsymbol{u}$, there must exist an equilibrium in which all firms enter, since revenue $R\left(\tilde{q}_L, \tilde{q}_H, Z_i\right)$ is strictly increasing in $\tilde{q}_L$ and $\tilde{q}_H$. To show uniqueness, suppose by contradiction that there is another equilibrium in which some A-firms do not enter under $\boldsymbol{\tau} + \varepsilon\boldsymbol{u}$. Denote by $S_{not} \subseteq \left\{1, \ldots, N_A\right\}$ the set of such firms. Then, $b^{-1}(1) \notin S_{not}$, since firm $b^{-1}(1)$ finds it strictly optimal to enter under $\boldsymbol{\tau} + \varepsilon\boldsymbol{u}$ when no other A-firm does. But then, $b^{-1}(2)$ cannot be in such set, since firm $b^{-1}(2)$ finds it optimal to enter when it believes $b^{-1}(1)$ enters. Continuing the process, we conclude that S_{not} is empty, a contradiction. □

A.5. Proof of Proposition 4

I start with some notation. For a given ranking $b(i)$, let $\hat{q}_{iL}(b)$ and $\hat{q}_{iH}(b)$ denote the number of low- and high-productivity firms, respectively, ranked before firm i. That is,

$$\hat{q}_{iL}(b) = \left\{i' : b\left(i'\right) < b(i) \text{ and } Z_i = (1-s)Z\right\}, \tag{A.4}$$

$$\hat{q}_{iH}(b) = \left\{i' : b\left(i'\right) < b(i) \text{ and } Z_i = Z\right\}. \tag{A.5}$$

Without loss of generality, I assume that firms $1, 2, \ldots, (1-r)N_A$ are the ones not hit by the shock in the remainder of this proof.

Consider an economy in a mild cascade. Let b^* be an optimal ranking, with $\boldsymbol{\tau}^* = \left\{\tau_1^*, \dots, \tau_{N_A}^*\right\}$ being the (optimal) subsidy scheme associated to it. Suppose by contradiction that a firm k_1 with low productivity receives a positive subsidy, $\tau_{k_1}^* > 0$. Consider the following alternative ranking: $b^{**}(i) = i$, that is, high-productivity firms are ranked first. Let $\boldsymbol{\tau}^{**} = \left\{\tau_1^{**}, \dots, \tau_{N_A}^{**}\right\}$ be the subsidy scheme associated to it. Under such subsidy scheme, low-productivity firms are not subsidized in a mild cascade: $\tau_i^{**} = 0$, for $i > (1-r)N_A$.

Take some arbitrary $x \in \left\{1, \dots, N_A\right\}$ and let k^* and k^{**} be A-firms such that $b^*(k^*) = x$ and $b^{**}(k^{**}) = x$, respectively. I want to show that $\tau_{k^{**}}^{**} \le \tau_{k^*}^*$. If $\tau_{k^{**}}^{**} = 0$, this is obviously true, so consider now $\tau_{k^{**}}^{**} > 0$, in which case k^{**} is a high-productivity firm. Note that

$$\hat{q}_{k^{**}L}\left(b^{**}\right) + \hat{q}_{k^{**}H}\left(b^{**}\right) = \hat{q}_{k^*L}\left(b^*\right) + \hat{q}_{k^*H}\left(b^*\right), \tag{A.6}$$

$$\hat{q}_{k^{**}H}\left(b^{**}\right) \ge \hat{q}_{k^*H}\left(b^*\right). \tag{A.7}$$

That is, firms k^* and k^{**} have the same number of firms ranked before them, but firm k^{**} have at least as many high-productivity ones. Since $R(a,b,c) \ge R(b,a,d)$, for all $b \ge a$ and $c \ge d$, we then have that

$$\begin{aligned}\tau_{k^{**}}^{**} &= \psi - R\left(\hat{q}_{k^{**}L}\left(b^{**}\right), \hat{q}_{k^{**}H}\left(b^{**}\right), Z\right)\\ &\le \max\left\{\psi - R\left(\hat{q}_{k^*L}\left(b^*\right), \hat{q}_{k^*H}\left(b^*\right), Z_{k^*}\right), 0\right\} = \tau_{k^*}^*.\end{aligned}$$

Now let $x = b^*(k_1)$, so $k^* = k_1$. By the definition of k_1, $\tau_{k^*} > 0$ and k^* is a low-productivity firm. I claim that $\tau_{k^{**}}^{**} < \tau_{k^*}^*$. Again, if $\tau_{k^{**}}^{**} = 0$ this is trivially satisfied. If $\tau_{k^{**}}^{**} > 0$, k^{**} is a high-productivity firm. Since (A.6) and (A.7) hold, and $R(a,b,Z) > R(b,a,(1-s)Z)$, for all $b \ge a$, we then have $\tau_{k^{**}}^{**} < \tau_{k^*}^*$. Hence, subsidy scheme $\boldsymbol{\tau}^{**}$ yields less spending than $\boldsymbol{\tau}^*$ and also satisfies the unique implementation constraint, which contradicts that $\boldsymbol{\tau}^*$ is optimal. □

A.6. Proof of Proposition 5

First statement. Suppose the economy is in a deep cascade, and hence any subsidy scheme that satisfies the unique implementation constraint pays strictly positive subsidies to all A-firms. Denote by l and h the indexes of a low- and high-productivity A-firm, respectively. Assume $\theta > 2$ and consider a ranking $b(i)$ such that $b(h) = b(l) + 1$, and hence l is ranked right before h. Denote by $\hat{q}_L$ and $\hat{q}_H$ the number of low- and high-productivity firms, respectively, ranked before l under the ranking b (that is, those are given by the right-hand side of (A.4) and (A.5), respectively, setting $i = l$).

Let $\boldsymbol{\tau} = \left\{\tau_1, \dots, \tau_{N_A}\right\}$ be the subsidy scheme associated to $b(i)$. Defining $Y\left(q_L, q_H\right)$ as in (A.2) and letting $X\left(q_L, q_H\right) = Y\left(q_L, q_H\right)^{\frac{1}{\theta}}$, we can then write

$$\tau_l = \psi - \left[(1-s)Z\right]^{\frac{\theta-1}{\theta}} X\left(\hat{q}_L + 1, \hat{q}_H\right), \tag{A.8}$$

$$\tau_h = \psi - Z^{\frac{\theta-1}{\theta}} X\left(\hat{q}_L + 1, \hat{q}_H + 1\right). \tag{A.9}$$

Now consider a ranking $\tilde{b}(i)$ that swaps the position of l and h: $\tilde{b}(i) = b(i)$, for $i \notin \{l,h\}$, $\tilde{b}(l) = b(h)$ and $\tilde{b}(h) = b(l)$. Let $\tilde{\boldsymbol{\tau}} = \left\{\tilde{\tau}_1, \dots, \tilde{\tau}_{N_A}\right\}$ be the subsidy associated to $\tilde{b}(i)$. Since for a firm $i \notin \{l,h\}$ the identity of the firms ranked below themselves has not changed, $\tilde{\tau}_i = \tau_i$ for $i \notin \{l,h\}$. For firms l and h we have

$$\tilde{\tau}_l = \psi - \left[(1-s)Z\right]^{\frac{\theta-1}{\theta}} X\left(\hat{q}_L + 1, \hat{q}_H + 1\right), \tag{A.10}$$

$$\tilde{\tau}_h = \psi - Z^{\frac{\theta-1}{\theta}} X\left(\hat{q}_L, \hat{q}_H + 1\right). \tag{A.11}$$

The difference in spending between $\tilde{\boldsymbol{\tau}}$ and $\boldsymbol{\tau}$ is given by

$$\Delta\tau \equiv \sum_{i=1}^{N_A} \tilde{\tau}_i - \sum_{i=1}^{N_A} \tau_i = \left(\tilde{\tau}_l - \tau_l\right) + \left(\tilde{\tau}_h - \tau_h\right),$$

which, using (A.8), (A.9), (A.10) and (A.11), can be written as

$$\begin{aligned}\Delta\tau = {} & \left[(1-s)Z\right]^{\frac{\theta-1}{\theta}}\left[X\left(\hat{q}_L + 1, \hat{q}_H\right) - X\left(\hat{q}_L + 1, \hat{q}_H + 1\right)\right]\\ & + Z^{\frac{\theta-1}{\theta}}\left[X\left(\hat{q}_L + 1, \hat{q}_H + 1\right) - X\left(\hat{q}_L, \hat{q}_H + 1\right)\right],\end{aligned} \tag{A.12}$$

which, by the fundamental theorem of calculus, yields

$$\begin{aligned}\Delta\tau = {} & -\left[(1-s)Z\right]^{\frac{\theta-1}{\theta}} \int_0^1 \frac{\partial X\left(\hat{q}_L + 1, \hat{q}_H + v\right)}{\partial q_H} dv + Z^{\frac{\theta-1}{\theta}}\\ & \cdot \int_0^1 \frac{\partial X\left(\hat{q}_L + v, \hat{q}_H + 1\right)}{\partial q_L} dv.\end{aligned}$$

Using the definition of $X\left(q_L, q_H\right)$, we have that $\frac{\partial X(q_L,q_H)}{\partial q_L} = \frac{1}{\theta-1} X\left(q_L, q_H\right)^{2-\theta} \left[(1-s)Z\right]^{\frac{\theta-1}{\theta}}$ and $\frac{\partial X(q_L,q_H)}{\partial q_H} = \frac{1}{\theta-1} X\left(q_L, q_H\right)^{2-\theta} Z^{\frac{\theta-1}{\theta}}$, which plugging in the equation above and simplifying yields

$$\begin{aligned}\Delta\tau = {} & \frac{Z^{\frac{2(\theta-1)}{\theta}}(1-s)^{\frac{\theta-1}{\theta}}}{\theta-1} \int_0^1 \Big[X\left(\hat{q}_L + v, \hat{q}_H + 1\right)^{2-\theta}\\ & - X\left(\hat{q}_L + 1, \hat{q}_H + v\right)^{2-\theta}\Big] dv.\end{aligned} \tag{A.13}$$

Since we are assuming $\theta > 2$, $X\left(\hat{q}_L + v, \hat{q}_H + 1\right)^{2-\theta} < X\left(\hat{q}_L + 1, \hat{q}_H + v\right)^{2-\theta}$, for all $v \in (0,1)$, and hence $\Delta\tau < 0$. Therefore, ranking the firms according to $b(i)$ and paying the associated subsidy scheme $\boldsymbol{\tau}$ is not optimal, which concludes the proof of the first statement.

Second statement. The proof is similar to that of the first statement, after the redefinition of some objects. Now, $b(i)$ is a ranking such that $b(l) = b(h) + 1$, with $\hat{q}_L$ and $\hat{q}_H$ representing the number of low- and high-productivity firms, respectively, ranked before h under the ranking b. Eqs. (A.8) and (A.9) are naturally replaced by

$$\tau_l = \psi - \left[(1-s)Z\right]^{\frac{\theta-1}{\theta}} X\left(\hat{q}_L + 1, \hat{q}_H + 1\right),$$

$$\tau_h = \psi - Z^{\frac{\theta-1}{\theta}} X\left(\hat{q}_L, \hat{q}_H + 1\right).$$

The considered deviation still swaps h and l in the ranking, but now Eqs. (A.10) and (A.11) should be replaced by

$$\tilde{\tau}_l = \psi - \left[(1-s)Z\right]^{\frac{\theta-1}{\theta}} X\left(\hat{q}_L + 1, \hat{q}_H\right),$$

$$\tilde{\tau}_h = \psi - Z^{\frac{\theta-1}{\theta}} X\left(\hat{q}_L + 1, \hat{q}_H + 1\right).$$

Following the same steps, one gets that the change in spending after the deviation is

$$\begin{aligned}\Delta\tau_{2\text{nd}} = {} & \left[(1-s)Z\right]^{\frac{\theta-1}{\theta}}\left[X\left(\hat{q}_L + 1, \hat{q}_H + 1\right) - X\left(\hat{q}_L + 1, \hat{q}_H\right)\right]\\ & + Z^{\frac{\theta-1}{\theta}}\left[X\left(\hat{q}_L, \hat{q}_H + 1\right) - X\left(\hat{q}_L + 1, \hat{q}_H + 1\right)\right].\end{aligned} \tag{A.14}$$

Denoting by $\Delta\tau_{1\text{st}}$ the expression in the right-hand side of (A.12), we have $\Delta\tau_{2\text{nd}} = -\Delta\tau_{1\text{st}}$. The result then follows.

Third statement. Start from a ranking $b(i)$ with an associated subsidy scheme $\boldsymbol{\tau}$. From the proofs of the first and second statements, we conclude that swapping the position of any two firms in the ranking and offering the respective associated subsidy scheme does not alter the total spending when $\theta = 2$. By doing a sequence of such swaps, one can arrive at any possible ranking, and hence any ranking is optimal. □

A.7. Proof of Proposition 6

First statement. Consider a ranking $b(i)$ and let $\boldsymbol{\tau} = \left\{\tau_1, \dots, \tau_{N_A}\right\}$ be the subsidy scheme associated to it. Denote by l and h the indexes of a low- and high-productivity A-firm, respectively, such that $b(h) = b(l) + 1$, $\tau_h > 0$ and $\tau_l > 0$. Denote by $\hat{q}_L$ and $\hat{q}_H$ the number of low- and high-productivity firms, respectively, ranked before l under the ranking b (that is, those are given by the right-hand side of (A.4) and (A.5), respectively, setting $i = l$).

Defining $Y\left(q_L, q_H\right)$ as in (A.2) and letting $X\left(q_L, q_H\right) = Y\left(q_L, q_H\right)^{\frac{1}{\theta}}$ we get that τ_l and τ_h are given by the expressions in (A.8) and (A.9), respectively. Now consider a ranking $\tilde{b}(i)$ that swaps the position of firms l and h: $\tilde{b}(l) = b(h)$, $\tilde{b}(h) = b(l)$ and $\tilde{b}(i) = b(i)$ for all $i \neq \{l,h\}$.

Letting $\tilde{\boldsymbol{\tau}} = \left\{\tilde{\tau}_1, \ldots, \tilde{\tau}_{N_A}\right\}$ be the subsidy scheme associated to $\tilde{b}(i)$, we then have $\tilde{\tau}_i = \tau_i$, for $i \neq \{l, h\}$ and

$$\tilde{\tau}_l = \max\left\{\psi - [(1-s)Z]^{\frac{\theta-1}{\theta}} X\left(\hat{q}_L + 1, \hat{q}_H + 1\right), 0\right\} = \psi - [(1-s)Z]^{\frac{\theta-1}{\theta}} X\left(\hat{q}_L + 1, \hat{q}_H + 1\right), \tag{A.15}$$

$$\tilde{\tau}_h = \max\left\{\psi - Z^{\frac{\theta-1}{\theta}} X\left(\hat{q}_L, \hat{q}_H + 1\right), 0\right\} = \psi - Z^{\frac{\theta-1}{\theta}} X\left(\hat{q}_L, \hat{q}_H + 1\right). \tag{A.16}$$

The second equality in (A.15) follows from the fact that

$$\psi - [(1-s)Z]^{\frac{\theta-1}{\theta}} X\left(\hat{q}_L + 1, \hat{q}_H + 1\right) > \psi - Z^{\frac{\theta-1}{\theta}} X\left(\hat{q}_L + 1, \hat{q}_H + 1\right) = \tau_h > 0.$$

The second equality in (A.16) follows from the fact that

$$\psi - Z^{\frac{\theta-1}{\theta}} X\left(\hat{q}_L, \hat{q}_H + 1\right) > \psi - Z^{\frac{\theta-1}{\theta}} X\left(\hat{q}_L + 1, \hat{q}_H + 1\right) = \tau_h > 0.$$

Therefore, we get the same expressions for τ_l, τ_h, $\tilde{\tau}_l$ and $\tilde{\tau}_h$ as in the proof of the first statement of Proposition 5. We can then follow the same steps as in that proof to show that with $\theta > 2$, $\tilde{\boldsymbol{\tau}}$ yields strictly lower spending than $\boldsymbol{\tau}$ and hence $\boldsymbol{\tau}$ is not optimal, which concludes the proof of the first statement.

Second statement. To prove the second statement, let $b^*(i)$ denote an optimal ranking, with associated subsidy scheme $\boldsymbol{\tau}^* = \left\{\tau_1^*, \ldots, \tau_{N_A}^*\right\}$. Suppose that, under b^* there is a low-productivity firm l' ranked right before a high-productivity one h': $b^*\left(h'\right) = b^*\left(l'\right) + 1$. Hereafter, $\hat{q}_L$ and $\hat{q}_H$ are redefined to denote the number of low- and high-productivity firms, respectively, ranked before l' under the ranking b^*. By the first statement, it must be that $\tau_{h'}^* = 0$, otherwise both firms would receive positive subsidies, contradicting that $b^*(i)$ is an optimal ranking.

Now consider a ranking $b^\dagger(i)$ that swaps the positions of h' and l': $b^\dagger(i) = b^*(i)$ for $i \neq \left\{l', h'\right\}$, $b^\dagger\left(h'\right) = b^*\left(l'\right)$ and $b^\dagger\left(l'\right) = b^*\left(h'\right)$. Denote by $\boldsymbol{\tau}^\dagger = \left\{\tau_1^\dagger, \ldots, \tau_{N_A}^\dagger\right\}$ the subsidy associated to $b^\dagger$, which implies that $\tau_i^\dagger = \tau_i^*$ for $i \neq \left\{l', h'\right\}$. I will show that spending under $\boldsymbol{\tau}^\dagger$ is not larger than spending under $\boldsymbol{\tau}^*$, implying that $b^\dagger$ is also an optimal ranking.

If $\tau_{l'}^* = 0$, and given that $\tau_{h'}^* = 0$, it must be that $\tau_{h'}^\dagger = 0$ and $\tau_{l'}^\dagger = 0$, and hence $\boldsymbol{\tau}^*$ and $\boldsymbol{\tau}^\dagger$ yield the same spending. Suppose now $\tau_{l'}^* > 0$ and $\tau_{l'}^\dagger = 0$. In this case, spending cannot increase, since $\tau_{l'}^\dagger = \tau_{h'}^* = 0$ and $\tau_{h'}^\dagger = \max\left\{\psi - Z^{\frac{\theta-1}{\theta}} X\left(\hat{q}_L, \hat{q}_H + 1\right), 0\right\} < \psi - [(1-s)Z]^{\frac{\theta-1}{\theta}} X\left(\hat{q}_L + 1, \hat{q}_H\right) = \tau_{l'}^*$. Finally, assume $\tau_{l'}^* > 0$ and $\tau_{l'}^\dagger > 0$. The difference in spending between $\boldsymbol{\tau}^\dagger$ and $\boldsymbol{\tau}^*$ is

$$\begin{aligned}
\Delta\tau &= \left(\tau_{l'}^\dagger - \tau_{l'}^*\right) + \left(\tau_{h'}^\dagger - \tau_{h'}^*\right) \\
&= [(1-s)Z]^{\frac{\theta-1}{\theta}} \left[X\left(\hat{q}_L + 1, \hat{q}_H\right) - X\left(\hat{q}_L + 1, \hat{q}_H + 1\right)\right] \\
&\quad + \max\left\{\psi - Z^{\frac{\theta-1}{\theta}} X\left(\hat{q}_L, \hat{q}_H + 1\right), 0\right\}.
\end{aligned}$$

If $\max\left\{\psi - Z^{\frac{\theta-1}{\theta}} X\left(\hat{q}_L, \hat{q}_H + 1\right), 0\right\} = 0$, we then have $\Delta\tau < 0$. Suppose now that $\max\left\{\psi - Z^{\frac{\theta-1}{\theta}} X\left(\hat{q}_L, \hat{q}_H + 1\right), 0\right\} > 0$. In this case, the expression above becomes

$$\begin{aligned}
\Delta\tau &= [(1-s)Z]^{\frac{\theta-1}{\theta}} \left[X\left(\hat{q}_L + 1, \hat{q}_H\right) - X\left(\hat{q}_L + 1, \hat{q}_H + 1\right)\right] \\
&\quad + \psi - Z^{\frac{\theta-1}{\theta}} X\left(\hat{q}_L, \hat{q}_H + 1\right) \\
&\leq [(1-s)Z]^{\frac{\theta-1}{\theta}} \left[X\left(\hat{q}_L + 1, \hat{q}_H\right) \right.\\
&\quad \left. - X\left(\hat{q}_L + 1, \hat{q}_H + 1\right)\right] + \psi - Z^{\frac{\theta-1}{\theta}} X\left(\hat{q}_L, \hat{q}_H + 1\right) \\
&\quad + \underbrace{Z^{\frac{\theta-1}{\theta}} X\left(\hat{q}_L + 1, \hat{q}_H + 1\right) - \psi}_{\geq 0, \text{ since } \tau_{h'}^* = 0} \\
&= [(1-s)Z]^{\frac{\theta-1}{\theta}} \left[X\left(\hat{q}_L + 1, \hat{q}_H\right) \right.\\
&\quad \left. - X\left(\hat{q}_L + 1, \hat{q}_H + 1\right)\right] \\
&\quad + Z^{\frac{\theta-1}{\theta}} \left[X\left(\hat{q}_L + 1, \hat{q}_H + 1\right) - X\left(\hat{q}_L, \hat{q}_H + 1\right)\right].
\end{aligned}$$

Note that the last expression above is identical to (A.12), and hence, following the same steps as in the proof of the first statement of Proposition 5, we can conclude that $\Delta\tau < 0$, provided $\theta > 2$. Hence, starting from any optimal ranking in which high-productivity firms are not always ranked before low-productivity ones, we can construct another ranking in which high-productivity ones are ranked first without increasing spending under the respective associated subsidies, which concludes the proof of the second statement. □

A.8. Proof of Proposition 7

This proof is similar to that of Proposition 4, with some adjustments.

First, note that the arguments in the proof of Lemma 1 still apply to the extension considered here, one only needs to write the revenue of a firm i as a function of the entry decisions of all A-firms and Z_i—and not only as a function $\tilde{q}_L$, $\tilde{q}_H$ and Z_i as in (A.1)—and then follow the same steps as in that proof. Hence, we can think of the problem of implementing $q = N_A$ as that of first choosing a ranking $b(i)$ and then finding the subsidy associated to that ranking. Without loss of generality, I assume that $Z_1 \geq Z_2 \geq \cdots \geq Z_{N_A}$ in this proof.

Let b^* be an optimal ranking, with $\boldsymbol{\tau}^* = \left\{\tau_1^*, \ldots, \tau_{N_A}^*\right\}$ being the (optimal) subsidy scheme associated to it. Suppose, by contradiction, that a firm $k_1 > \bar{k}$ receives a positive subsidy, $\tau_{k_1}^* > 0$. Consider the following alternative ranking: $b^{**}(i) = i$. Let $\boldsymbol{\tau}^{**} = \left\{\tau_1^{**}, \ldots, \tau_{N_A}^{**}\right\}$ be the subsidy scheme associated to it. Note that $\tau_i^{**} = 0$, for all $i > \bar{k}$, given the assumptions of the proposition.

Take some arbitrary $x \in \left\{1, \ldots, N_A\right\}$ and let k^* and k^{**} be A-firms such that $b^*\left(k^*\right) = x$ and $b^{**}\left(k^{**}\right) = x$, respectively. I want to show that $\tau_{k^{**}}^{**} \leq \tau_{k^*}^*$. If $\tau_{k^{**}}^{**} = 0$, this is obviously true, so consider now $\tau_{k^{**}}^{**} > 0$, which implies that $Z_{k^{**}} = Z^*$. By definition, both firms have the same number of firms ranked above them. The subsidies are given by

$$\begin{aligned}
\tau_{k^{**}}^{**} &= \psi - Z_{k^{**}}^{\frac{\theta-1}{\theta}} \left(\sum_{j=1}^{x} Z_{b^{**-1}(j)}^{\frac{\theta-1}{\theta}}\right)^{\frac{1}{\theta-1}} \\
&\leq \max\left\{\psi - Z_{k^*}^{\frac{\theta-1}{\theta}} \left(\sum_{j=1}^{x} Z_{b^{*-1}(j)}^{\frac{\theta-1}{\theta}}\right)^{\frac{1}{\theta-1}}, 0\right\} = \tau_{k^*}^*,
\end{aligned} \tag{A.17}$$

where the inequality follows from the fact $\sum_{j=1}^{x} Z_{b^{**-1}(j)}^{\frac{\theta-1}{\theta}} = \sum_{i=1}^{x} Z_i^{\frac{\theta-1}{\theta}} \geq \sum_{k=1}^{x} Z_{b^{-1}(j)}^{\frac{\theta-1}{\theta}}$, for any ranking $b(i)$, and $Z_{k^{**}} = Z^* \geq Z_{k^*}$.

Now let $x = b^*\left(k_1\right)$, so $k^* = k_1$. By the definition of k_1, $\tau_{k^*} > 0$ and $Z_{k^*} < Z^*$. I claim that $\tau_{k^{**}}^{**} < \tau_{k^*}^*$. Again, if $\tau_{k^{**}}^{**} = 0$ this is trivially satisfied. If $\tau_{k^{**}}^{**} > 0$, $Z_{k^{**}} = Z^* > Z_{k^*}$, which then implies that (A.17) must hold with strict inequality. □

A.9. Algebraic conditions for each type of cascade and for their efficiency

Throughout this section, $Y\left(q_L, q_H\right)$ refers to the expression in (A.2).

For a cascade ($q = 0$) not to be an equilibrium, we need an A-firm not hit by the shock to be strictly preferring to enter under the belief that it is the only A-firm operating. Therefore, a cascade is not an equilibrium if, and only if,

$$Z^{\frac{\theta-1}{\theta}} Y(0,1)^{\frac{1}{\theta}} > \psi. \tag{A.18}$$

Hence, for any shock (s, r) that satisfies (A.18), the economy is in the "no cascades" region depicted in Fig. 1.

All operating ($q = N_A$) is socially optimal if, and only if, both conditions below hold:

$$Y\left(rN_A,(1-r)N_A\right) - N_A\psi \geq Y(0,0), \tag{A.19}$$

$$Y\left(rN_A,(1-r)N_A\right) - N_A\psi \geq Y\left(0,(1-r)N_A\right) - (1-r)N_A\psi. \tag{A.20}$$

One can easily verify that the second derivatives of $Y\left(q_L, q_H\right)$ are strictly positive, and so the planner will never choose an interior q_L and/or q_H: $q_L \in \left\{0, rN_A\right\}$ and $q_H \in \left\{0,(1-r)N_A\right\}$ under the planner's choice.[14] Moreover, it can never be optimal to make low-productivity firms enter ($q_L > 0$) and high-productivity firms not enter ($q_H = 0$). Therefore, we only need to compare three options: (i) To make all enter; (ii) To make no A-firm enter; (iii) To make only high-productivity A-firms enter. This is what conditions (A.19) and (A.20) do. If both (A.19) and (A.20) hold, then it is socially optimal for all firms to enter.

To have a deep cascade, we need high-productivity firms not to be willing to enter under the belief that all firms will enter. Hence, the necessary and sufficient condition to be in a deep cascade is

$$Z^{\frac{\theta-1}{\theta}} Y\left(rN_A,(1-r)N_A\right)^{\frac{1}{\theta}} < \psi. \tag{A.21}$$

Any shock satisfying (A.21) throws the economy in a deep cascade, and if (A.19) holds strictly, this is inefficient.

To have a mild cascade as an equilibrium outcome, we need (A.18) not to hold and low-productivity firms to be willing to enter once the entry of all high-productivity firms is guaranteed, that is

$$[(1-s)Z]^{\frac{\theta-1}{\theta}} Y\left(1,(1-r)N_A\right)^{\frac{1}{\theta}} \geq \psi. \tag{A.22}$$

If (A.18) is violated and (A.22) holds, we have an equilibrium with a mild cascade.

Finally, whenever (A.18) is violated and a cascade cannot be classified as deep or mild, according to the criteria above, we classify it as an intermediate case.

A.10. Expressions for the optimal subsidy in a deep cascade

Without loss of generality, and to ease notation, assume throughout this section that firms $1,\ldots,(1-r)N_A$ are not hit by the shock and that firms $(1-r)N_A+1,\ldots,N_A$ are hit by the shock. In what follows, assume the economy is in a deep cascade before the subsidy scheme is implemented.

First consider $\theta > 2$. From Proposition 5, we have that the ranking $b(i) = i$ is optimal. Using (5) and the fact that all A-firms get paid positive subsidies, the optimal subsidy associated to that ranking for a firm i can be written as

$$\tau_i^* = \begin{cases} \psi - Z\left[i + (1-r)N_B + rN_B(1-s)^{\frac{\theta-1}{\theta}}\right]^{\frac{1}{\theta-1}}, & \text{if } i \leq (1-r)N_A, \\ \psi - (1-s)^{\frac{\theta-1}{\theta}} Z\Big[(1-r)N + \big(i-(1-r)N_A & \\ \quad +rN_B\big)(1-s)^{\frac{\theta-1}{\theta}}\Big]^{\frac{1}{\theta-1}}, & \text{if } i > (1-r)N_A. \end{cases}$$

Now consider $\theta < 2$. In that case, firms hit by the shock are ranked first, and hence $b(i) = N_A + 1 - i$ is optimal. Using (5) and the fact that all get paid positive subsidies, the optimal subsidy associated to that ranking for a firm i can be written as

$$\tau_i^* = \begin{cases} \psi - Z\left[(1-r)N - i + 1 + rN(1-s)^{\frac{\theta-1}{\theta}}\right]^{\frac{1}{\theta-1}}, & \text{if } i \leq (1-r)N_A, \\ \psi - (1-s)^{\frac{\theta-1}{\theta}} Z\Big[(1-r)N_B & \\ \quad +\left(N_A - i + 1 + rN_B\right)(1-s)^{\frac{\theta-1}{\theta}}\Big]^{\frac{1}{\theta-1}}, & \text{if } i > (1-r)N_A. \end{cases}$$

[14] The benefit of adding an additional firm increases as the planner adds more firms, while the cost is constant and equal to ψ.

Appendix B. A model with labor

Unless otherwise stated, the assumptions and notation are the same as in the model of the main text. The only difference is that now firms use labor to produce their varieties, which is supplied by the household. Household utility is given by

$$U = C - \xi L,$$

where L denotes the total amount of labor supplied by the household and $\xi > 0$. The labor market is competitive, which implies that in equilibrium the wage w (in units of the composite good) is equal to the constant marginal disutility of labor, $w = \xi$.

Upon entry, each firm i has access to the following production technology:

$$y_i = Z_i l_i^{\alpha}, \tag{B.1}$$

where $\alpha \in (0,1)$, l_i denotes the amount of labor used by firm i, and Z_i is the firm's productivity. Firms still need to pay the fixed cost ψ (in units of the composite good) to operate. If they do not operate, they do not hire labor and produce zero.

As before, we can split A-firms' problems in two stages: In the first stage, they decide whether or not to enter; (ii) In the second stage, those that entered decide how much labor to use.

In the model of the main text, firms' production decisions, after they have paid the fixed operating cost and the number of operating firms was fixed, were trivial and coincided with the first best: they produced up to their limit Z_i, regardless of their beliefs about what other firms that entered would do, since the marginal production cost was zero up to the limit. Now, in a decentralized equilibrium, firms will equal their marginal revenue to their marginal cost, which, as usual, leads to inefficient hiring decisions due to market power. In other words, even after a social planner has ensured the efficient entry of firms, there would still be inefficiencies in the intensive margin (labor demand).

Since the focus of my paper is on how to fix inefficiencies in the extensive margin (firms' operating decisions), I will abstract from any inefficiency in the intensive margin by assuming that, in the second stage, operating firms demand the amount of labor dictated by a social planner that maximizes the welfare of the household. One can think that once firms have entered, they are subject to price regulations (antitrust policies) and therefore a social planner can achieve without costs his desired level of production for each firm by restricting the price they can charge. Still, the social planner cannot force firms to enter and is still going to need to rely on subsidies to achieve efficient entry in the first stage. Of course, the assumption that it is costless for a regulator to implement the efficient allocation in the second stage is not realistic, but it makes the problem in the second stage more tractable and allows me to concentrate on the inefficiency that is the focus of my paper (the extensive margin).[15]

I start characterizing the efficient allocation in the second stage, taking as given the entry decision of each firm. Then, I solve the Nash equilibrium of the entry game of firms in the first stage, assuming firms anticipate they will have to produce according to the efficient allocation in the second stage.

[15] Since I have a finite number of firms, in the decentralized equilibrium in the second stage firms do not take Y as given (as is usually the case in monopolistic competition models with a continuum of firms), which complicates the algebra when solving for decentralized equilibrium. Of course, for large N this would have little impact on the decentralized equilibrium outcomes.

B.1. (Efficient) production decisions in the second stage

Fix A-firms' entry decisions $e_1, e_2, \ldots, e_{N_A}$, and denote by $\mathcal{E} = \{i \in \{1, 2, \ldots, N_A\} : e_i = 1\} \cup \{N_A + 1, \ldots, N\}$ the set of operating firms. The social planner must choose the labor demand of each firm in $\mathcal{E}$, subject to the production functions (B.1) and (1), to maximize household welfare

$$W = Y - q\psi - \xi \sum_{i \in \mathcal{E}} l_i,$$

where q denotes the number of A-firms that entered in the first stage (which is fixed at this point). Replacing $l_i = \left(\frac{y_i}{Z_i}\right)^{\frac{1}{\alpha}}$ above, which comes from (B.1), and using (1), we can write the problem in terms of choosing y_i, for $i \in \mathcal{E}$, to maximize

$$W = \left(\sum_{i \in \mathcal{E}} y_i^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}} - q\psi - \xi \sum_{i \in \mathcal{E}} \left(\frac{y_i}{Z_i}\right)^{\frac{1}{\alpha}}.$$

The first order condition implies that

$$y_i = \left(\frac{\alpha}{\xi}\right)^{\frac{\alpha\theta}{(1-\alpha)\theta+\alpha}} Z_i^{\frac{\theta}{(1-\alpha)\theta+\alpha}} Y^{\frac{\alpha}{(1-\alpha)\theta+\alpha}}. \tag{B.2}$$

Using (1), we get

$$Y = \left\{\sum_{i \in \mathcal{E}} \left[\left(\frac{\alpha}{\xi}\right)^{\frac{\alpha\theta}{(1-\alpha)\theta+\alpha}} Z_i^{\frac{\theta}{(1-\alpha)\theta+\alpha}} Y^{\frac{\alpha}{(1-\alpha)\theta+\alpha}}\right]^{\frac{\theta-1}{\theta}}\right\}^{\frac{\theta}{\theta-1}},$$

which after solving for Y yields

$$Y = \left(\frac{\alpha}{\xi}\right)^{\frac{\alpha}{1-\alpha}} \left\{\sum_{i \in \mathcal{E}} Z_i^{\frac{\theta-1}{(1-\alpha)\theta+\alpha}}\right\}^{\frac{(1-\alpha)\theta+\alpha}{(\theta-1)(1-\alpha)}}. \tag{B.3}$$

Plugging it back in (B.2), it yields

$$y_i = \left(\frac{\alpha}{\xi}\right)^{\frac{\alpha}{1-\alpha}} Z_i^{\frac{\theta}{(1-\alpha)\theta+\alpha}} \left(\sum_{j \in \mathcal{E}} Z_j^{\frac{\theta-1}{(1-\alpha)\theta+\alpha}}\right)^{\frac{\alpha}{(\theta-1)(1-\alpha)}}. \tag{B.4}$$

B.2. Equilibrium in the first stage

Using (2), (B.3) and (B.4), we can write the revenue of operating firms as

$$R_i = \left(\frac{\alpha}{\xi}\right)^{\frac{\alpha}{1-\alpha}} Z_i^{\frac{\theta-1}{(1-\alpha)\theta+\alpha}} \left(\sum_{j \in \mathcal{E}} Z_j^{\frac{\theta-1}{(1-\alpha)\theta+\alpha}}\right)^{\frac{1}{(\theta-1)(1-\alpha)}}. \tag{B.5}$$

Profits of operating firms, net of fixed costs, are given by $\Pi_i = R_i - \xi l_i = R_i - \xi\left(\frac{y_i}{Z_i}\right)^{\frac{1}{\alpha}}$, which, using (B.4) and (B.5), yield

$$\Pi_i = (1-\alpha)\left(\frac{\alpha}{\xi}\right)^{\frac{\alpha}{1-\alpha}} Z_i^{\frac{\theta-1}{(1-\alpha)\theta+\alpha}} \left(\sum_{j \in \mathcal{E}} Z_j^{\frac{\theta-1}{(1-\alpha)\theta+\alpha}}\right)^{\frac{1}{(\theta-1)(1-\alpha)}}. \tag{B.6}$$

Having characterized how firms' profits are determined for a given entry decision of firms, I can now characterize the equilibrium in the first stage, before and after the shock.

B.2.1. Equilibrium before the shock

As in the main model, before the shock $Z_i = Z$, for all i. Using $Z_i = Z$, (B.6) becomes

$$\Pi_i = (1-\alpha)\left(\frac{\alpha}{\xi}\right)^{\frac{\alpha}{1-\alpha}} Z^{\frac{1}{1-\alpha}} \left(q + N_B\right)^{\frac{1}{(\theta-1)(1-\alpha)}}.$$

As one can see, Π_i is strictly increasing in Z and q. Therefore, if

$$(1-\alpha)\left(\frac{\alpha}{\xi}\right)^{\frac{\alpha}{1-\alpha}} Z^{\frac{1}{1-\alpha}} \left(1 + N_B\right)^{\frac{1}{(\theta-1)(1-\alpha)}} - \psi \le 0,$$

an A-firm is willing not to enter if no other A-firm does so, which is the required condition for $q = 0$ to be an equilibrium outcome. The condition above can be rewritten as

$$Z \le \Omega\left(1 + N_B\right)^{-\frac{1}{\theta-1}} \equiv \widetilde{Z},$$

where $\Omega \equiv \psi^{1-\alpha}\left(\frac{\xi}{\alpha}\right)^{\alpha}(1-\alpha)^{\alpha-1}$.

If

$$(1-\alpha)\left(\frac{\alpha}{\xi}\right)^{\frac{\alpha}{1-\alpha}} Z^{\frac{1}{1-\alpha}} N^{\frac{1}{(\theta-1)(1-\alpha)}} - \psi \ge 0,$$

an A-firm is willing to enter if it believes all others will do so, which is the condition needed for $q = N_A$ to be an equilibrium outcome. The condition above can be rewritten as

$$Z \ge \Omega N^{-\frac{1}{\theta-1}} \equiv \underset{\sim}{Z}.$$

Note that $\widetilde{Z} > \underset{\sim}{Z}$. From the previous discussion, one can conclude that: If $Z > \widetilde{Z}$, $q = N_A$ is the unique equilibrium outcome; If $Z < \underset{\sim}{Z}$, $q = 0$ is the unique equilibrium outcome; If $Z \in \left[\underset{\sim}{Z}, \widetilde{Z}\right]$, both $q = N_A$ and $q = 0$ are equilibrium outcomes. Hence, the results in Proposition 1 hold (up to the redefinition of the bounds $\underset{\sim}{Z}$ and $\widetilde{Z}$, of course). As before, whenever $Z > \widetilde{Z}$, I say that the economy is in *good times*.

B.2.2. Equilibrium after the shock

The definition of an uneven shock is the same as in the main model (productivity Z falls by a factor s for a fraction r of A- and B-firms). I now show a result similar to Proposition 2 for the model with labor.

Proposition B.1. *Consider the model with labor. Suppose the economy is in good times and is hit by an uneven shock* (s, r)*. Then, there are shocks and parameters such that an equilibrium in which all A-firms exit* $(q = 0)$ *exists, and that may be the unique equilibrium. In particular,* $q = 0$ *is an equilibrium outcome if*

$$Z\left[1 + (1-r) N_B + r N_B (1-s)^{\frac{\theta-1}{(1-\alpha)\theta+\alpha}}\right]^{\frac{1}{\theta-1}} \le \Omega, \tag{B.7}$$

and it is the unique equilibrium outcome if

$$Z\left[(1-r) N + r N (1-s)^{\frac{\theta-1}{(1-\alpha)\theta+\alpha}}\right]^{\frac{1}{\theta-1}} < \Omega. \tag{B.8}$$

Proof. By (B.6), one can see that whenever low-productivity firms are willing to enter, high-productivity firms strictly prefer to do so. Conversely, if high-productivity firms are willing not to enter, then it is strictly optimal for the low-productivity ones not to enter. Hence, an equilibrium with $q = 0$ exists if, and only if, it is optimal for high-productivity firms not to enter when they are the only A-firm entering, which, using (B.6), yields condition (B.7). For an equilibrium with $q = 0$ to be the unique equilibrium, it is sufficient to ensure that high-productivity firms strictly prefer not to enter when they believe $q = N_A$, which, after using (B.6), yields (B.8).

To give an example of when (B.7) holds, assume $r = 1/2$, in which case (B.7) becomes

$$Z\left[1 + \frac{N_B}{2} + \frac{N_B}{2}(1-s)^{\frac{\theta-1}{(1-\alpha)\theta+\alpha}}\right]^{\frac{1}{\theta-1}} \le \Omega.$$

As $s \to 1$, this inequality becomes $Z \le \Omega\left(1 + \frac{N_B}{2}\right)^{-\frac{1}{\theta-1}} \equiv \overline{Z}$. Note that $\overline{Z} > \widetilde{Z}$, and hence, the economy can be in good times before the shock and have $q = 0$ as an equilibrium outcome after a shock that hits half of the firms, provided that s is sufficiently large. Now, I construct an example in which (B.8) holds, with $r = 1/2$. In that case, (B.8) becomes:

$$Z\left[\frac{N}{2} + \frac{N}{2}(1-s)^{\frac{\theta-1}{(1-\alpha)\theta+\alpha}}\right]^{\frac{1}{\theta-1}} < \Omega,$$

and as $s \to 1$ it becomes $Z < \Omega(N/2)^{-\frac{1}{\theta-1}} \equiv \underline{Z}$. As in the main model, it is not obvious that $\underline{Z} > \widetilde{Z}$, and for that to be true, one needs the same additional restriction that $N_B > N_A - 2$. □

Note that the left-hand side of both (B.7) and (B.8) are strictly decreasing in s and r, and hence, whenever any of those conditions is satisfied for some uneven shock (s, r), the condition is also satisfied for uneven shocks larger than (s, r).

B.3. The first best after an uneven shock

The social planner maximizes the welfare of the household $W = Y - q\psi - \xi L$. Combining $l_i = \left(\frac{y_i}{Z_i}\right)^{\frac{1}{\alpha}}$, (B.3) and (B.4), after some algebra, one can show that the total disutility of labor is given by $\xi \sum_{i=1}^{N_A} l_i = \alpha Y$, and hence, household welfare is given by

$$W = (1-\alpha) Y - q\psi. \tag{B.9}$$

Summing across all firms' profits, using (B.6), and then using (B.3), not surprisingly, we can show that the sum of firms' profits equals the expression for welfare above. If an equilibrium with $q = N_A$ exists, it must be that all firms make profits that are equal or larger than zero when all enter. Moreover, operating firms' profits are strictly increasing in the number of operating firms (see (B.6)). Therefore, whenever an equilibrium with $q = N_A$ exists, it is the social optimum, as firms will have larger aggregate profits in that scenario than in any other with $q < N_A$. That is, the first statement of Proposition 3 holds.

To show that cascades can be inefficient even when they are the unique equilibrium outcome, I provide a numerical example. Let $\theta = 2$, $Z = 1$, $s = 0.7$, $r = 0.5$, $\psi = 330$, $\alpha = 0.5$, $\xi = 1$, $N_A = 10$, $N_B = 40$. One can verify that $Z > \tilde{Z}$ for those parameters, and hence the economy is good times before the shock. Also, (B.8) holds, and so $q = 0$ is the unique equilibrium outcome after the shock. Finally, one can use (B.3) and (B.9) to compute welfare when: (i) all enter; (ii) no A-firm enters; (iii) only high-productivity A-firms enter. This comparison reveals that $q = N_A$ is the social optimum.[16]

As in the main text, I focus on inefficient business closure cascades in what follows, that is, situations in which $q = 0$ is an equilibrium outcome after the shock, but the social optimum is $q = N_A$.

B.4. Optimal subsidies

The definition of deep and mild cascades is the same as in the main text.

Denote by $\Pi\left(\tilde{q}_L, \tilde{q}_H, Z_i\right)$ the profits, net of the fixed cost and after the shock, of a firm i with productivity $Z_i \in \{(1-s)Z, Z\}$, when it enters and a quantity $\tilde{q}_L$ and $\tilde{q}_H$ of *other* firms, with low and high productivity, respectively, enter. Using (B.6), it is given by

$$\begin{aligned}\Pi\left(\tilde{q}_L, \tilde{q}_H, Z_i\right) &= (1-\alpha)\left(\frac{\alpha}{\xi}\right)^{\frac{\alpha}{1-\alpha}} Z_i^{\frac{\theta-1}{(1-\alpha)\theta+\alpha}} \\ &\quad \cdot \left\{ Z_i^{\frac{\theta-1}{(1-\alpha)\theta+\alpha}} + \left[\tilde{q}_H + (1-r)N_B\right] Z^{\frac{\theta-1}{(1-\alpha)\theta+\alpha}} + \left[\tilde{q}_L + rN_B\right] \right. \\ &\quad \left. \cdot \left[Z(1-s)\right]^{\frac{\theta-1}{(1-\alpha)\theta+\alpha}} \right\}^{\frac{1}{(\theta-1)(1-\alpha)}}.\end{aligned}$$

I now claim that Lemma 1 and Proposition 4 (optimal subsidy in a mild cascade) still hold in the model with labor. The proofs are identical to those of the main text results, one only needs to replace $R\left(\tilde{q}_L, \tilde{q}_H, Z_i\right)$ by $\Pi\left(\tilde{q}_L, \tilde{q}_H, Z_i\right)$ in the corresponding proofs. Regarding deep cascades, the relevant bound on θ for the optimal ranking is no longer 2, but instead depends on α. Yet, the main message is the same as in Proposition 5:

Proposition B.2. *Consider a deep cascade in the model with labor. Then, under the optimal policy:*

1. *If $\theta > \frac{2-\alpha}{1-\alpha}$, firms not hit by the supply shock are ranked first;*
2. *If $\theta < \frac{2-\alpha}{1-\alpha}$, firms hit by the supply shock are ranked first;*
3. *If $\theta = \frac{2-\alpha}{1-\alpha}$, any ranking is optimal.*

Proof. Denote by q_L and q_H the number low- and high-productivity A-firms that enter, respectively, and let $X\left(q_L, q_H\right)$ be

$$\begin{aligned}X\left(q_L, q_H\right) &= (1-\alpha)\left(\frac{\alpha}{\xi}\right)^{\frac{\alpha}{1-\alpha}} \\ &\quad \cdot \left\{ \left[q_H + (1-r)N_B\right] Z^{\frac{\theta-1}{(1-\alpha)\theta+\alpha}} + \left[q_L + rN_B\right] \right. \\ &\quad \left. \cdot \left[Z(1-s)\right]^{\frac{\theta-1}{(1-\alpha)\theta+\alpha}} \right\}^{\frac{1}{(\theta-1)(1-\alpha)}}.\end{aligned} \tag{B.10}$$

Using (B.6), note that the profits of a firm with productivity Z_i that enters, net of fixed costs, can be written as

$$\Pi_i = Z_i^{\frac{\theta-1}{(1-\alpha)\theta+\alpha}} X\left(q_L, q_H\right). \tag{B.11}$$

Also, note that

$$\begin{aligned}\frac{\partial X}{\partial q_L} &= \frac{1}{\theta-1}(1-\alpha)^{(1-\alpha)\theta+\alpha-2}\left(\frac{\alpha}{\xi}\right)^{(\theta-1)\alpha} X\left(q_L, q_H\right)^{2-\alpha-(1-\alpha)\theta} \\ &\quad \cdot \left[(1-s)Z\right]^{\frac{\theta-1}{(1-\alpha)\theta+\alpha}}\end{aligned} \tag{B.12}$$

and

$$\frac{\partial X}{\partial q_H} = \frac{1}{\theta-1}(1-\alpha)^{(1-\alpha)\theta+\alpha-2}\left(\frac{\alpha}{\xi}\right)^{(\theta-1)\alpha} X\left(q_L, q_H\right)^{2-\alpha-(1-\alpha)\theta} Z^{\frac{\theta-1}{(1-\alpha)\theta+\alpha}}. \tag{B.13}$$

The proof of all three statements then follow the same steps as the proof of Proposition 5, with some straightforward changes: We now assume $\theta > \frac{2-\alpha}{1-\alpha}$ when proving the first statement, $\theta < \frac{2-\alpha}{1-\alpha}$ when proving the second statement and $\theta = \frac{2-\alpha}{1-\alpha}$ for the third one. Moreover, all equations with exponents $\frac{\theta-1}{\theta}$ should have it replaced by $\frac{\theta-1}{(1-\alpha)\theta+\alpha}$ (given (B.11)), $X\left(q_L, q_H\right)$ and its derivatives are now given by (B.10), (B.12) and (B.13), and after those adjustments, Eq. (A.13), in the proof of the first statement, becomes

$$\begin{aligned}\Delta\tau &= \frac{1}{\theta-1}(1-\alpha)^{(1-\alpha)\theta+\alpha-2}\left(\frac{\alpha}{\xi}\right)^{(\theta-1)\alpha} Z^{\frac{2(\theta-1)}{(1-\alpha)\theta+\alpha}}(1-s)^{\frac{\theta-1}{(1-\alpha)\theta+\alpha}} \\ &\quad \cdot \int_0^1 \left[X\left(\hat{q}_L + v, \hat{q}_H + 1\right)^{2-\alpha-(1-\alpha)\theta} - X\left(\hat{q}_L + 1, \hat{q}_H + v\right)^{2-\alpha-(1-\alpha)\theta}\right] dv,\end{aligned}$$

which implies that $\Delta\tau < 0$, when $2-\alpha-(1-\alpha)\theta < 0$, that is, when $\theta > \frac{2-\alpha}{1-\alpha}$. After making the aforementioned changes, the proof of Proposition 5 applies to Proposition B.2. □

[16] Using (B.3), one can write Y as a function of the number of low- and high-productivity A-firms operating, denoted by q_L and q_H, respectively:

$$\begin{aligned}Y\left(q_L, q_H\right) &= \left(\frac{\alpha}{\xi}\right)^{\frac{\alpha}{1-\alpha}} Z^{\frac{1}{1-\alpha}} \left[q_H + (1-r)N_B + \left(q_L + rN_B\right)\right. \\ &\quad \left. \cdot (1-s)^{\frac{\theta-1}{(1-\alpha)\theta+\alpha}}\right]^{\frac{(1-\alpha)\theta+\alpha}{(\theta-1)(1-\alpha)}}.\end{aligned}$$

One can verify that all the second derivatives of this function are strictly positive, and hence, it suffices to check the welfare function in (B.9) in the scenarios mentioned to find the social optimum.

Data availability

No data was used for the research described in the article.

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