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On the cardinality of the message space in sender–receiver games[☆]

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ABSTRACT

We study sender–receiver games in which a privately informed sender sends a message to N receivers, who then take an action. The sender's type space T has finite cardinality (i.e., $|T| < \infty$). We show that every equilibrium payoff vector (resp. every Pareto efficient equilibrium payoff vector) is achieved by an equilibrium in which the sender sends at most $|T| + N$ (resp. $|T| + N - 1$) messages with positive probability. We also show that such bounds do not exist when two privately informed senders simultaneously send a message to a receiver.

1. Introduction

We study a model that consists of a sender and N receivers. The sender has a type in T, where $|T| < \infty$. The sender sends a public message to the receivers, who then take action. We allow for the possibility that the receivers also have private information, which we call the receivers' type. The players' payoff is a function of the sender's type, the message sent by the sender, the receivers' type and the actions taken by the receivers. For a given strategy profile, a payoff vector $w \in \mathbb{R}^{N+|T|}$ specifies the ex ante expected utility of the receivers and the interim utility of the sender. The *equilibrium* payoff vectors are the set of payoff vectors that can be achieved by a strategy profile that is a perfect Bayesian equilibrium.

We show that every equilibrium payoff vector can be achieved by a perfect Bayesian equilibrium in which the sender sends at most $|T| + N$ messages with positive probability. The proof of both bounds uses Carathéodory's theorem to show that for any sender strategy, there exists an alternative sender strategy that has support on only $|T| + N$ messages that (i) induces the same expected payoff for all agents assuming the receivers' strategies remain the same and (ii) conditional on any message, the belief about the sender's type remains the same. The second condition guarantees that the receivers' best-response function remains the same under the alternative strategy. We also show that every Pareto optimal equilibrium payoff vector can be achieved by a perfect Bayesian equilibrium in which the sender sends at most $|T| + N - 1$ messages with positive probability. One less message is needed to attain all Pareto optimal payoff vectors because the set of Pareto optimal payoffs has a dimensionality of one less the dimensionality of all the payoff vectors.

We then give an example to show that the bounds are tight. In this example, there are $N - 1$ passive receivers who do not take any action in the game, while the remaining receiver takes an action. In this game, every payoff vector can be achieved only if the sender sends at least $|T| + N$ messages with positive probability; every Pareto optimal payoff vector can be achieved only if the sender sends at least $|T| + N - 1$ messages with positive probability.

Finally, it is natural to ask whether the bounds could be extended to situations in which there are multiple informed senders. We provide an example to show that such bounds do not exist in games with two privately informed senders who simultaneously send a message to a receiver. The intuition for why the result cannot be extended is that Sender 1's best-response function depends on the distribution of posterior beliefs induced by Sender 2's strategy profile. Hence, to guarantee that Sender 1's best response function remains the same, it is not enough for the posterior belief after each message sent by Sender 2 to remain the same. This is in contrast with the receivers' best response function, which only depends on the posterior belief induced by any given message (but not on the ex ante distribution over messages). Hence, in general, when two senders simultaneously send a message to a receiver, each sender requires a message space equal to the simplex of her type space because a sender can potentially have one message for each possible belief of her type space.

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Literature review. Bester and Strausz (2001) study a principal–agent model where the principal has limited commitment.[1] They show that all Pareto efficient equilibrium payoffs can be achieved by mechanisms in which the agent sends only $|T|$ messages with positive probability. They use Carathéodory's theorem to show that for any fixed strategy (possibly with infinite support), there is always another strategy in which the sender's strategy has support on at most $|T|$ messages, and the receiver's posterior belief after each message is the same as in the original strategy. They then show that, since the original strategy was part of a Pareto efficient equilibrium, the new strategy cannot yield a lower payoff for the receiver.

Our model can be reinterpreted as a principal–agent model with limited commitment.[2] Our paper makes progress by giving bounds when there are multiple uninformed agents and when we are interested in all equilibrium payoffs (including Pareto inefficient ones). We remark that in many applications, it is natural to select Pareto inefficient equilibria, for example, the risk-dominant equilibrium (cf. Carlsson and Van Damme, 1993). We use a strategy similar to that in Bester and Strausz (2001) to prove our results, where the main novelty is that we consider an augmented vector that includes both the receivers' posterior beliefs after each message and the receivers' expected payoffs. Hence, when using Carathéodory's theorem, the receivers' payoffs do not change under the new strategy.

Bester and Strausz (2000) show that in a model such as that in Bester and Strausz (2001), but with more uninformed receivers, it is necessary to have more available messages to achieve all equilibrium payoff vectors. Our example in Section 4 follows a similar intuition to that provided in Bester and Strausz (2000). In fact, when there is only one uniformed receiver and we focus on Pareto optimal payoffs, our example gives the same bound as the example given by Bester and Strausz (2000). Our example generalizes the construction to accommodate any number of additional uninformed receivers and payoff vectors that are not Pareto optimal.

Evans and Reiche (2008) study a model such as that in Bester and Strausz (2000), but in which agents have transferable utilities. They show that transferable utilities, and ex ante transfers, allow a reduction in the number of messages when only one agent is privately informed. They also show that in a model with two privately informed agents, in general, the message space must be larger than that in the benchmark case where only one sender is privately informed (even with transferable utility). We show that such bounds do not exist when there are multiple privately informed senders.

Hart (1985) studies long cheap talk games. He uses Carathéodory's theorem to provide bounds on the number of messages that the sender must send with positive probability to achieve all equilibrium payoff vectors in finitely repeated cheap talk games. We allow for multiple receivers and for the possibility that the messages are payoff-relevant.

Kamenica and Gentzkow (2011) and Bester and Strausz (2007) use Carathéodory's theorem to show that a semi-infinite linear program has a solution with a finite number of non-zero entries. The former paper studies the problem of a sender–receiver game in which the sender can commit ex ante to a strategy; the latter paper studies a sender–receiver game in which communication is mediated. In contrast to their work, the set of Nash equilibria in our model is not characterized as the solution to a semi-infinite linear program.

The rest of the paper proceeds as follows. Section 2 provides the model. Section 3 presents bounds on the number of messages that must be sent with positive probability to achieve all equilibrium payoff vectors. In Section 4, we show that our bounds are tight. In Section 5, we provide an example in which two informed senders need a message space with infinite cardinality to span all equilibrium payoffs vectors. Section 6 discusses our paper's main assumptions. Section 7 concludes.

2. Model

There is a sender and N receivers. The sender has a type $t \in T$, with $|T| < \infty$, and the receivers have prior belief $\pi \in \Delta T$ over T.[3] Receiver i has a type $\theta_i \in \Theta_i$, with Θ_i being a metric space, and the common prior belief over $\Theta \triangleq \times_{i \in N} \Theta_i$ is given by $\mu \in \Delta\Theta$. We additionally assume that $t \in T$ is independent of $\theta \in \Theta$; this assumption simplifies the notation, but the results hold if it is relaxed.

The game is as follows. First, the sender observes $t \in T$ and sends a message $m \in M$. Second, each receiver $i \in \{1, \ldots, N\}$ privately observes θ_i and additionally observes message m. Finally, every receiver $i \in \{1, \ldots, N\}$ takes an action $a_i \in A_i$ simultaneously. We assume that A_i and M are metric spaces and that all functions defined throughout the paper are measurable with respect to the (respective) Borel σ-algebra. We remark that every receiver observes message m; hence, m is a public message.

The payoffs are a function of the sender's type t, the message sent by the sender m, the receivers' type θ, and the action taken by the receivers $a \in A \triangleq \times_{i \in N} A_i$:

$$\forall i \in \{1, \ldots, N, S\} \quad u^i : M \times A \times T \times \Theta \to \mathbb{R}. \tag{1}$$

Note that we allow for the possibility that the utility functions depend directly on the message sent by the sender; this is necessary to accommodate some of the applications.[4]

We denote by ΔM the set of probability measures on M. We denote by S the set of all functions $s : T \to \Delta M$.[5] A sender strategy is an element in $s \in S$. We denote by X_i the set of all functions $x : M \times \Theta_i \to A_i$. A strategy for receiver i is an element $x_i \in X_i$.[6] A strategy profile for the receivers is denoted by:

$$x(m, \theta) = (x_1(m, \theta_1), \ldots, x_N(m, \theta_N)).$$

We denote by X the set of all strategy profiles. Similarly, $(a_i, x_{-i}(m, \theta))$ denotes the strategy profile in which every receiver

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1 We can think of their model as follows. The allocation is given by two actions $(a, a') \in A \times A'$. The principal designs a mechanism that determines what action $a' \in A'$ the principal will take as a function of the message m sent by the sender. In other words, the principal *commits* to a mapping $m \mapsto a'$, which determines the action $a' \in A'$ taken after observing message m. Additionally, after the agent sends a message in this mechanism, the principal takes an action $a \in A$, which must be sequentially rational. In other words, the principal *cannot commit* to taking the particular action $a \in A$ after observing any message m.

2 The mechanism is the game (as described in Section 2) played by the sender and the receiver. First, the agent (sender) sends a message $m \in M$, which could directly affect the players' payoffs if the principal has a partial commitment; then, the principal (receiver) takes an action $a \in A$.

3 The fact that the receivers have a common prior belief is inconsequential for our analysis. Relaxing the common prior assumption would require introducing additional notation, without changing the analysis.

4 For example, suppose we interpret our model as a principal–agent model with limited commitment, as described in Footnote 1. The message sent by the sender (agent) will be payoff relevant because it will determine the action a'. We are not interested in analyzing the set of all possible mechanisms but simply in determining the number of messages that must be sent with positive probability to attain all payoff vectors for a fixed mechanism. Since there is a mapping $m \mapsto a'$ that ultimately determines the players' payoffs, we simply write the utility function of the players directly as a function of the message m.

5 All functions in this paper are assumed to be measurable with respect to the Borel σ-algebra.

6 Note that we have assumed, without loss of generality, that the receivers use pure strategies, as we can always redefine the action space of $i \in \{1, \ldots, N\}$ by $\tilde{A}_i \triangleq \Delta A_i$ and redefine the utility functions u^i appropriately.

other than i uses the strategy specified by $x(m,\theta)$ while receiver i takes action a_i.

A function $p : M \to \Delta T$ is a valid conditional distribution for strategy $s \in S$ if p is computed according to Bayes' rule whenever possible. The conditional distribution p is the players' posterior belief of T after observing $m \in M$. We denote by $\mathbb{E}^{\pi}_{t}[\cdot]$ the expected value over the random variable t using measure π and denote by $\mathbb{E}^{s(t)}_{m}[\cdot]$ the expected value over the random variable m using measure $s(t) \in \Delta M$. The support of strategy $s \in S$ is denoted by $supp(s)$.[7]

We study the set of perfect Bayesian Nash equilibria (henceforth, PBE), which we now formally define.

Definition 1 (*Perfect Bayesian Equilibrium*)**.** A strategy profile $(s,x) \in S \times X$ and a conditional distribution $p : M \to \Delta T$ forms a PBE if:

1. The sender maximizes her expected payoff given the receivers' strategies $x \in X$. That is,

$$\forall\,(t \in T,\ \ m \in supp\,(s\,(t))) \quad m \in \arg\max_{m' \in M} \mathbb{E}^{\mu}_{\theta}[u^{S}(m', x(m',\theta), t, \theta)].$$

2. For all $m \in M$ and for all $\theta_i \in \Theta$, each receiver maximizes his expected profits using probability measure $p(m) \in \Delta T$. That is, for every receiver $i \in \{1,\dots,N\}$:

$$\forall (m \in M,\ \ \theta_i \in \Theta_i)$$
$$x_i(m,\theta_i) \in \arg\max_{a_i \in A_i} \mathbb{E}^{p(m)}_{t}[\mathbb{E}^{\mu}_{\theta}[u^{i}(m,(a_i, x_{-i}(m,\theta_{-i})), t, \theta)|\theta_i]].$$

3. p is a valid conditional distribution for strategy s.

In the following section, we define the set of payoffs that can be attained in this game and give an upper bound on the number of elements that $supp\,(s\,(t))$ must have to attain all payoffs.

3. Bound on message space

We denote by $\mathcal{W} \subset \mathbb{R}^{N+|T|}$ the set of all payoff vectors that are attained by some PBE , where the first N coordinates refer to the ex ante utility of the receivers and the last $|T|$ coordinates denote the interim utility of the $|T|$ types of sender.[8] We denote by $\mathcal{V} \subset \mathbb{R}^{N}$ the projection of $\mathcal{W}$ onto the payoffs of the receivers. We denote by $\mathcal{PV}$ the set of Pareto optimal payoffs in $\mathcal{V}$:

$$\mathcal{PV} = \left\{ v \in \mathcal{V} : \begin{array}{c} \nexists v' \in \mathcal{V} \text{ such that } v' \text{ is weakly larger than} \\ v \text{ in all components,} \\ \text{and } v' \text{ is strictly larger than } v \text{ in some component.} \end{array} \right\}.$$

That is, $\mathcal{PV}$ is the set of all vectors in $v \in \mathcal{V}$ such that there is no other vector $v' \in \mathcal{V}$ with the property that vector v' is weakly larger than v in all components and strictly larger in at least one component. We denote by $\mathcal{PW} \subset \mathcal{W}$ the set of payoffs on $\mathcal{W}$ such that their projection onto the payoffs of the receivers lies in $\mathcal{PV}$. We note that $\mathcal{PW}$ is a weak sense of Pareto optimality, as it does not include the utility of the sender. Proposition 2 also holds if we consider the standard notion of Pareto optimality, which includes the sender's utility. We study this weaker form of Pareto optimality because it is the notion of optimality studied by Bester and Strausz (2001) (they call $\mathcal{PW}$ the set of *incentive efficient* payoff vectors); thus, our results are directly comparable. The objective is to characterize the minimum number of messages that need to be sent in equilibrium to allow attaining all payoffs in $\mathcal{W}$ and $\mathcal{PW}$.

We begin by giving a preliminary result. Let $((s,x),p)$ be a PBE and define $f : M \times T \to \mathbb{R}^{N}$ as follows:

$$f(m,t) \triangleq \mathbb{E}^{\mu}_{\theta}[(u^{1}(m,x(m,\theta),t,\theta),\dots,u^{N}(m,x(m,\theta),t,\theta))]. \quad (2)$$

That is, $f(m,t)$ is the receivers' expected payoff when the sender sends message m and she is type t. We now show that the same payoff vectors can be attained by an alternative "equivalent" strategy s'.

Proposition 1 (*Change of Measure*)**.** *For every $s \in S$ and $p : M \to \Delta T$, such that p is a valid conditional distribution for strategy s, there exists $s' \in S$ such that*

1. *The support of s' is finite:*

$$|supp(s')| \le |T| + N. \quad (3)$$

2. *Conditional distribution p is also a valid conditional distribution for strategy s'.*
3. *The expectation of $f(m,t)$ remains the same:*

$$\mathbb{E}^{\pi}_{t}[\mathbb{E}^{s(t)}_{m}[f(m,t)]] = \mathbb{E}^{\pi}_{t}[\mathbb{E}^{s'(t)}_{m}[f(m,t)]]. \quad (4)$$

Proposition 1 shows that for any strategy $s \in S$, we can find an "equivalent" strategy $s' \in S$ in which the number of messages sent is at most $|T| + N$. The sense in which these strategies are equivalent is as follows. First, the conditional distribution when the strategy is s is also a valid conditional distribution under s'. Second, the expected value of $f(m,t)$ is the same under s as it is under s'. The first property guarantees that the receivers' best-response function is the same for strategies s and s'; the second property states that changing the sender's strategy from s to s' will not change the receivers' payoff. Note that the sender's expected payoff is not included in the vector $f(m,t)$. However, because s is an equilibrium strategy, every message that is sent with positive probability when the sender is type t yields the same expected payoff. Hence, the sender's optimality condition will guarantee that changing the sender's strategy from s to s' will not change the sender's payoff.

We now give a brief overview of the proof. Fix some strategy s and conditional probability p. We denote by $v(m)$ the expectation of f conditional on m. That is:

$$v(m) \triangleq \mathbb{E}^{p(m)}_{t}[f(m,t)]. \quad (5)$$

We define the vector $r(m) \in \mathbb{R}^{|T|+N}$ and its expected value as follows

$$r(m) \triangleq \begin{pmatrix} p(m) \\ v(m) \end{pmatrix} \quad \text{and} \quad \bar{r} \triangleq \mathbb{E}^{q}_{m}[r(m)], \quad (6)$$

where q denotes the pushforward measure (or image measure) on M induced by s (i.e., for any measurable set $A \subset M$, $q(A) = \pi(s^{-1}(A))$). The pushforward measure q is closely related to the distribution of posterior beliefs; if every message induces a different posterior belief $\{p(m)\}_{m\in M}$, then the pushforward measure on M is the same as the distribution of posterior beliefs $\{p(m)\}_{m\in M}$.

Now we can use Carathéodory's theorem to prove Proposition 1. Carathéodory's theorem states that there exists a different pushforward measure over M, say q', that places positive weight on only $|T| + N$ messages, and the expected value of $r(m)$ remains the same (i.e., $\mathbb{E}^{q'}_{m}[r(m)] = \bar{r}$). Because $\mathbb{E}^{q'}_{m}[p(m)] = \pi$ (note that the first T components of $\bar{r}$ are equal to π), we

[7] That is, $supp(s)$ is a subset $M' \subset M$ such that M' has probability one of occurring and there is no open subset of M' that has probability zero of occurring.

[8] For any strategy profile (s,x), the sender interim expected utility when her type is t is given by:

$\mathbb{E}^{s(t)}_{m}[\mathbb{E}^{\mu}_{\theta}[u^{S}(m,x(m,\theta),t,\theta)]]$;

receiver i's ex ante utility is given by:

$\mathbb{E}^{\pi}_{t}[\mathbb{E}^{s(t)}_{m}[\mathbb{E}^{\mu}_{\theta}[u^{i}(m,x(m,\theta),t,\theta)]]]$.

can construct a sender strategy s' that induces this pushforward measure. Because $\mathbb{E}_m^{q'}[v(m)] = \mathbb{E}_m^{q}[v(m)]$, this alternative strategy profile will satisfy (4).

We can now give our paper's main result.

Proposition 2 (*Message Space Needed to Achieve $\mathcal{W}$ and $\mathcal{PW}$*). *For all $w \in \mathcal{W}$ (resp. $w \in \mathcal{PW}$), there exists a PBE $((s, x), p)$ in which the equilibrium payoff vector is w and $|supp(s)| \leq N + |T|$ (resp. $|supp(s)| \leq N + |T| - 1$).*

This proposition shows that all equilibrium payoff vectors can be achieved by a perfect Bayesian Nash equilibrium in which the sender sends at most $|T| + N$ messages with positive probability in equilibrium. We prove this by starting with an equilibrium in which the sender uses strategy s. Proposition 1 is then used to show that there exists another strategy s' in which the sender sends at most $|T| + N$ messages in equilibrium and the expected utilities of all players remain the same. Moreover, because the conditional probability over the sender's type remains the same, the continuation game after the sender sends m does not change; that is, the incentives of all the other players remain the same.

The reason that one fewer message is required to achieve all equilibrium payoffs in $\mathcal{PV}$ is that the dimensionality is $N - 1$, whereas $\mathcal{V}$ has dimensionality N. Because the dimensionality of $\mathcal{PW}$ is smaller than the dimensionality of $\mathcal{W}$, one less message is needed to span $\mathcal{PW}$. A simple way to see this is to observe that the first $N - 1$ components of $f(m, t)$ are sufficient to specify a point in $\mathcal{PW}$.

The following two sections show that our bounds are tight and cannot be extended to models with two privately informed senders.

4. Tightness of the bounds

We now show via an example that the bounds in Proposition 2 are tight. Let the type space and the action space be as follows:

$$T = \{0, 1\} \quad ; \quad \forall i \neq N,\ A_i = \{\emptyset\} \quad ; \quad A_N = [0, 1]. \tag{7}$$

In other words, the sender has two types and only receiver N takes an action in the game. The prior distribution of the sender's type assigns equal probability to both types. The sender's message space is $M = \{m_1, \ldots, m_{N+2}\}$. The sender's payoff is always equal to zero (i.e., $u^s = 0$), so her incentives play no role in the construction of the equilibrium.

The utility function of receiver N is given by:

$$u^N(a, t) \triangleq -a^2 - t + 2 \cdot t \cdot a + \frac{1}{4}. \tag{8}$$

Clearly, if receiver N assigns probability $\tilde{p} \in [0, 1]$ that the sender is type $t = 1$, then his best response is[9]:

$$\tilde{p} = \arg\max_{a \in [0,1]} \mathbb{E}_t^{\tilde{p}}[-a^2 - t + 2 \cdot t \cdot a + \frac{1}{4}]. \tag{9}$$

The utility function of receiver $i \in \{1, \ldots, N - 1\}$ depends on only the action taken by receiver N (that is, it does not depend on the sender's type). To construct the utility function of receiver $i \in \{1, \ldots, N - 1\}$, we first define

$$\forall i \in \{1, \ldots, N + 2\}, \quad p_i \triangleq \begin{cases} 1/2 + \varepsilon(N - i) & i \in \{1, \ldots, N\}; \\ 1 & i = N + 1; \\ 0 & i = N + 2. \end{cases}$$

Here, ε is a sufficiently small number in the sense specified in the proof of Proposition 3. The utility function of receivers $i \in \{1, \ldots, N - 1\}$ is defined as follows,

$$\forall i \in \{1, \ldots, N - 1\}, \quad u^i(a) = \begin{cases} 1 & a = p_i; \\ 0 & a \in \{p_1, \ldots, p_{N+2}\} \backslash \{p_i\}; \\ -1 & \text{otherwise}. \end{cases} \tag{10}$$

The utility of receiver $i \in \{1, \ldots, N - 1\}$ is equal to 1 if and only if receiver N takes action $a = p_i$. We now formally state that the bounds in Proposition 2 are tight.

Proposition 3 (*Bounds are Tight*). *There exists a sufficiently small $\varepsilon > 0$ such that the game defined by (7)–(10) has a payoff vector $w \in \mathcal{W}$ (resp. $w \in \mathcal{PW}$) that can be achieved as a PBE only if the sender sends at least $|T| + N$ (resp. $|T| + N - 1$) messages with positive probability.*

We now provide an intuitive account for Proposition 3. Receiver i receives a positive payoff only if action $a \in \{p_1, \ldots, p_{N-1}\}$ is taken with positive probability. Therefore, any equilibrium in which every receiver $i \in \{1, \ldots, N - 1\}$ receives a positive payoff requires that each action in $\{p_1, \ldots, p_{N-1}\}$ be taken with positive probability. However, each action in $\{p_1, \ldots, p_{N-1}\}$ occurs only when receiver N's posterior belief assigns similar probability to both of the sender's types; in other words, this only happens when receiver N is almost completely uninformed. Therefore, when receiver N takes an action $a \in \{p_1, \ldots, p_{N-1}\}$, his expected utility is close to zero. The maximum utility receiver N can attain is 1/4, which is achieved only when he knows the sender's type. Thus, receiver N can obtain an expected payoff close to 1/4 only if the sender signals his type perfectly with sufficiently high probability — this requires two messages. It follows that there is some payoff in $\mathcal{PV}$ that requires the sender to send $|T|+N-1$ messages with positive probability in equilibrium; that is, one message for each action in $\{p_1, \ldots, p_{N-1}\}$ and two additional messages so that receiver N's payoff is sufficiently close to 1/4. Finally, the sender can reduce the payoff of all receivers by sending a message that induces action a_N with positive probability. Thus, there are some payoffs in $\mathcal{W}$ that require the sender to send one additional message.

5. Two informed senders

We now examine a game in which two privately informed senders simultaneously send a message to a receiver. We show that it is necessary to use an infinite-cardinality message space to achieve all equilibrium payoff vectors. In other words, we give an example in which for all message spaces M_1, M_2 such that $|M_1|, |M_2| < \infty$, there exists an equilibrium payoff vector that can be attained only if the message space's cardinality is larger than $|M_1|$ and $|M_2|$. This game illustrates that there does not exist a finite bound on the number of messages needed to achieve all equilibrium payoff vectors when there are two informed senders who simultaneously send a message to a receiver.

There are two informed senders and one receiver. The senders' type space is $T_1 = T_2 = \{0, 1\}$, and the two types are independently distributed, with equal prior distributions. Throughout this section, we denote by π the probability that sender i is type $t = 1$.

The game is as follows. Each sender $i \in \{1, 2\}$ simultaneously sends a message in M_i to a receiver. After observing the messages sent by the senders, the receiver takes action $a \in [0, 1]^2$. The receiver's utility function is given by:

$$u^R(a_1, a_2, t_1, t_2) \triangleq t_1 \cdot a_1 - \frac{a_1^2}{2} + t_2 \cdot a_2 - \frac{a_2^2}{2}.$$

[9] To check this, note that $\mathbb{E}_t^p[-a^2 - t + 2 \cdot t \cdot a + \frac{1}{4}] = -a^2 - p + 2 \cdot p \cdot a + \frac{1}{4}$. This is a quadratic function over a with a unique maximum at $a = p$.

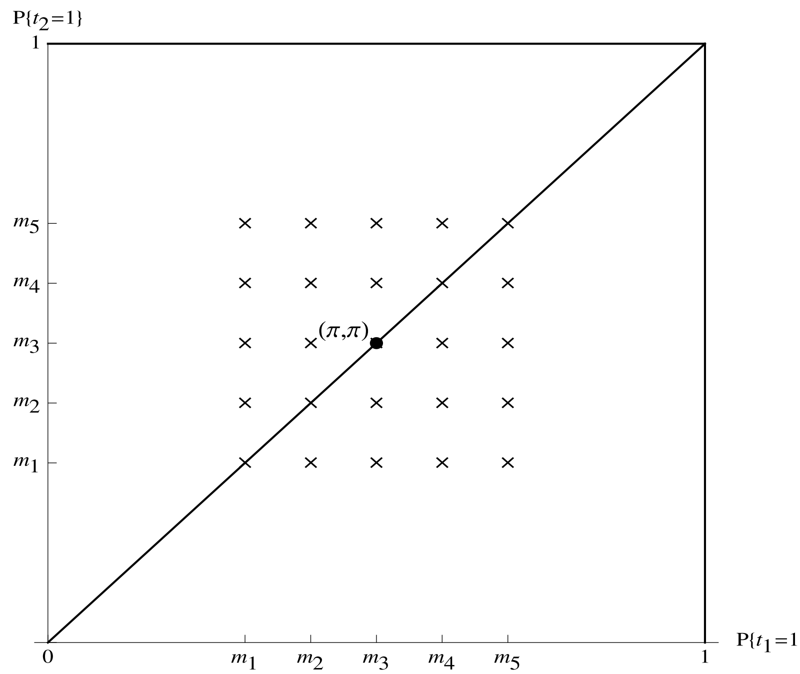

Fig. 1. PBE using 5 messages per player.

For any $\tilde{p}_1, \tilde{p}_2 \in \Delta\mathbb{R}^2$, the receiver's best response is given by:

$$(\tilde{p}_1, \tilde{p}_2) = \underset{(a_1,a_2)\in[0,1]^2}{\arg\max}\ \mathbb{E}^{\tilde{p}_1,\tilde{p}_2}_{t_1,t_2}[t_1 \cdot a_1 - \frac{a_1^2}{2} + t_2 \cdot a_2 - \frac{a_2^2}{2}] \tag{11}$$

The senders' payoff function is given by:

$$(u^1(a_1, a_2), u^2(a_1, a_2)) \triangleq \begin{cases} (1, -1) & a_1 = a_2; \\ (0, 0) & \text{otherwise.} \end{cases} \tag{12}$$

Sender 1 receives a payoff of 1 if the receiver takes an action (a_1, a_2) such that $a_1 = a_2$, and she obtains a payoff equal to zero otherwise. Sender 2 receives a payoff equal to the negative of the payoff of Sender 1, so this is a zero-sum game for the senders.

We denote by $\mathcal{U} \subset \mathbb{R}^2$ the set of ex ante payoffs of the senders that are attained by an equilibrium strategy that is a PBE. We now formally state that it is not possible to bound the number of messages that each sender sends with positive probability.

Proposition 4 (*Unbounded Message Space is Needed*)**.** *For all $n > 0$, $(+1/n, -1/n) \in \mathcal{U}$ if and only if $|M_1|, |M_2| \geq n$.*

The details of the proof are given in the Appendix; here, we briefly present an intuitive account of the result. In Fig. 1, we illustrate an equilibrium in which each sender sends five messages with positive probability. The x-axis represents the receiver's posterior belief about the type of Sender 1, and the y-axis represents the receiver's posterior belief about the type of Sender 2. Each message induces a different posterior belief for the receiver.

The strategies are constructed such that each message is sent with an equal probability *ex ante*. The probability with which each message is sent depends on the sender's type, which is why they induce different posterior beliefs for the receiver. However, the strategies are appropriately constructed such that from an ex ante perspective, each message is sent with probability 1/5.

We can now heuristically verify that these strategies conform to an equilibrium. For this, we simply need to check that each message yields an equal expected probability for each sender. From the perspective of Sender 1, all messages sent by Sender 2 have equal probability, so all messages induce a payoff of 1 with a probability of 1/5. Hence, Sender 1 is indifferent across all messages. Since Sender 1 is indifferent to all messages, he randomizes between messages in such a way that each message induces a posterior as suggested by the equilibrium. The argument for Sender 2 is analogous.

Following similar arguments, one can construct equilibria in which each sender uses n messages and the senders' expected payoff is $(1/n, -1/n)$. Moreover, we show that the expected payoff vector $(1/n, -1/n)$ cannot be attained if the senders cannot send at least n different messages. This example illustrates that with two privately informed senders, it is not possible to bound the cardinality of the message space to achieve all equilibrium payoff vectors.

When two senders simultaneously send a message to a receiver, each sender requires a message space equal to the simplex of her type space because a sender can potentially have one message for each possible belief of her type space. Our example illustrates that games with two senders simultaneously sending messages are a particular class of two players playing an *infinite* normal-form game. This result follows from the fact that (in general) the receiver can take one action for every possible belief he may have, and the sender's utility function may be any function of the actions taken by the receiver. By changing the senders' payoff function (12), we can replicate any two-player normal-form game in which players' actions space is equal to [0, 1]. Thus, bounding the message space of the senders is similar to the problem of finding finite approximations of infinite normal-form games.[10]

6. Discussion of assumptions

Finally, we discuss some of our model's most important assumptions.

Consider first the assumption that the sender sends a public message. One could alternatively consider a model in which the sender can send a private message to each receiver. As shown by Aumann (1987), in this case the sender can send correlated

10 For example, see Reny (2011) for a paper analyzing this problem.

signals that can help to attain payoffs that cannot be achieved when the signals are public (more precisely, it is possible to attain the set of all correlated equilibria).[11] The sender may enlarge the set of equilibrium payoffs by sending private signals even if she has no private information (i.e., even if T is a singleton). Hence, allowing for private messages presents additional complexities that are not present when the message m is a public message. Therefore, our results clearly do not apply to the case in which the sender can send private messages. Moreover, the number of messages sent by the sender in equilibrium is not bounded by the cardinality of the sender's type space but instead depends on the number of actions available to the receivers.

A second important assumption in our model is that the sender communicates with the receivers directly. There is a rich literature studying mediated communication. In this literature, the sender sends a message to a mediator, who later sends messages to the receivers. Mediated communication can enlarge the set of equilibrium payoffs. Bester and Strausz (2007) study a principal–agent model in which the principal does not have full commitment but in which the principal and agent communicate via a noisy communication device. The bounds on the message space in Bester and Strausz (2007) are found using linear programming techniques.[12] When the communication is direct (as in our model), it is not possible to use techniques from linear programming. Therefore, the results from this literature cannot be applied to models of direct communication.

Finally, we have assumed that the sender sends a message to the receiver once, i.e., there is only one stage of communication. Allowing for multiple stages of communication may augment the set of equilibrium payoffs (cf. Aumann and Hart, 2003). However, the bounds on the number of messages that the sender must send with positive probability to attain all equilibrium payoffs remains the same as long as the messages sent by the sender are public (see Bester and Strausz, 2001 for a discussion).

7. Conclusions

We showed that in a broad class of sender–receiver games, it is possible to restrict attention to sender strategies in which the number of messages sent with positive probability is bounded. The general principle behind the result is that it is possible to change the sender's strategy without changing the optimality conditions of the receivers or the expected payoffs. We have provided a general analysis that allows understanding when this technique can and cannot be used. We hope that this paper will bring this technical result to the attention of a broader audience and simplify the analysis of a wide range of problems in game theory.

Appendix

In the appendix, we provide the proofs of all the results in the main text. We begin by presenting Carathéodory's theorem.

Theorem 1 (*Carathéodory's Theorem*)**.** *If $\bar{r}$ lies in the convex hull of a set $R \subset \mathbb{R}^N$, then $\bar{r}$ can be written as a linear combination of $N+1$ elements of R.*

Proof. See Rockafellar (1970, p. 155). ■

We now present the proofs of the results in our paper.

Proof of Proposition 1. Throughout the proof, $s_m(t)$ denotes the probability that function $s : T \to \Delta M$ assigns to $m \in M$ when the type is $t \in T$; π_t is the probability that the sender is type t according to the prior $\pi \in \Delta T$; and $p_t(m)$ is the probability that the sender is type t according to the conditional distribution p when the observed message is m.

We now note that $p(m)$ lies in $\Delta\mathbb{R}^{|T|}$, so we can write $p(m)$ as an element in $\mathbb{R}^{|T|-1}$. Additionally, $v(m)$ lies in $\mathbb{R}^N$. Therefore, we can write $r(m)$ (defined in (6)) as an element of $\mathbb{R}^{|T|+N-1}$. Since $\bar{r}$ lies in the convex hull of $\{r(m)\}_{m \in supp(s)}$, by Carathéodory's theorem, we know that there exists $K \subset supp(s)$, with $|K| \le N+T$ and $\alpha \in \mathbb{R}_+^{|T|+N}$, such that

$$\bar{r} = \sum_{m \in K} \alpha(m) r(m). \tag{13}$$

Consider strategy s' with support in K, where the probability that the sender sends message $m \in K$ when she is type $t \in T$ is equal to:

$$s'_m(t) = \frac{\alpha(m) p_t(m)}{\pi_t},$$

and $s'_m(t) = 0$ if $m \notin K$. We make two observations. First, the definitions of s' and (13) imply that:

$$\sum_{m \in K} s'_m(t) = \sum_{m \in K} \frac{\alpha(m) p_t(m)}{\pi_t} = \frac{1}{\pi_t} \sum_{m \in K} \alpha(m) p_t(m) = \frac{\pi_t}{\pi_t} = 1.$$

Hence, the probabilities assigned to each message add up to one. This result implies that the strategy profile is well defined. Second, by Bayes' rule:

$$\forall m \in supp(s') \quad \mathbb{P}\{t | s(m)\} = \frac{s'_t(m)\pi_t}{\sum_{\tilde{t} \in T} s'_{\tilde{t}}(m)\pi_{\tilde{t}}} = \frac{\alpha(m) p_t(m)}{\alpha(m) \sum_{\tilde{t} \in T} p_{\tilde{t}}(m)} = p_t(m).$$

Hence, for all $m \in supp(s')$, the posterior distribution over T conditional on $m \in M$ is equal to $p(m)$. Additionally, all messages $m \notin supp(s')$ have zero probability of being sent. Therefore, Bayes' rule imposes no restrictions on the probability over T conditional on $m \notin supp(s')$, which implies that p is a valid conditional distribution when the sender uses strategy s'. Therefore, by construction, items 1 and 2 of Proposition 1 are satisfied.

Finally, we observes that $\{\alpha(m)\}_{m \in M}$ is the pushforward measure induced by strategy s'. Thus, we have that[13]:

$$\begin{aligned}
\mathbb{E}_t^{\pi}[\mathbb{E}_m^{s'(t)}[f(m,t)]] &= \sum_{m \in M} \alpha(m) \mathbb{E}_t^{p(m)}[f(m,t)] && (14)\\
&= \sum_{m \in M} \alpha(m) v(m) && (15)\\
&= \mathbb{E}_m^{q}[\mathbb{E}_t^{p(m)}[f(m,t)]] && (16)\\
&= \mathbb{E}_t^{\pi}[\mathbb{E}_m^{s(t)}[f(m,t)]]. && (17)
\end{aligned}$$

The first equality follows from the fact that p is a valid conditional distribution for s'. The second equality follows from the definition of $v(m)$. The third equality is implied by (13) (more precisely, by the last N components of (13)). Finally, the fourth equality follows from the fact that p is a valid conditional distribution for s. It follows that point 3 of Proposition 1 is also satisfied. This concludes the proof. ■

[11] Aumann (1987) studies a model similar to ours but where sender and receivers do not have private information and the sender's payoff function is constant. The assumption that the sender's payoff function is constant means that it is not necessary to consider the sender's incentives.

[12] Pollrich (2017) studies a model of mediated communication between an agent and a principal that cannot be studied by the techniques developed by Bester and Strausz (2007). Salamanca (2016) studies a model of Bayesian persuasion where the sender has limited commitment but the communication between sender and receiver is mediated.

[13] For all measurable functions $f(m,t)$ and s, if p is a valid conditional distribution, then $\mathbb{E}_t^{\pi}[\mathbb{E}_m^{s(t)}[f(m,t)]] = \mathbb{E}_m^{q}[\mathbb{E}_t^{p(m)}[f(m,t)]]$, where q is the pushforward measure induced by s. See, for example, Pollard (2002) for a proof.

Proof of Proposition 2. (Part I: Payoff Vector in $\mathcal{W}$) Proposition 1 states that there exists $s' \in S$ such that $|supp(s')| \leq |T|+N$, p is a valid conditional distribution when the sender uses strategy s', and

$$\mathbb{E}_t^{\pi}[\mathbb{E}_m^{s(t)}[f(m,t)]] = \mathbb{E}_t^{\pi}[\mathbb{E}_m^{s'(t)}[f(m,t)]], \tag{18}$$

where $f(m,t)$ is defined in (4). We now check that $((s',x),p)$ is a PBE and that the expected utility of the sender and the receivers is the same under PBE $((s,x),p)$ and PBE $((s',x),p)$.

By construction, $((s',x),p)$ satisfies condition 3 in Definition 1. We can also check that $((s',x),p)$ satisfies condition 2 in Definition 1. To verify this, note that by construction, p is a valid conditional distribution for both strategies s and s'. The receivers' best response after each message depends only on the distribution of T conditional on m. Thus, since $((s,x),p)$ is a PBE, x satisfies condition 2 in Definition 1. Finally, since $((s,x),p)$ forms a PBE, we have that:

$$\forall\,(t\in T,\ m\in supp\,(s\,(t)))\quad m\in\arg\max_{m'\in M}\mathbb{E}_{\theta}^{\mu}[u^S(m',x(m',\theta),t,\theta)].$$

From the proof of Proposition 1, it is clear that, for all $t \in T$, $supp(s'(t)) \subset supp(s(t))$. In other words, under strategy s', the sender uses a subset of the messages she uses under strategy s. This implies that:

$$\forall\left(t\in T,\ m\in supp\left(s'\left(t\right)\right)\right)\quad m\in\arg\max_{m'\in M}\mathbb{E}_{\theta}^{\mu}[u^S(m',x(m',\theta),t,\theta)].$$

Therefore, s' is an optimal strategy, so condition 1 in Definition 1 is also satisfied. Therefore, $((s',x),p)$ is a PBE.

By construction, the expected utility of the receivers stays the same (i.e., s' is constructed so that (18) is satisfied). Moreover, since the receivers' strategy profile has not changed and both s and s' are sender optimal strategies, they must yield the same expected payoff to the sender. Thus, the sender's interim expected utility is the same under (s,x) as it is under (s',x). Hence, we prove the first part of Proposition 2.

(Part II: Payoff Vector in $\partial\mathcal{W}$) We now fix the payoff vector as $w \in \mathcal{PW}$. We already proved that for all $w \in \mathcal{W}$ there exists a PBE $((s,x),p)$ that induces this payoff vector and $|supp(s)| \leq N+|T|$. Thus, we assume without loss of generality that the PBE $((s,x),p)$ satisfies $|supp(s)| = N+|T|$ (if the inequality were strict then the result would be trivially satisfied, so we assume that $|supp(s)| = N+|T|$ to address the nontrivial case). We prove the results in two steps.

(Step 1) Step 1 of the proof consists of showing that if $w \in \mathcal{PW}$, then $\{r(m)\}_{m\in supp(s)}$ – defined as in (5), (6) and (4) – contains strictly less than $N+|T|-1$ linearly independent vectors. We proceed by contradiction. That is, we assume that $\{r(m)\}_{m\in supp(s)}$ contains $N+|T|-1$ linearly independent vectors and then reach a contradiction. Before we proceed we remind the reader that $\{r(m)\}_{m\in supp(s)}$ contains $N+|T|$ elements but at most $N+|T|-1$ linearly independent vectors because $p(m)$ lies in $\Delta\mathbb{R}^{|T|}$ (this was explained in the proof of Proposition 1). Thus, the objective of this part of the proof is to show that when w belongs to $\mathcal{PW}$ instead of $\mathcal{W}$, then $\{r(m)\}_{m\in supp(s)}$ contains one less linearly independent vector.

We first define $\hat{M} \subset supp(s)$ as follows:

$$\hat{M} \triangleq \{m\in supp(s)\mid s_t(m)=1 \text{ for some } t\in T\}.$$

In other words, $\hat{M}$ is the set of messages such that at least one type sends a message $m \in \hat{M}$ with probability one. Similarly, we denote by $\hat{T}$ the set of types that send a message $m \in \hat{M}$ with probability one:

$$\hat{T} \triangleq \{t\in T\mid s_t(m)=1 \text{ for some } m\in\hat{M}\}.$$

We note that for all $t \in \hat{T}$ and $m \notin \hat{M}$, $p_t(m) = 0$. In other words, the conditional probability of being a type $t \in \hat{T}$ conditional on *not* observing a message in $\hat{M}$ is equal to zero.

Note that for every $m \in supp(s)\backslash\hat{M}$:

$$\forall t\in\hat{T},\quad p_t(m)=0. \tag{19}$$

Since we assumed that $\{r(m)\}_{m\in supp(s)}$ contains $N+|T|-1$ linearly independent vectors, the vectors $\{r(m)\}_{m\in supp(s)\backslash\hat{M}}$ contain at least $N+|T|-1-|\hat{T}|$ linearly independent vectors.

Let

$$e \triangleq \begin{pmatrix}\vec{0}\\ \vec{1},\end{pmatrix}$$

where $\vec{0}\in\mathbb{R}^{|T|}$ and $\vec{1}\in\mathbb{R}^N$. Since $\{r(m)\}_{m\in supp(s)\backslash\hat{M}}$ consists of $|T|+N-1-|\hat{T}|$ linearly independent vectors, these vectors completely span the subspace of $\mathbb{R}^{|T|+N}$ that satisfies the constraints (19) and $p(m)\in\Delta\mathbb{R}^{|T|}$. Clearly, e also satisfies these constraints. Thus, we can write e as a linear combination of the vectors $\{r(m)\}_{m\in supp(s)\backslash\hat{M}}$. Thus, we can find $\{\beta(m)\}_{m\in supp(s)\backslash\hat{M}}$ such that

$$e=\sum_{m\in supp(m)\backslash\hat{M}}\beta(m)r(m). \tag{20}$$

Note that β is defined only for messages $m \notin \hat{M}$. We extend the definition of β to messages $m \in \hat{M}$ by defining $\beta(m) = 0$. Clearly:

$$e=\sum_{m\in supp(m)}\beta(m)r(m).$$

This expression differs from (20) because here we are summing over all messages (not only over messages in $supp(m)\backslash\hat{M}$); since $\beta(m) = 0$ for the extra messages, we are only adding null terms.

Consider strategy s' defined as follows:

$$s'_m(t)=\begin{cases}\frac{(q(m)+\varepsilon\cdot\beta(m))p_t(m)}{\pi_t} & m\in supp(s);\\ 0 & \text{otherwise},\end{cases}$$

where $p_t(m)$ is the conditional distribution of the PBE, q is the pushforward measure on M induced by s, and ε is sufficiently small. We first observe that:

$$\begin{aligned}\sum_{m\in supp(s)}s'_m(t)&=\sum_{m\in supp(s)}\frac{(q(m)+\varepsilon\beta(m))p_t(m)}{\pi_t}\\ &=\frac{1}{\pi_t}\sum_{m\in supp(s)}(q(m)+\varepsilon\beta(m))p_t(m) &(21)\\ &=\frac{1}{\pi_t}\sum_{m\in supp(s)}q(m)p_t(m)+\frac{\varepsilon}{\pi_t}\sum_{m\in supp(s)}\beta(m)p_t(m)\\ & &(22)\\ &=1. &(23)\end{aligned}$$

Hence, the probabilities assigned to each message sum to one. Additionally, we consider the case in which strategy s has full support over $N+|T|$ messages, so $s_m(t) > 0$ if ε is sufficiently small. Finally, we note that by Bayes' rule, $s_t(m) = q(m)p_t(m)/\pi_t$ and $\beta(m) \neq 0$ only if $s_t(m) < 1$, so $s'_t(m) < 1$ for a sufficiently small ε. This result implies that strategy profile s' is well defined.

Second, by Bayes' rule:

$$\forall m\in supp(s'),\quad \frac{s'_t(m)\pi_t}{\sum_{t'\in T}s'_{t'}(m)\pi_{t'}}=\frac{(q(m)+\varepsilon\beta(m))p_t(m)}{(q(m)+\varepsilon\beta(m))\sum_{t'\in T}p_{t'}(m)}=p_t.$$

Hence, for all $m \in supp(s)$, the distribution over T conditional on $m \in supp(s')$ is equal to $p(m)$. Additionally, all messages $m \notin supp(s')$ have zero probability of being sent. Thus, any distribution over T conditional on $m \notin supp(s')$ conforms to Bayes' rule. This result implies that p is a valid conditional distribution when the sender uses strategy s'. Following the same arguments as in Part I of the proof, we conclude that $((s', x), p)$ is a Nash equilibrium.

Finally, following the same steps as in (14)–(17), we obtain that:

$$\mathbb{E}_t^{\pi}[\mathbb{E}_m^{s'(t)}[f(m,t)]] = \sum_{m \in supp(s)} (q(m) + \varepsilon\beta(m))f(m,t) \quad (24)$$

$$= \mathbb{E}_t^{\pi}[\mathbb{E}_m^{s(t)}[f(m,t)]] + \varepsilon \sum_{m \in supp(s)} \beta(m)f(m,t) \quad (25)$$

$$= \mathbb{E}_t^{\pi}[\mathbb{E}_m^{s(t)}[f(m,t)]] + (1,\ldots,1) \quad (26)$$

$$> \mathbb{E}_t^{\pi}[\mathbb{E}_m^{s(t)}[f(m,t)]] \quad (27)$$

Thus, strategy s' yields a higher expected payoff to every receiver. This leads to a contradiction with the assumption that $w \in \mathcal{PW}$, so we conclude that $\{r(m)\}_{m\in supp(s)}$ – defined as in (5), (6) and (4) –contains strictly less than $N + |T| - 1$ linearly independent vectors.

(Step 2) From the previous part of the proof, we know that for all $w \in \partial\mathcal{W}$, there exists a PBE $((s,x),p)$ that induces this payoff vector and $|supp(s)| \le N + |T|$. Moreover, the previous step of the proof shows that if $|supp(s)| = N + |T|$, then $\{r(m)\}_{m\in supp(s)}$ has at most $|T| + N - 2$ linearly independent vectors. Hence, $\{r(m)\}_{m\in supp(s)}$ lies in a space of dimension $|T| + N - 2$. We can repeat the same arguments as in Part I of this proof and use Caratheodory's theorem to show that there exists a strategy with support cardinality of at most $|T| + N - 1$. From this point onwards, the proof follows in the same way as that in Part I, which concludes the proof. ■

Proof of Proposition 3. We first prove the case that $w \notin \mathcal{PW}$ and then prove the case that $w \in \mathcal{PW}$.

Part I. To show that a payoff $w \notin \mathcal{PW}$ requires all messages $\{m_1,\ldots,m_{N+2}\}$, we consider the following payoff vector in $\mathcal{V}$:

$$v = (\underbrace{\varepsilon^2,\ldots,\varepsilon^2}_{N-1 \text{ times}}, (1-\varepsilon)/4),$$

where ε is sufficiently small. In other words, we want to achieve a payoff vector that yields $1/4 - \varepsilon$ to receiver N and ε^2 to all other receivers. The proof now proceeds in two steps. We first show that this payoff vector can be achieved by a PBE and then show it can be achieved only if the message space has more than $N + |T|$ messages.

Step A. We first show that the payoff vector $v = (\varepsilon^2,\ldots,\varepsilon^2, (1-\varepsilon)/4)$ can be achieved by a PBE. We construct the equilibrium assuming that the prior distribution is $\pi = (1/2, 1/2)$, but the construction can be modified accordingly to accommodate any prior. We relabel the message space as $M = \{1,\ldots,N+2\}$. Each message $i \in \{1,\ldots,N+2\}$ is associated with the posterior belief p_i. We denote by $s_i(t)$ the probability that the sender sends message i when she is type t and consider the following strategy for the sender:

$$s_i(1) = \begin{cases} \varepsilon^2(1+2\varepsilon(N-i)) & i \in \{1,\ldots,N-1\}; \\ \varepsilon - (N-1)\varepsilon^2 + 4\sum_{i=1}^{N-1}\varepsilon^4(N-i)^2 & i = N; \\ 1 - \varepsilon - \sum_{i=1}^{N-1}\varepsilon^3 2(N-i) - 4\sum_{i=1}^{N-1}\varepsilon^4(N-i)^2 & i = N+1; \end{cases}$$

$$s_i(0) = \begin{cases} \varepsilon^2(1-2\varepsilon(N-i)) & i \in \{1,\ldots,N-1\}; \\ \varepsilon - (N-1)\varepsilon^2 + 4\sum_{i=1}^{N-1}\varepsilon^4(N-i)^2 & i = N; \\ 1 - \varepsilon + \sum_{i=1}^{N-1}\varepsilon^3 2(N-i) - 4\sum_{i=1}^{N-1}\varepsilon^4(N-i)^2 & i = N+2; \end{cases}$$

We now verify that this strategy profile induces the payoff vector $v = (\varepsilon^2,\ldots,\varepsilon^2,(1-\varepsilon)/4)$.

We first note that:

$$\forall i \in \{1,\ldots,N-1\}, \quad \mathbb{P}\{t = 1 | m = i\} = (\frac{1}{2} + \varepsilon(N-i)) = p_i.$$

In other words, receiver N's posterior belief of the sender's type after observing $m = i$ is equal to p_i. It is easy to check that the posterior belief after observing message $m = N$ is equal to p_N and the posterior belief after observing messages $m \in \{N+1, N+2\}$ is equal to $p \in \{0, 1\}$.

We now note that:

$$\forall i \in \{1,\ldots,N-1\}, \quad \mathbb{P}\{m = i\} = \varepsilon^2.$$

That is, each message $m \in \{1,\ldots,N-1\}$ is sent with probability ε^2 ex ante. Since only posterior beliefs in $\{p_1,\ldots,p_N,p_{N+1},p_{N+2}\}$ are indices with positive probability, we have that each receiver $i \in \{1,\ldots,N-1\}$ obtains a payoff of zero when message p_i is not sent. Thus, the expected utility of each receiver $i \in \{1,\ldots,N-1\}$ is equal to the ex ante probability that message p_i is sent, which is equal to ε^2.

We now note that:

$$\forall i \in \{1,\ldots,N\}, \quad \max \mathbb{E}_t^{p_i}[-a^2 - t + 2\cdot t\cdot a + \frac{1}{4}] = -p_i + p_i^2 + \frac{1}{4} = \varepsilon^2(N-i)^2 \quad (28)$$

In other words, receiver N's expected utility when he observes $m \in \{1,\ldots,N-1\}$ is equal to $\varepsilon^2(N-i)^2$ (note that for p_N, this is equal to zero). We also note that:

$$\forall i \in \{N+1, N+2\}, \quad \max \mathbb{E}_t^{p_i}[-a^2 - t + 2\cdot t\cdot a + \frac{1}{4}] = \frac{1}{4}.$$

In other words, receiver N's expected utility when he observes $m \in \{N+1, N+2\}$ is equal to $1/4$. The total probability with which message $m \in \{N+1, N+2\}$ is sent is equal to:

$$(\mathbb{P}\{m = N+1\} + \mathbb{P}\{m = N+2\}) = 1 - \varepsilon - 4\sum_{i=1}^{N-1}\varepsilon^4(N-i)^2.$$

Finally, we compute receiver N's expected utility, which is equal to:

$$\sum_{i=1}^{N-1}\mathbb{P}\{m = i\}\cdot\varepsilon^2(N-i)^2 + (\mathbb{P}\{m = N+1\} + \mathbb{P}\{m = N+2\})\cdot\frac{1}{4} \quad (29)$$

$$= (1-\varepsilon)/4. \quad (30)$$

Thus, receiver N's expected utility is equal to $(1-\varepsilon)/4$. This proves that the payoff vector $v = (\varepsilon^2,\ldots,\varepsilon^2, 1/4 - \varepsilon)$ can be achieved by a PBE.

Step B. We now show that it is necessary to use $N + |T|$ messages to induce this payoff vector. Whenever an action in $\{p_1,\ldots,p_N\}$ is taken, the expected utility of receiver N is of order ε^2 (see (28)). Thus, for receiver N to obtain an expected payoff of order $(1-\varepsilon)$, the sender must induce with probability of order ε of an action not in $\{p_1,\ldots,p_N\}$ (otherwise receiver N would obtain a payoff lower than $(1-\varepsilon)/4$). Moreover, with probability of order $1-\varepsilon$, she must induce exactly actions $(p_{N+1}, p_{N+2}) = (1, 0)$ (instead of some actions $(a_1, a_2) \approx (1, 0)$) (otherwise the sender would be inducing an action that yields negative payoff to all other receivers with probability of order 1, so receivers $i \in \{1,\ldots,N-1\}$ would obtain negative expected payoff). Thus, actions $(a_1, a_2) = (p_{N+1}, p_{N+1})$ must be induced in equilibrium. Additionally, since all passive receivers obtain a payoff of ε^2, actions $\{p_1,\ldots,p_{N-1}\}$ must be taken with positive probability. Thus, the sender must induce actions $\{p_1,\ldots,p_{N-1},p_{N+1},p_{N+2}\}$, which requires $N - 1$ messages. However, a passive receiver obtains a payoff of order ε^2 and not of order ε. From this, it follows that the sender must be inducing an action that yields receiver N an expected payoff lower than 1/4 and the rest of the receivers a payoff less than or equal to 0. Thus, an action not in $\{p_1,\dots,p_{N-1},p_{N+1},p_{N+2}\}$ must be taken with positive probability, which requires one additional message. It follows that $|T|+N$ messages are needed. ■

Part II. We now repeat the same arguments as those in Part I of the proof but consider the following payoff vector:

$$v=(\underbrace{\varepsilon^2,\dots,\varepsilon^2}_{N-1\text{ times}},\frac{1}{4}+\frac{\varepsilon^2}{12}(3-N-6N^2+4N^3)), \tag{31}$$

where ε is sufficiently small. As before, the proof proceeds in two steps. We first show that this payoff vector can be achieved by a PBE and then show it can be achieved only if the message space has more than $N+|T|-1$ messages.

Step A. We first show that the payoff vector

$$v=(\varepsilon^2,\dots,\varepsilon^2,\frac{1}{4}+\frac{\varepsilon^2}{12}(3-N-6N^2+4N^3))$$

can be achieved by a PBE. As before, we construct the equilibrium assuming that the prior distribution is $\pi=(1/2,1/2)$, but the construction can be modified accordingly to accommodate any prior. We relabel the message space as $M=\{1,\dots,N+2\}$. Each message $i\in\{1,\dots,N+2\}$ is associated with posterior belief p_i. We denote by $s_i(t)$ the probability that the sender sends message i when she is type t and consider the following strategy for the sender:

$$s_i(1)=\begin{cases}\varepsilon^2(1+2\varepsilon(N-i)) & i\in\{1,\dots,N-1\};\\ 1-(N-1)\varepsilon^2-\sum_{i=1}^{N-1}\varepsilon^3 2(N-i) & i=N+1;\end{cases}$$

$$s_i(0)=\begin{cases}\varepsilon^2(1-2\varepsilon(N-i)) & i\in\{1,\dots,N-1\};\\ 1-(N-1)\varepsilon^2+\sum_{i=1}^{N-1}\varepsilon^3 2(N-i) & i=N+2;\end{cases}$$

Note that the probability with which messages $i=1,\dots,N-1$ are sent remains the same as that in Part I of the proof. Thus, as before:

$$\forall i\in\{1,\dots,N-1\},\quad \mathbb{P}\{t=1|m=i\}=(\frac{1}{2}+\varepsilon(N-i))=p_i; \tag{32}$$

$$\forall i\in\{1,\dots,N-1\},\quad \mathbb{P}\{m=i\}=\varepsilon^2. \tag{33}$$

In other words, receiver N's posterior belief of the sender's type after observing $m=i$ is equal to p_i. Message $i=N$ is sent with zero probability, so any posterior beliefs are consistent with Bayes' rule (this fact will play no role in the analysis). Each message $m\in\{1,\dots,N-1\}$ is sent with probability ε^2 ex ante. Thus, the expected utility of each receiver $i\in\{1,\dots,N-1\}$ is equal to ε^2.

Recall that receiver N's expected utility when he observes $m\in\{1,\dots,N-1\}$ is equal to $\varepsilon^2(N-i)^2$; when he observes $m\in\{N+1,N+2\}$, his expected utility is equal to 1/4. The total probability with which message $m=N+1$ or $m=N+2$ is sent is equal to:

$$(\mathbb{P}\{m=N+1\}+\mathbb{P}\{m=N+2\})=1-\varepsilon^2(N-1).$$

We now compute receiver N's expected utility, which is equal to:

$$\sum_{i=1}^{N-1}\mathbb{P}\{m=i\}\cdot\varepsilon^2(N-i)^2+(\mathbb{P}\{m=N+1\}+\mathbb{P}\{m=N+2\})\cdot\frac{1}{4} \tag{34}$$

$$=\frac{1}{4}+\frac{\varepsilon^2}{12}(3-N-6N^2+4N^3). \tag{35}$$

Thus, receiver N's expected utility is equal to $\frac{1}{4}+\frac{\varepsilon^2}{12}(3-N-6N^2+4N^3)$. This proves that the payoff vector (31) can be achieved by a PBE.

Step B. To achieve a payoff vector (31), the sender has to induce every action $a\in\{p_1,\dots,p_{N+2}\}\backslash\{p_N\}$, which requires sending $N+1$ different messages. To see this, note that a payoff that yields strictly positive payoff to all receivers $i\in\{1,\dots,N-1\}$ must induce actions $\{p_1,\dots,p_{N-1}\}$. Note that each of these actions yields a payoff of order ε^2 for receiver N. Therefore, the payoff of receiver N is sufficiently close to 1/4 only if the sender signals her type perfectly with sufficiently high probability, which requires two additional messages. Thus, $N+|T|-1$ messages are needed.

Proof of Proposition 4. **Part I.** We first prove that if the message space $M_1=M_2=M$ is such that $|M|\geq n$, then $(+1/n,-1/n)\in\mathcal{U}$. Without loss of generality, we assume that $|M|=n$ and we label the messages as follows[14]:

$$M=\{1,\dots,n\}.$$

We construct a symmetric equilibrium, so we omit the senders' label on the strategy profile. A strategy profile is a function $s:\{0,1\}\to[0,1]^n$ that specifies the probability with which each message is sent conditional on each type of sender. We denote by $s^k(t)$ the probability with which message k is sent by a sender of type t. We consider the following strategy profile

$$s^k(1)=\frac{1}{n}+(k-\frac{n+1}{2})\cdot\frac{\varepsilon}{\pi}; \tag{36}$$

$$s^k(0)=\frac{1}{n}-(k-\frac{n+1}{2})\cdot\frac{\varepsilon}{1-\pi}, \tag{37}$$

where ε is a sufficiently small number such that $s^k(1),s^k(0)\in[0,1]$. Note that the probabilities sum to one:

$$\sum_{k=1}^{n}s^k(1)=\sum_{k=1}^{n}s^k(0)=1.$$

Thus, all the specified probabilities are between zero and one and sum to one, so the strategy profile is well defined.

We first make some observations regarding strategy (36)–(37). First, the total probability with which sender i sends message k is given by:

$$\mathbb{P}\{m=k\}=\pi s^k(1)+(1-\pi)s^k(0)=\frac{1}{n}.$$

In other words, every message has equal probability ex ante of being sent. Second, we compute the probability that sender i is type t_i conditional on the probability that he sends message $m=k$:

$$\mathbb{P}\{t_i=1|m=k\}=\frac{\pi s^k(1)}{\pi s^k(1)+(1-\pi_i)s^k(0)}=\pi(1+n(k-\frac{n+1}{2})\cdot\frac{\varepsilon}{\pi}).$$

Thus, for all $k\neq k'$, $\mathbb{P}\{t_i=1|m=k\}\neq\mathbb{P}\{t_i=1|m=k'\}$.

We now argue that (36)–(37) is a Nash equilibrium. First, note that in the conjectured equilibrium, Sender 1 obtains a payoff of 1 and Sender 2 obtains a payoff of -1 if and only if they send the same message in equilibrium. This result is easy to verify because the receiver's belief assigns equal probability to both senders being type $t=1$ only when both senders send the same

[14] If $|M|>n$, then it is possible to ignore the extra messages by assuming that the sender never sends any of these additional messages in equilibrium. In the case (off-path) that the receiver observes one of these extra messages, his beliefs will be the same as when observing one of the messages that are sent with positive probability. Thus, no sender has a strict incentive to send one of the extra message as this yields the same payoff as one of the message that is sent on-path.

message. Since every message is sent with equal probability ex ante, both senders are indifferent among all messages. Hence, it is optimal to follow the strategy specified by (36)–(37). Finally, the payoff vector induced by this equilibrium is $(1/n, -1/n)$.

Part II. We now prove that if $|M_1|, |M_2| \leq n-1$, then $(1/n, -1/n) \notin \mathcal{U}$. We assume that there exists an equilibrium that yields payoff $(1/n, -1/n)$ and $|M_1|, |M_2| \leq n-1$, and find a contradiction. For a given strategy profile (s_1, s_2), we can simply check that this is not an equilibrium by proving that one of the senders can send one message that guarantees herself a payoff higher than the conjectured payoff vector $(1/n, -1/n)$. We consider two cases.

Case 1. Suppose first that there exists a message $m_2 \in M_2$ such that for all $m_1 \in M_1$, the action induced by the message profile (m_1, m_2) yields a payoff of zero to Sender 2. In this case, Sender 2 can guarantee herself a payoff of zero by simply sending message m_2 with probability one. Thus, $(1/n, -1/n)$ would not be an equilibrium payoff.

Case 2. Suppose now that for every message $m_2 \in M_2$, there exists $m_1 \in M_1$, such that the action induced by the message profile (m_1, m_2) yields a payoff $(1, -1)$. We know that there must exist a message $\hat{m}_2 \in M_2$ that Sender 2 sends with probability *at least* $1/(n-1)$ (as $|M_2| \leq n-1$) and there exists a message $\hat{m}_1$ such that message profile $(\hat{m}_1, \hat{m}_2)$ yields a payoff $(1, -1)$. Therefore, Sender 1 can guarantee herself a payoff $1/(n-1)$ by simply sending message $\hat{m}_1$ with probability one. Thus, $(1/n, -1/n)$ would not be an equilibrium payoff. ∎

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