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Signaling in dynamic contests with heterogeneous rivals

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ABSTRACT

We study signaling in dynamic contests where a privately informed challenger faces multiple rivals. When the challenger can choose between simultaneous and sequential contests, the optimal choice hinges on the degree of player heterogeneity. This is because the strength of future opponents affects the incentive to signal, whereas the strength of current opponents affects the extent and cost of signaling. We show that against homogeneous rivals, the challenger may prefer to reveal information through sequential contests. However, against heterogeneous opponents, the challenger prefers simultaneous contests to avoid information revelation. Additionally, in sequential contests with heterogeneous rivals, we characterize the equilibrium choice of rivals' *order* (e.g., weak first and strong second) and show that only pooling and partial-pooling equilibria emerge.

1. Introduction

We study signaling in dynamic contests, where one player ("the challenger") chooses *how* to compete against two rivals ("the incumbents"). If the challenger chooses the *timing* of the contests, will he opt for simultaneous or sequential contests, and if sequential, will he choose to face stronger or weaker opponents first?

The answer depends on whether the challenger benefits from information revelation. Simultaneous contests do not reveal information, while sequential contests reveal some information to the second rival. Furthermore, the cost of information revelation through sequential contests is *endogenously* determined by the characteristics of the players in the first contest. This is a crucial distinction from other settings where the cost of signaling is exogenous (see, e.g., Corchón and Yıldızparlak, 2013; Fu et al., 2013; Denter et al., 2021).

Learning and signaling in dynamic contests has been underexplored in the literature, partly due to the technical difficulties of the setting. To keep our analysis tractable, our model assumes that the challenger is privately informed about his type, while the rivals' types are known but can be heterogeneous. A player's marginal cost of effort is his/her type: A stronger (weaker) player has a lower (higher) marginal cost. We follow Siegel (2014) and use all-pay contests with discrete types for two reasons. First, equilibrium payoffs are analytically simpler than in other contest settings. Second, discrete types simplify belief updating since it is enough to keep track of a single-dimensional parameter (the beliefs that the challenger is strong). Lastly, we assume that only the result of the first contest is observable (see, e.g., Moldovanu and Sela, 2001), so the challenger's type is never fully revealed because players use mixed strategies.

We first present results for *static* contests with incomplete information. These results follow Siegel (2014) and also relate to recent work by Chen and Chen (2024). When facing a strong opponent, a strong challenger benefits from "appearing weak" to induce beliefs that assign a lower probability to stronger types, which relaxes competition. Contrarily, a weak challenger benefits from "appearing strong" to discourage a weak opponent.

Thus, the challenger benefits from a signal that misleads the second incumbent. However, both the benefit and cost of signaling depend on the type of incumbents the challenger faces. The strength of future rivals determines the value of signaling, whereas the strength of current rivals determines its cost. For instance, facing a strong incumbent in the second contest incentivizes the challenger to signal weakness through the result of the first contest. This incentive to signal triggers a strategic response by the first incumbent, which creates an endogenous cost for the challenger.

Our framework provides insights for multiple environments, including a political campaign where a new candidate faces two established politicians, a corporate takeover where a company seeks to acquire two firms, or a multi-product startup planning to enter two distinct markets

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dominated by two firms. In these scenarios, the challenger has private information about their capabilities – whether political skills, financial strength, or technological potential – and must decide whether to engage in simultaneous or sequential contests (e.g., primaries, takeover bids, or product launches) and, if sequential, in what order to face its rivals. Our results show that the heterogeneity of rivals (e.g., differences in experience, firm characteristics, or market shares) is a key factor affecting the challenger's decision.

Specifically, we show that equilibrium behavior depends on the degree of heterogeneity – how different the strong and the weak types are – and, notably, on whether the incumbents' types are different. When facing incumbents of different types, we show that the challenger prefers to avoid information revelation. That is, whenever possible, the challenger chooses to face the incumbents simultaneously rather than in any sequential order. Instead, the challenger may prefer to play sequentially when facing incumbents of the same type.

To understand these results, consider a situation where the challenger faces a weak incumbent in the second contest. Here, a weak challenger has an incentive to signal strength to discourage the second incumbent from competing aggressively. The only way for the weak challenger to signal strength is by winning the first contest. Therefore, he puts in extra effort to win the first contest, more than he would in a single, isolated contest. In contrast to the weak challenger, the strong challenger does not benefit from signaling. The first incumbent is uncertain about whether he is facing a weak or strong challenger. Therefore, she responds by exerting more effort as she is potentially facing a more aggressive weak challenger. This intensified effort by the first incumbent reduces the weak challenger's expected payoff in the first contest compared to his payoff against the same opponent in a static (single) contest. This payoff reduction in the first contest caused by the incentive to signal strength is what we term the *endogenous cost* of signaling.

Moreover, although the strong challenger does not benefit from signaling, the strategic response of the first incumbent imposes a negative externality on him: the strong challenger is forced to exert more effort because the first incumbent is acting more aggressively.

Critically, when incumbents are *heterogeneous*, the challenger's gain in the second contest is smaller than his loss in the first one. Hence, the challenger gets trapped in a situation where sequential rationality demands him to signal, but the first incumbent's response makes signaling very expensive. The challenger would prefer not to reveal information by playing his rivals simultaneously if possible.

On the other hand, if both incumbents are *homogeneous* and weak, the challenger may benefit from signaling through sequential contests. By increasing his effort in the first contest to discourage the second incumbent, the challenger also discourages the first incumbent, who is also weak. However, facing a tamer rival in the first contest diminishes the value of signaling: beating a discouraged, weak rival does not convey much information, and losing against such an opponent is too revealing. As a result of this tradeoff, the challenger may prefer sequential contests.

Similar economic insights arise when the challenger faces the strong incumbent in the second contest, although the roles are reversed. The strong challenger benefits from signaling weakness for the second incumbent to act tamer. To appear weaker, the challenger reduces his effort in the first contest, giving the first incumbent a better chance of winning, which the first incumbent capitalizes by exerting more effort. This generates a negative externality on the challenger type with no incentive to signal (in this case, the weak type). Moreover, when the first incumbent is weak, the challenger's gain in the second contest is smaller than his loss in the first contest, so the challenger prefers to face the incumbents simultaneously. However, when both incumbents are strong, and the challenger signals weakness, the first incumbent reduces her effort, giving the challenger better chances of winning, making facing opponents sequentially potentially optimal.

Thus, the general insight is that whenever incumbents are heterogeneous, the challenger is worse off from facing them sequentially rather than simultaneously. On the contrary, if incumbents are homogeneous, the challenger's signaling effort always makes the first incumbent less competitive, so the signal is less informative, so sequential contests may or may not benefit the challenger.

Simultaneity is advantageous to the challenger because it allows him to commit not to signal. However, in some situations, the challenger cannot choose the timing and must play sequentially. In such situations, he may be able to choose the order in which it plays the incumbents. The mere choice of a particular order can reveal information about the challenger. Therefore, choosing the incumbent's order creates an additional signaling stage *before* the first contest takes place.

The same underlying economic force is present here: the challenger prefers to avoid information revelation when choosing the order in which he will face the (heterogeneous) incumbents. We show that there is no separating equilibrium where different challenger types face the incumbents in different orders because the challenger would fully reveal its type. An equilibrium always exists, and it is either pooling or partial pooling. We provide a novel, constructive proof that fully characterizes the equilibrium for all parameter values.

In a pooling equilibrium, each challenger's type faces the incumbents in the same order, which does not reveal any information about the challenger's type; in a partial pooling, each challenger type randomizes between each possible order of incumbents. This strategy again limits information revelation. Which equilibrium arises depends on the degree of player heterogeneity and the prior belief that the challenger is strong.

For instance, when the prior is low (i.e., likely weak challenger), both challenger types pool at facing the weak incumbent first. This is because defeating a weak incumbent is likely even for the weak challenger, so the posterior is only slightly higher than the prior after a challenger's victory. Instead, if the challenger loses against the weak incumbent, he signals weakness and the second incumbent relaxes. By contrast, when the prior is high (i.e., likely strong challenger), the relative strength matters because losing against a weak incumbent is too revealing if the strength difference is substantial. Hence, both challenger types face the strong incumbent first when heterogeneity is low (which is when signaling weakness is not too costly). In contrast, they pool at facing the weak incumbent in the first contest when heterogeneity is high (to avoid the cost of signaling weakness).

Related Literature. Static all-pay contests have been extensively used to model economic issues such as lobbying (Hillman and Riley, 1989) and litigation (Baye et al., 2005). While earlier works like Amann and Leininger (1996) and Siegel (2014) established foundational results on the existence and uniqueness of equilibrium in static settings, recent literature has focused on issues related to information disclosure (see, e.g. Chen and Serena, 2023; Chen and Chen, 2024; Lemus and Temnyalov, 2023), dynamics, and learning.

Dynamic contests, where each stage is an all-pay contest, have been studied by Konrad and Kovenock (2009) and Krumer et al. (2017), but these works do not consider learning between stages. We contribute to the recent literature examining learning in sequential contests. For instance, Barbieri and Serena (2022) investigate how players in a sequence of battles can build reputations for rational or all-in (behavioral) strategies, while Beccuti and Möller (2020) explore learning about a common valuation in multi-stage contests.

The role of signaling and pre-play communication in contests has also garnered attention, typically assuming an *exogenous cost* of signaling. Corchón and Yıldızparlak (2013) analyze the incentives for declaring war as a signal of a country's private value from winning, and Farmer and Pecorino (2013) study a two-stage litigation game involving settlement offers. Other works, such as Fu et al. (2013) and Ifergane et al. (2017), examine how players publicize their confidence to win or communicate before the contest begins. Denter et al.

(2021) consider the signaling behavior of a privately-informed entrant competing against an incumbent. In contrast to these studies, signaling in our setting emerges *endogenously* as a (stochastic) result of a contest, which relates to the work of Bernhardt and Lee (2015) on sequential litigation under uncertainty.

Unlike our focus on player heterogeneity, other studies have examined signaling in sequential contests with homogeneous players. For instance, Münster (2009) shows that high types may mimick low types when two ex-ante identical, privately informed players compete in sequential Tullock contests. Kubitz (2023) expand on these findings using an all-pay contests with more general cost functions with ex-ante homogeneous players. However, he does not compare sequential and simultaneous contests nor the equilibrium selection of the order of opponents' types when contests are sequential. Also, in contrast to our setting, players observe first stage effort rather than only contest outcomes.

Our study emphasizes the role of player heterogeneity in sequential contests, a factor that has been explored in different contexts. For instance, Beviá and Corchón (2013) study sequential contests where the strength of a player in the second contest depends on the first contest's result. Antsygina (2022) consider multi-dimensional rewards with heterogeneous players. Also related is the literature investigating how to arrange matches among heterogeneous players, as Groh et al. (2012) and Kräkel (2014). However, our focus diverges by allowing a privately-informed strategic player to choose the order of opponents, where signaling incentives are prominent, an aspect not considered in prior work.

2. Static contest with incomplete information

As a benchmark, we follow Siegel (2014) and study a static all-pay contest with incomplete information. The results in this section are the building blocks for our novel results in subsequent sections.

Two players (A and B) simultaneously choose their effort and compete for a prize of 1. Player A ("he") is privately informed of his marginal cost of effort (his type), $c_A \in \{c_L, c_H\}$, with $0 < c_L < c_H$. We say that type c_L is *strong* and type c_H is *weak*.

Player B ("she") holds a belief $p = \Pr(c_A = c_L) \in (0,1)$ and her marginal cost of effort, $c_B > 0$, is public information. Player $i \in \{A, B\}$ chooses a non-negative effort, e_i, and obtains an expected payoff of $1(e_i > e_{-i}) + \frac{1}{2} \cdot 1(e_i = e_{-i}) - c_i e_i$, where $1(\cdot)$ is the indicator function. We denote the contest result by $R \in \{W, D\}$, where W means that player A won and D means he was defeated.[1] We define the *efficiency ratio* by $\gamma = c_L/c_H \in (0,1)$, which measures the degree of similarity between types.

Definition 1. The static all-pay contest described above is denoted $C \equiv C(p, \{c_L, c_H\}, c_B)$. We denote by $\pi^j(C)$ player A's expected payoff when his type is c_j and $\Pr^j(R|C)$ the probability that he obtains result R in contest C, for $j \in \{L, H\}$.

The equilibrium of $C(p, \{c_L, c_H\}, c_B)$ is unique in mixed strategies (see, e.g., Siegel, 2014), and entails uniform mixing on at most two intervals, no gaps, and at most one player places an atom at 0. We present the equilibrium strategies, payoffs, and probability of winning in the proof of Lemma 1.

Lemma 1. *Consider contest $C \equiv C(p, \{c_L, c_H\}, c_B)$. Player A's equilibrium expected payoff when he is strong (type c_L), $\pi^L(C)$, is weakly decreasing in p and when he is weak (type c_H), $\pi^H(C)$, is weakly increasing in p.*

Fig. 1 illustrates the equilibrium expected payoff for each type of player A as a function of the prior belief p when $c_B \leq c_L$ or $c_H \leq c_B$.

In the classic book on military strategy, "The Art of War", Sun Tzu, writes: "appear weak when you are strong, and strong when you are weak". Lemma 1 provides a rationale for this claim. Player A weakly benefits when player B assigns a low probability to his actual type. The strong type, c_L, is better off when p is small, and the weak type, c_H, is better off when p is large.

Fig. 1(a) shows the expected payoff of each type of player A when player B is stronger than any of them ($c_B \leq c_L$). Player A's expected payoff is zero when he is weak. When he is strong, however, his payoff decreases linearly in p and is zero for $p > c_B/c_L$. Intuitively, player B is uncertain about her relative strength, so she is more *complacent* for smaller values of p because she is more likely to face the weaker opponent.[2] For $p > c_B/c_L$, player B simply plays as if she was facing type c_L for sure, and even if player A is strong, his expected payoff is zero. Thus, if the strong type could freely choose his opponent's belief, he would choose $p = 0$ ("act weak when you are strong").

Fig. 1(b) shows the expected payoff of each type of player A when player B is weaker than any type of player A ($c_H \leq c_2$). When player A is strong, his payoff is constant in p. When player A is weak, however, his payoff increases linearly in p. Intuitively, as p grows, player B gets *discouraged* because she is more likely facing a stronger opponent, so she reduces her effort, which softens competition. Thus, if player A is weak and could freely choose his opponent's belief, he would choose $p = 1$ ("act strong when you are weak").

3. Sequential contests

In this section, we study a dynamic game where one player (the "challenger") sequentially competes against two rivals, "Incumbent-1" and "Incumbent-2". The challenger is privately informed about his type $c \in \{c_L, c_H\}$, with $0 < c_L < c_H$. The publicly-known marginal costs of the incumbents are c_1 and c_2, respectively, with $c_1, c_2 \in \{c_L, c_H\}$. The incumbents share a common prior belief $p = \Pr(c = c_L) \in (0,1)$. Incumbent-2 observes the result of the first contest, R, and forms a belief that the challenger is strong denoted by $q(R)$. The second contest corresponds to the static contest $C(q(R), \{c_L, c_H\}, c_2)$.

Fixing Incumbent-2's beliefs, the challenger considers the impact of his effort on sending signal R. The following result shows that the challenger's equilibrium effort in the first contest of the *dynamic* game equals the equilibrium effort of a privately-informed player in a *static* contest with distorted ("virtual") costs. This observation simplifies the equilibrium analysis because the virtual costs fully capture dynamic considerations.

Proposition 1. *The equilibrium strategies in the first contest of the dynamic game are the equilibrium strategies of the static contest $C(p, \{\tilde{c}_L, \tilde{c}_H\}, c_1)$, where*

$$\tilde{c}_j \equiv \frac{c_j}{1 + \pi^j(C_W) - \pi^j(C_D)}, \quad \text{for } j \in \{L, H\}, \tag{1}$$

where $C_R = C(q(R), \{c_L, c_H\}, c_2)$ is the second contest after result $R \in \{W, D\}$.

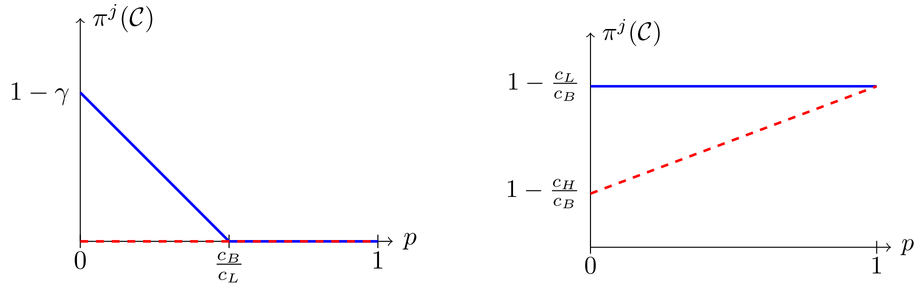
Intuitively, the challenger of type j receives $1 + \pi^j(C_W)$ for winning and $\pi^j(C_D)$ for losing. Since effort increases the probability of winning, its marginal value is the payoff difference between winning and losing, $1 + \pi^j(C_W) - \pi^j(C_D)$. We can rescale the challenger's payoff, which yields an equivalent static all-pay contest where the prize is 1 and a marginal cost of effort of $\tilde{c}_j$ for the challenger.[3] The virtual costs $\{\tilde{c}_L, \tilde{c}_H\}$ are key for our analysis since they are one-dimensional objects capturing the challenger's signaling incentives.

Definition 2. $\mathcal{S}(p, \{c_L, c_H\}, j \rightarrow k)$ is the sequence of contest: $C(p, \{\tilde{c}_L, \tilde{c}_H\}, c_j)$ is the first contest, and after observing result R, the second contest is $C(q(R), \{c_L, c_H\}, c_k)$.

[1] We ignore ties because they are zero-probability events in equilibrium.

[2] Equilibrium strategies generally involve mixing. In this case, for $p < p'$ player B's equilibrium distribution for p' FOSD the distribution for p.

[3] With a non-linear, strictly increasing cost function $C_j(e) = c_j \cdot k(e)$ the result in the result goes through after a change of variables, $y = k(e)$.



(a) $c_B \le c_L$ **(b)** $c_H \le c_B$

Fig. 1. Equilibrium expected payoff for each type of Player A as a function of the opponent's belief $p = \Pr(c_A = c_L)$. In each panel, the solid line corresponds to the payoff of type c_L, $\pi^L(\mathcal{C})$, and the dashed line corresponds to the payoff of type c_H, $\pi^H(\mathcal{C})$.

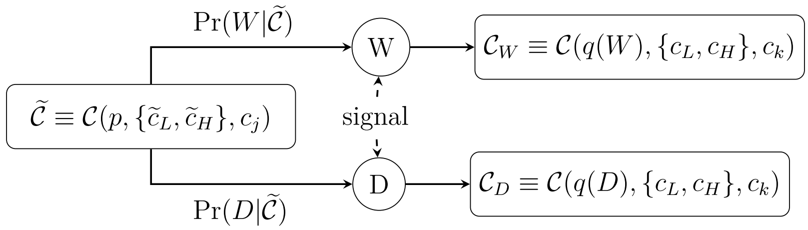


Fig. 2. The challenger's effort in the first contest affects $\Pr(R|\widetilde{\mathcal{C}})$, the probability of result $R \in \{W, D\}$. Equilibrium strategies in the first contest of the dynamic game are the same as in the static contest $\widetilde{\mathcal{C}} = \mathcal{C}(p, \{\widetilde{c}_L, \widetilde{c}_H\}, c_j)$. $\widetilde{c}_L$ and $\widetilde{c}_H$ capture signaling incentives.

The notation $j \to k$ highlights that the challenger faces the incumbent of type c_j in the first contest, and the incumbent of type c_k in the second one, with $c_j, c_k \in \{c_L, c_H\}$.

Fig. 2 illustrates the timing of the sequential contests $\mathcal{S}(p, \{c_L, c_H\}, j \to k)$. We study perfect-Bayesian equilibria (PBE) of this dynamic game.

Definition 3. The PBE of the sequential contest satisfies:

1. The second incumbent holds beliefs $q(R)$, for $R \in \{D, W\}$.
2. Equilibrium effort strategies:
   (a) First contest: The equilibrium effort strategies in $\widetilde{\mathcal{C}} \equiv \mathcal{C}(p, \{\widetilde{c}_L, \widetilde{c}_H\}, c_j)$.
   (b) Second contest: The equilibrium effort strategies in $\mathcal{C}_R \equiv \mathcal{C}(q(R), \{c_L, c_H\}, c_k)$.
3. $\widetilde{c}_j$ is defined in (1), for $j \in \{L, H\}$.
4. $q(R)$, for $R \in \{W, D\}$, is consistent with Bayes' rule:

$$q(R) = \frac{\Pr^L(R|\widetilde{\mathcal{C}})p}{\Pr^L(R|\widetilde{\mathcal{C}})p + \Pr^H(R|\widetilde{\mathcal{C}})(1-p)}. \tag{2}$$

We have the following result:

Proposition 2. *There exists a unique PBE satisfying Definition 3. In the equilibrium we have $q(D) < q(W)$, and the virtual costs satisfy $c_L \le \widetilde{c}_L < \widetilde{c}_H \le c_H$.*

In equilibrium, Incumbent-2 updates her belief upwards after observing $R = W$, and downwards after $R = D$. In the first contest, the strong challenger plays as a weakly more inefficient player ($c_L \le \widetilde{c}_L$) and the weak challenger plays as a weakly more efficient one ($\widetilde{c}_H \le c_H$).

To understand the challenger's signaling incentives, note that the challenger's payoff in the second contest depends on the belief $q = (q(W), q(D))$ and, crucially, on the *type* of Incumbent-2.

When facing a weak Incumbent-2 the challenger's virtual costs are

$$\widetilde{c}_L = c_L \quad \text{and} \quad \widetilde{c}_H = \frac{c_H}{1 + (q(W) - q(D))(1-\gamma)}. \tag{3}$$

These virtual costs reflect that the weak challenger wants to project strength when facing a weak Incumbent-2. The only way to do that is to beat Incumbent-1 because only $R \in \{W, D\}$ is observable. Thus, the weak challenger plays more aggressively against Incumbent-1, compared to a static contest against the same rival, acting as a "more efficient" type (i.e., $\widetilde{c}_H < c_H$). Signaling is irrelevant for the strong challenger since his payoff is constant for any belief $q(R)$ and, therefore, $\widetilde{c}_L = c_L$.

In contrast, when facing a strong Incumbent-2 the challenger's virtual costs are

$$\widetilde{c}_L = \frac{c_L}{1 - (q(W) - q(D))(1-\gamma)} \quad \text{and} \quad \widetilde{c}_H = c_H. \tag{4}$$

The strong challenger now has an incentive to signal weakness, which he can only achieve by losing the first contest.[4] Thus, the strong type acts "tamer" by mimicking the strategy of a weaker type, $\widetilde{c}_L > c_L$.

4. Simultaneous versus sequential

When contests are simultaneous, there are no signaling incentives. When they are sequential, the incentive to signal depends on the *type* of rival the challenger will face in the second contest. Signaling, however, can benefit or harm the challenger since the first incumbent responds strategically. In other words, the *cost* of signaling is endogenous since the challenger's payoff in the first contest can increase or decrease as a consequence of the strategic response of the first incumbent. We compare sequential and simultaneous contests to assess under what conditions the challenger benefits from signaling.

Proposition 3. *Relative to simultaneous contests, in the perfect-Bayesian equilibrium of sequential contests:*

[4] In the first contest, the challenger's equilibrium effort equals that of a privately-informed player in the static contest $\widetilde{\mathcal{C}} \equiv \mathcal{C}(p, \{\widetilde{c}_L, c_H\}, c_j)$.

1. *Facing a weak Incumbent-2, signaling reduces the challenger's payoff in the first contest when Incumbent-1 is strong and increases it when Incumbent-1 is weak.*
2. *Facing a strong Incumbent-2, signaling reduces the challenger's payoff in the first contest when Incumbent-1 is weak and it is ambiguous when Incumbent-1 is strong.*

The equilibrium impact of signaling on the second contest is generally ambiguous. The reason is that the challenger can influence the probability of winning for any fixed belief. However, in equilibrium, this belief is endogenously determined by the strategic response of Incumbent-1, i.e., a PBE requires consistency between (1) and (2). As such, signaling could be detrimental to the challenger. Fig. 6, in the Appendix, numerically illustrates the equilibrium intensity of signal jamming, measured as the difference between the challenger's winning probability in the first contest under signaling and no signaling, $\Pr(W|\widetilde{C}) - \Pr(W|C)$, where $\widetilde{C} = C(p, \{\widetilde{c}_L, \widetilde{c}_H\}, c_j)$ and $C = C(p, \{c_L, c_H\}, c_j)$. The figure shows that signaling is effective only for a small set of parameters, hinting that the strategic reaction of Incumbent-1 can "undo" much of the signaling effort by the challenger.

We now explore what contest format favors the challenger: simultaneous or sequential. Simultaneous contests do not reveal information about the challenger's type, so the challenger's payoff is the sum of the payoffs of two static contests.

Proposition 4. *We have two cases depending on the heterogeneity of rivals:*

1. *If the incumbents are heterogeneous (i.e., one strong and one weak), the challenger is better off facing them simultaneously rather than sequentially.*
2. *If the incumbents are homogeneous (i.e., both strong or weak), the challenger prefers to face them simultaneously for some parameter values.*

Proposition 4 shows that signaling hurts the challenger when incumbents are heterogeneous but can be beneficial when they are homogeneous. To understand this result, recall that the weak challenger benefits from signaling strength to a weak Incumbent-2, and the strong challenger benefits from signaling weakness to a strong Incumbent-2.

Consider first the incentives to signal of the strong challenger. Suppose that Incumbent-1 is strong and Incumbent-2 is weak. Here, a *weak* challenger has incentives to signal strength to discourage Incumbent-2, and therefore acts more aggressively in the first contest. As a consequence Incumbent-1 also acts more aggressively, creating a negative externality on the *strong* challenger, making him worse off.

Now, consider the opposite case, i.e., Incumbent-1 is weak and Incumbent-2 is strong. Here, in sequential contests, the strong challenger plays less aggressively in the first contest to signal weakness, which is costly because it increases the probability of losing. We show that he only partially recoups this loss, by manipulating the second incumbent's belief, leading to an overall loss. This overall negative effect can be explained by the strategic effect induced in the weak Incumbent-1, who is encouraged by the actions of the challenger.

For the weak challenger, the logic is similar. If the contests are sequential and he faces a weak Incumbent-2, he signals strength by playing more aggressively in the first contest. However, this is costly since the reaction from a strong Incumbent-1 leads to an overall loss compared to a simultaneous contest. When the challenger faces a weak Incumbent-1, and a strong Incumbent-2, he suffers from the negative externality generated by the strong challenger type, who wants to signal weakness, which encourages Incumbent-1.

The challenger cannot commit not to manipulate his effort to influence the result of the first contest. Then, he gets trapped in an equilibrium where Incumbent-2 expects his effort distortion, and Incumbent-1 reacts accordingly. The challenger would be better off committing to a myopically-optimal effort, which he can accomplish by playing simultaneously.

Next, consider the case of homogeneous rivals. In this case, signaling could benefit the challenger because the incentive to signal aligns in the two contests. When both rivals are weak, the weak challenger projects strength, which discourages both rivals. Similarly, when both rivals are strong, the strong player pretends to be weak, which induces complacency in both rivals. However, in both cases, the strategic reaction of Incumbent-1 diminishes the value of signaling. To understand why, suppose that both incumbents are weak. If the first incumbent is too discouraged, then beating her is not too informative (i.e., both a weak and strong challenger are likely to beat a weak and discouraged incumbent). Therefore, signaling strength becomes harder. When the value of signaling does not compensate for the cost, the challenger prefers to play simultaneously. Fig. 7, in the Appendix, shows that signaling benefits the challenger when facing weak (strong) incumbents for low (high) priors, since that is when the result of the first contest influences belief updating the most.

In the next section, we examine situations in which the challenger cannot choose the timing of the contests. That is, the two contests must be played sequentially. However, the challenger can choose the order of his opponents.

5. Choosing the order of your battles

In some situations, it is impossible to play against both incumbents simultaneously. However, the challenger may be able to choose the order of his opponents. This decision matters only if the incumbents are heterogeneous, so we focus on this case. In such situations, does the challenger prefer to face the "weak" or the "strong" rival first? The main strategic consideration is that, since the challenger is privately informed about his type, the mere act of selecting an order reveals information.

To study these situations, we add an additional stage at the beginning of the game where the challenger chooses the *order* of his rivals. Both incumbents observe this choice and update their beliefs about the challenger's type.

We introduce some notation to facilitate the description of different games, each indexing a different order of rivals.

Definition 4. For the unique PBE of the dynamic game $\mathcal{S}(p, \{c_L, c_H\}, j \to k)$, where $j \neq k \in \{L, H\}$, we define the following equilibrium objects:

1. $q_{j\to k}(R)$, the second incumbent's belief after result R in the first contest.
2. $\pi^i_{j\to k}(p)$, the expected payoff of challenger of type c_i (considering both contests).

We now describe the PBE of the augmented game that adds an additional signaling stage as a consequence of the challenger choosing the order of the incumbents.

Definition 5. A perfect Bayesian equilibrium when the challenger of type c_i chooses the order of his battles against incumbents of types c_j and c_k is characterized by: choice probabilities $a_{j\to k}$, beliefs $\mu_{j\to k}$, equilibrium actions and beliefs in the two contests such that:

1. With probability $a_{j\to k}(c_i) \in [0,1]$ the challenger of type c_i chooses order $j \to k$.
2. After observing the challenger's order, $j \to k$, the incumbents form a belief $\mu_{j\to k} = \Pr(c_i = c_L | j \to k)$. The continuation game is $\mathcal{S}(\mu_{j\to k}, \{c_L, c_H\}, j \to k)$. The equilibrium of this continuation game is characterized in Section 3.
3. $\mu_{j\to k}$ satisfies Bayes' rule on path:
$$\mu_{j\to k} = \frac{a_{j\to k}(c_L)p}{a_{j\to k}(c_L)p + a_{j\to k}(c_H)(1-p)}$$

4. $a_{j\to k}$ satisfies sequential rationality:
$$a_{j\to k} \in \arg\max_{a\in[0,1]} a \cdot \pi^i_{j\to k}(\mu_{j\to k}) + (1-a)\cdot \pi^i_{k\to j}(\mu_{k\to j})$$

A key result that makes the equilibrium characterization easier is that the strong (weak) challenger's expected payoff is increasing (decreasing) in the incumbent's belief, p.

Proposition 5. *For any order of incumbents, the equilibrium payoffs are continuous, $\pi^H_{j\to k}(\cdot)$ are strictly increasing in p, $\pi^L_{j\to k}(\cdot)$ are decreasing in p, and for both challenger's types $\pi^i_{j\to k}(0) = \pi^i_{k\to j}(0)$ and $\pi^i_{j\to k}(1) = \pi^i_{k\to j}(1)$.*

Proposition 5 shows that Sun Tzu's intuition "look weak when strong, and strong when weak" holds for dynamic battles, regardless of the order of opponents, even when they are heterogeneous. While this result is known for *static* contests, we show that it also holds for *dynamic* environments where the cost of signaling is endogenous and players are heterogeneous. Fig. 3 shows the payoff of the strong challenger (left panel) and the weak challenger (right panel) for different orders of incumbents, when $\gamma = 0.6$.

For degenerate beliefs (i.e., either $p = 0$ or $p = 1$), the ordering of contests is irrelevant because there is complete information. However, when $p \in (0,1)$, the challenger cares about the beliefs induced by his choice of which opponent to face first.

Definition 6. For $i, j, k \in \{L, H\}$, with $j \neq k$, let $\hat{p}^i_{j\to k}(p)$ be the belief such that the challenger of type c_i is indifferent between order $j \to k$ under beliefs p and order $k \to j$ under beliefs $\hat{p}^i_{j\to k}(p)$. Formally,
$$\hat{p}^i_{j\to k}(p) = [\pi^i_{k\to j}]^{-1}\left(\pi^i_{j\to k}(p)\right), \quad \text{for all } p \in (0,1). \tag{5}$$

By Proposition 5, $\hat{p}^i_{j\to k}(\cdot)$ is well defined and strictly increasing. Furthermore, it is easy to see that
$$[\hat{p}^i_{j\to k}]^{-1} = \hat{p}^i_{k\to j}. \tag{6}$$
By the monotonicity of the challenger's payoffs, the weak challenger prefers order $j \to k$ under belief p to order $k \to j$ under any belief *above* $\hat{p}^H_{j\to k}(p)$. Instead, the strong challenger prefers order $j \to k$ under belief p to order $k \to j$ under any belief *below* $\hat{p}^L_{j\to k}(p)$.

These functions are central for our equilibrium construction. Consider a pooling equilibrium where both challenger types choose order $j \to k$. On the equilibrium path, the first incumbent's belief is p. By definition, $\hat{p}^i_{j\to k}(\cdot)$ describes the threshold off-path belief that prevents the challenger of type c_i from deviating to the reverse order of incumbents, $k \to j$. Any off-path belief in $(\hat{p}^L_{j\to k}(p), 1]$ prevents the strong challenger deviating, and any belief $[0, \hat{p}^H_{j\to k}(p))$ does the same for the weak challenger. A pooling equilibrium exists if there is a range of off-path beliefs that simultaneously prevents both types from deviating. Specifically, when $\hat{p}^L_{j\to k}(p) \le \hat{p}^H_{j\to k}(p)$, any off-path belief in $[\hat{p}^L_{j\to k}(p), \hat{p}^H_{j\to k}(p)]$ makes the deviation $k \to j$ unprofitable for both types, supporting a pooling equilibrium at order $j \to k$. The same logic applies for pooling equilibrium at the reverse order. Note that the off-path beliefs sustaining the pooling equilibrium differ qualitatively from those in standard signaling models (see, e.g., Spence, 1973), where it is possible to simultaneously punish deviations for all types with "extreme beliefs" (e.g., 0 or 1).

As we show next, there is no separating equilibrium in this game (i.e., an equilibrium where different challenger types choose different orders), so the challenger's type is never fully revealed. This is because the weak type wants to look strong, and the strong type wants to look weak. However, we show that an equilibrium always exists and it is either pooling or partial-pooling. In other words, both challenger types choose the same order or randomize among them.

Proposition 6. *There is no separating equilibrium. However, an equilibrium always exists, either pooling at the same order or partial-pooling.*

To gain intuition on Proposition 6, we provide the equilibrium construction for any prior p for the parameter $\gamma = 0.6$, beginning with pooling equilibria at the order $H \to L$.

Fig. 3 (left panel) shows the strong challenger's payoffs from choosing different orders. The figure highlights four points: $p = 0.2$, $\hat{p}^L_{H\to L}(0.2) = 0.16$, $p = 0.7$, and $\hat{p}^L_{H\to L}(0.7) = 0.78$. At $p = 0.2$, the strong challenger strictly prefers order $H \to L$ (in the figure, the solid line lies above the dashed line). Any off-path belief *above* $\hat{p}^L_{H\to L}(0.2) = 0.16$ prevents him from deviating to $L \to H$.

Fig. 3 (right panel) shows the weak challenger's payoffs from choosing different orders. The figure highlights four points: $p = 0.2$, $\hat{p}^H_{H\to L}(0.2) = 0.42$, $p = 0.7$, and $\hat{p}^H_{H\to L}(0.7) = 0.73$. At $p = 0.2$, the weak challenger also prefers order $H \to L$ (the solid line lies above the dashed line). Any off-path belief *below* $\hat{p}^H_{H\to L}(0.2) = 0.42$ prevents him from deviating to order $L \to H$.

These results together imply that when the prior that the challenger is strong is $p = 0.2$, there exists a pooling equilibrium where both challenger types choose the order $H \to L$. Any off-path belief in $[0.16, 0.42]$ sustains this equilibrium because it renders a deviation to $L \to H$ unprofitable for both types.

By contrast, it is not possible to construct a pooling equilibrium at $H \to L$ when $p = 0.7$. To see why, note that any off-path belief above $\hat{p}^L_{H\to L}(0.7) = 0.78$ prevents the strong challenger from deviating to $H \to L$. However, a deviation by the weak challenger can be prevented only with an off-path belief below $\hat{p}^H_{H\to L}(0.7) = 0.73$. Hence, it is impossible to prevent deviations simultaneously for each type.

The key to characterizing equilibria is to figure out for which values of the prior belief, p, it is possible to find off-path beliefs that simultaneously prevent deviations for *both* types. These regions are characterized by *threshold beliefs* (see Definition 6) shown in Fig. 4.

A pooling equilibrium at $H \to L$ exists whenever $\hat{p}^H_{H\to L}(\cdot) \ge \hat{p}^L_{H\to L}(\cdot)$, i.e., when the solid line in Fig. 4 is above the dashed line, which happens in the interval $I_{H\to L} = [0, 0.59]$. On the other hand, pooling equilibrium at $L \to H$ exists whenever $\hat{p}^H_{L\to H}(\cdot) \ge \hat{p}^L_{L\to H}(\cdot)$, which, by the identity in Eq. (6), is the same as $[\hat{p}^H_{H\to L}]^{-1}(\cdot) \ge [\hat{p}^L_{H\to L}]^{-1}(\cdot)$. In Fig. 4, this happens in the interval $I_{L\to H} = [0.65, 1]$. In the interval $I_M = (0.59, 0.65)$, where none of these two conditions hold, there is a partial-pooling equilibrium where both challenger types randomize between orders $H \to L$ and $L \to H$. If, as the result of the randomization, the challenger chooses the weak incumbent first (order $H \to L$), the first incumbent updates her belief from $p \in (0.59, 0.65)$ to 0.59, and to 0.65 if the strong incumbent is chosen for the first contest.

In general, the location and size of the three regions identified in Fig. 4 ($I_{H\to L}$, M, and $I_{L\to H}$) depend on the values of p and γ. When p is in the interval $I_{H\to L}$, the only equilibrium is pooling at $H \to L$. Since both challenger types choose the same order, there is no information revelation so the first incumbent believes the incumbent is strong with probability p. When p is in the interval $I_{L\to H}$, the only equilibrium is pooling at $L \to H$. When p is in the interval $I_M \equiv [p^M_0, p^M_1]$, the only equilibrium is partial pooling: both challenger types mix between orders $H \to L$ and $L \to H$ such that the induced beliefs from observing each order are given by p^M_0 and $p^M_1 = \hat{p}^L_{H\to L}(p^M_0)$. For these posterior beliefs both challenger types are indifferent between the two orders.

Generally, there is no closed-form solution for the functions $\hat{\pi}^i_{j\to k}$ nor their intersection for different values of γ. Nonetheless, we can apply the algorithm described in the proof of Proposition 6 to numerically find all the equilibria of the game for different values of $p \in (0,1)$ and $\gamma \in (0,1)$, as shown in Fig. 5. For instance, the figure shows that for $\gamma = 0.6$, there is pooling at $H \to L$ for $p \le 0.59$, partial pooling for $p \in [0.59, 0.65]$, and pooling at $L \to H$ for $p \ge 0.65$, which is the illustrative case explained in Fig. 4.

Fig. 5 shows that when the prior is low both challenger types pool at facing the weak incumbent first. The strong challenger type has an incentive to signal weakness but signaling is detrimental. By pooling at the weak incumbent first, the strong challenger type can "hide" his type

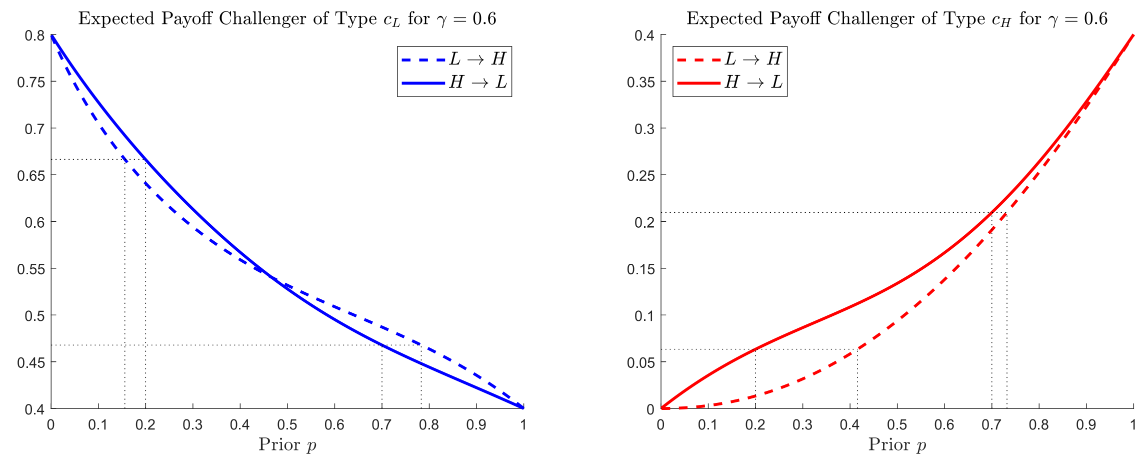


Fig. 3. Expected payoff for each challenger type, for a given order of incumbents. The dashed (solid) lines are the payoffs of choosing order $L \to H$ ($H \to L$). The left (right) panel shows the expected payoff of challenger of type c_L (c_H) when $\gamma = 0.6$.

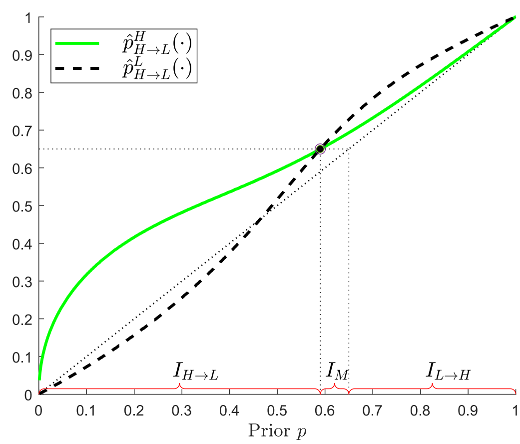


Fig. 4. Functions $\hat{p}^i_{H\to L}(p)$, for $\gamma = 0.6$.

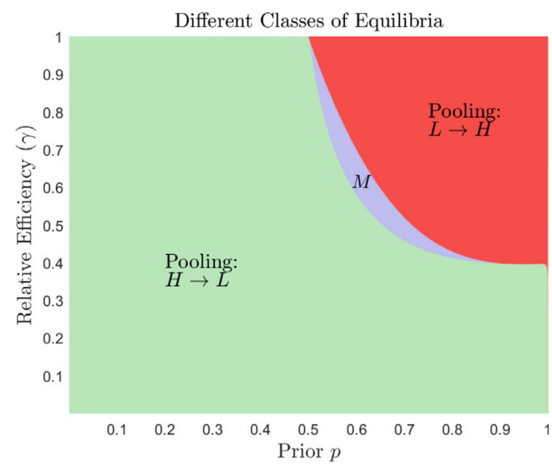


Fig. 5. Perfect Bayesian Equilibrium choice of incumbents' order for given of p and γ. In the green region, the only equilibrium is pooling at order $H \to L$ (i.e., weak incumbent first and strong incumbent second). In the red region, the PBE is pooling at $L \to H$ (i.e., strong incumbent first and weak incumbent second). In the blue region, M, the unique PBE is partial pooling, i.e., both challenger types randomize over the incumbents' order.

by winning. This is because defeating a weak incumbent is likely even for the weak challenger, so the posterior is only slightly higher than the prior after a challenger's victory. Instead, if the challenger loses against the weak incumbent, he signals weakness and the second incumbent relaxes.

By contrast, when the prior is high, the relative strength matters because losing against a weak incumbent is too revealing if the strength difference is substantial. Hence, both challenger types face the strong incumbent first when heterogeneity is low (which is when signaling weakness is not too costly). In contrast, they pool at facing the weak incumbent in the first contest when heterogeneity is high (to avoid the cost of signaling weakness).

6. Conclusion

We study the impact of information revelation and signaling in dynamic contests. We examine scenarios where the challenger, privately informed about his type, faces two opponents of known strength and can choose the *timing* of the contests—either simultaneously or sequentially. We then consider the case where the timing is necessarily sequential, but the challenger can choose the *order* of his opponents. A key economic force in our results is that belief manipulation of future opponents drives signaling incentives, and the extent and cost of signaling depend on the strategic response of current opponents. The relative importance of each of these forces varies with the *heterogeneity* of the players, leading to different equilibrium predictions.

In Section 2, we present benchmark results that characterize which type benefits from belief manipulation in a static setting. When facing a weak incumbent, the weak challenger benefits from looking strong. On the other hand, when facing a strong opponent, the strong challenger benefits from looking weak.

In Section 3, we study sequential contests and show that equilibrium strategies in the first contest are equivalent to those in a static contest replacing the actual type of the challenger by a "virtual" type. These virtual types capture all dynamic considerations, simplifying the equilibrium analysis. Importantly, what virtual types to use depend on the strength of the second opponent. When facing a weak incumbent in the second period, the weak challenger acts as a stronger type to win more often and look stronger in the second period. Analogously, if facing a strong incumbent in the second period, the strong challenger acts as a weaker type.

In Section 4, we show how these incentives interact, in equilibrium, with the strength of the first incumbent, who is aware of the challenger's signaling motivation. When facing a strong incumbent first and a weak incumbent second, the challenger tries to look strong to discourage the second incumbent, which prompts the first incumbent to exert more effort. This reaction turns out to be detrimental to the challenger. In fact, whenever there is heterogeneity in the strength of the incumbents, the challenger's cost of signaling outweigh the benefit, leading to a payoff loss. These insights are summarized in the first part of Proposition 4, which establishes that when facing heterogeneous incumbents, the challenger loses from the information revelation and, therefore, would prefer to face incumbents simultaneously. On the other hand, when facing a weak incumbent in each contest, trying to look strong makes the first incumbent tamer, which benefits the challenger. However, the value of signaling is diminished, because beating a tamer opponent is not as informative as beating an opponent that exerts more effort. The second part of Proposition 4 establishes that when facing homogeneous incumbents the challenger may benefit from information revelation through sequential contests.

In Section 5, we study situations in which the challenger must face the incumbents sequentially but can choose their order, which introduces an additional signaling stage. We show that the equilibrium is either pooling or partial pooling, and there does not exist separating equilibria. We characterize the order of rivals chosen by the challenger in equilibrium for different parameters. When the strong type is significantly stronger than the weak type, and the prior is that the challenger is weak, both challenger types pool at facing the weak incumbent first. By contrast, types pool at facing the strong incumbent first when type heterogeneity is mild and the prior is that the challenger is strong (see Fig. 5).

CRediT authorship contribution statement

Jorge Catepillán: Writing – original draft, Conceptualization. **Nicolás Figueroa:** Writing – original draft. **Jorge Lemus:** Writing – original draft.

Declaration of competing interest

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Appendix A. Additional figures

Fig. 6 shows the intensity of signal jamming in equilibrium. Specifically, it shows the difference in the probability of observing W in a sequential and a simultaneous contest, i.e., $\Pr(W|\widetilde{C}) - \Pr(W|C)$. The figures at top row correspond to cases where Incumbent-2 is weak.[5] The figures at the bottom row correspond to cases where Incumbent-2 is strong.[6]

[5] The plot shows the unconditional probability, $\Pr(W|\widetilde{C}) - \Pr(W|C) = (1-p)(\Pr^H(W|\widetilde{C}) - \Pr^H(W|C))$.

[6] The plot shows the unconditional probability, $\Pr(W|\widetilde{C}) - \Pr(W|C) = -(2-p)(\Pr^L(W|\widetilde{C}) - \Pr^L(W|C))$.

Appendix B. Proofs

Proof of Lemma 1. We proof follows from the algorithm in Siegel (2014) to find the unique equilibrium of the game. Chen and Chen (2024) also derives this result. The proof details are available in our Online Appendix.

Here, we summarize the equilibrium strategies, probabilities of winning, expected efforts, and expected payoffs as a function of p in three cases:

1. If $c_B \le pc_L$:
   (a) *Player A, High cost*: Plays zero with probability 1. Wins the contest with probability zero. Her expected effort is 0. Her expected payoff is 0.
   (b) *Player A, Low cost*: Places an atom of size $1-\frac{c_B}{pc_L}$ at zero and mixes with density $\frac{c_B}{p}$ on the interval $\left[0, \frac{1}{c_L}\right]$. Wins with probability $\frac{c_B}{2pc_L}$. Her expected effort is $\frac{c_B}{2pc_L^2}$. Her expected payoff is 0.
   (c) *Player B*: Mixes with density c_L on $\left[0, \frac{1}{c_L}\right]$. Her expected effort is $\frac{1}{2c_L}$.

2. If $pc_L < c_B \le pc_L + (1-p)c_H$, let $e_1 = \left(1-\frac{pc_L}{c_B}\right)\frac{1}{c_H}$ and $e_2 = \frac{p}{c_B}$.
   (a) *Player A, High cost*: Places an atom of size $1-\frac{\ell_1 c_B}{1-p}$ at zero and mixes with density $\frac{c_B}{1-p}$ on the interval $[0, \ell_1]$. Wins with probability $\frac{c_H c_B \ell_1^2}{2(1-p)}$. Her expected effort is $\frac{\ell_1^2 c_B}{2(1-p)}$. Her expected payoff is 0.
   (b) *Player A, Low cost*: Mixes with density $\frac{1}{\ell_2}$ on the interval $\left[\ell_1, \ell_1+\ell_2\right]$. Wins with probability $1-\frac{pc_L}{2c_B}$. Her expected effort is $\ell_1+\frac{\ell_2}{2}$. Her expected payoff is $(c_H - c_L)\ell_1$.
   (c) *Player B*: Mixes with density c_H on the interval $[0, \ell_1]$ and with density c_L on the interval $[\ell_1, \ell_1+\ell_2]$. Her expected effort is $\frac{c_H \ell_1^2}{2} + c_L \ell_2\left(\ell_1+\frac{\ell_2}{2}\right)$

3. If $pc_L + (1-p)c_H < c_B$, let $\ell_1 = \frac{1-p}{c_B}$ and $\ell_2 = \frac{p}{c_B}$.
   (a) *Player A, High cost*: Mixes with density $\frac{1}{\ell_1}$ on the interval $\left[0, \ell_1\right]$. Wins with probability $1-\frac{c_H\ell_1}{2} - c_L\ell_2$. Her expected effort is $\frac{\ell_1}{2}$. Her expected payoff is $1-\frac{pc_L+(1-p)c_H}{c_B}$.
   (b) *Player A, Low cost*: Mixes with density $\frac{1}{\ell_2}$ on the interval $\left[\ell_1, \ell_1+\ell_2\right]$. Wins with probability $1-\frac{c_L\ell_2}{2}$. Her expected effort is $\ell_1+\frac{\ell_2}{2}$. Her expected payoff is $1-\frac{c_L}{c_B}$.
   (c) *Player B*: Places an atom of size $1-\ell_1 c_H - \ell_2 c_L$ at zero and mixes with density c_H on the interval $[0, \ell_1]$ and with density c_L on the interval $[\ell_1, \ell_1+\ell_2]$. Her expected effort is $\frac{c_H\ell_1^2}{2} + c_L\ell_2\left(\ell_1+\frac{\ell_2}{2}\right)$.

Using these results, we can derive the payoffs illustrated in Fig. 1.

1. When $c_B \le c_L$, $\pi^H(C) = 0$ for all p and

$$\pi^L(C) = \begin{cases} (1-\gamma)\left(1-pc_L/c_B\right) & \text{, if } p \le c_B/c_L, \\ 0 & \text{, otherwise.} \end{cases} \tag{7}$$

2. When $c_H \le c_B$,

$$\pi^L(C) = 1 - c_L/c_B, \tag{8}$$

$$\pi^H(C) = 1 - c_H/c_B + p(c_H - c_L)/c_B. \tag{9}$$

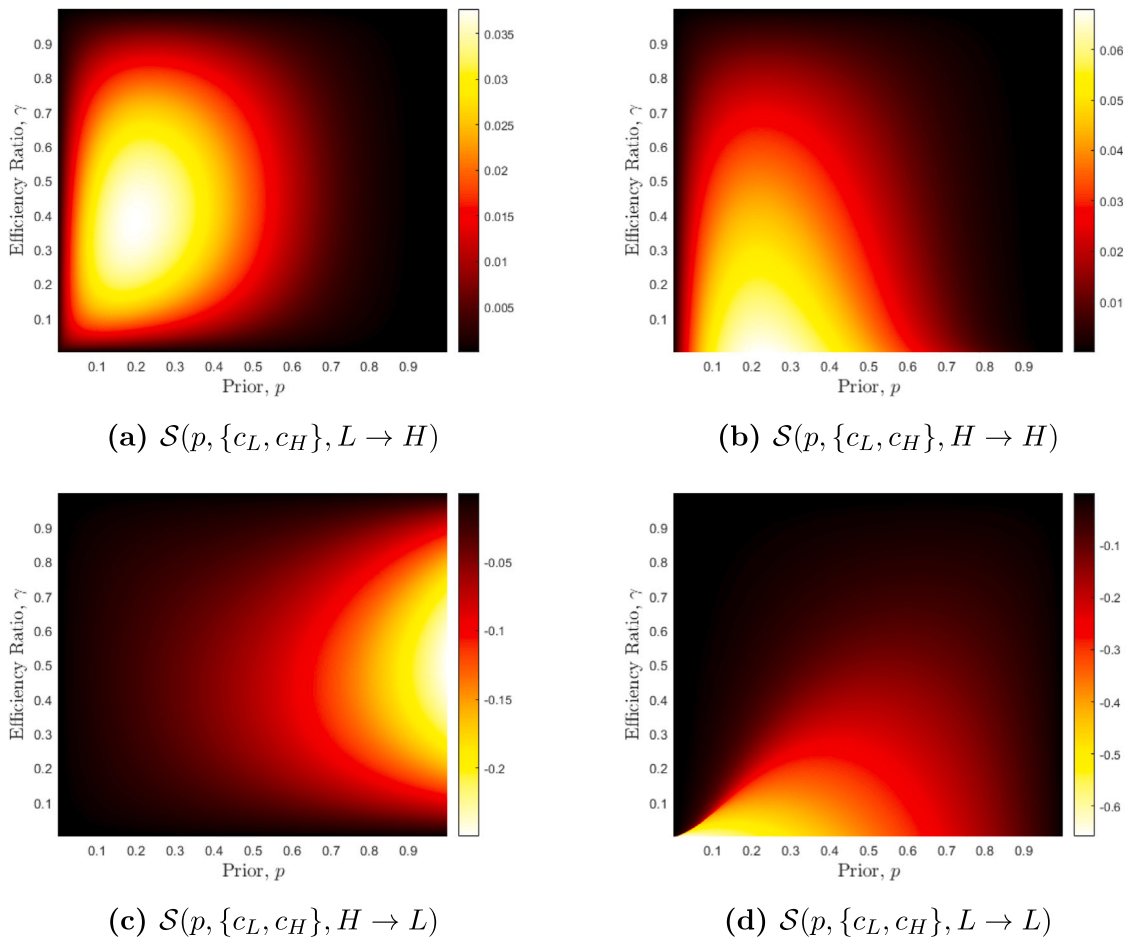


Fig. 6. Intensity of signal-jamming in equilibrium, $\Pr(W|\widetilde{C}) - \Pr(W|C)$.

Proof of Proposition 1

Proof. Equilibrium strategies in the second contest depend on the first contest result, R, the second incumbent's belief as a function of this result, $q(R)$, player A's types $\{c_L, c_H\}$, and the strength of the second's incumbent, c_2. For the sake of notation, we denote the second contest by $C_R \equiv C(q(R), \{c_L, c_H\}, c_2)$. Let $\pi^j(C_R)$ be the equilibrium payoff of a challenger of type $j \in \{L, H\}$ in contest C_R. Let e_1 be the incumbent's effort in the first contest of the dynamic game. The challenger of type j chooses his effort e to solve:[7]

$$\max_{e\geq 0}\,(1+\pi^j(C_W))\Pr(e,e_1)+\pi^j(C_D)(1-\Pr(e,e_1))-c_j e, \tag{10}$$

where $\Pr(e, e_1) = 1(e > e_1) + \frac{1}{2}\cdot 1(e = e_1)$. The solution to problem (10) is equivalent to the solution of

$$\max_{e\geq 0}\,\Pr(e,e_1)-\widetilde{c}_j e, \tag{11}$$

where the *virtual costs*, $\tilde{c}_j$, is defined by

$$\widetilde{c}_j \equiv \frac{c_j}{1+\pi^j(C_W)-\pi^j(C_D)}, \qquad \text{for } j \in \{L, H\}. \tag{12}$$

We note that Chen and Chen (2024) contains results that are similar to this lemma. □

[7] We ignore ties for notational convenience because they are zero-probability events in equilibrium.

Proof of Proposition 2

Proof. Proposition 1 and Lemma 1, characterize equilibrium strategies in the second contest, conditional on beliefs $q(R)$. It remains to show that a unique belief system satisfies Eq. (2) (Bayes Rule), for any given set of incumbents.

Eqs. (3) and (4) show that virtual costs are either independent of $q(W)$ and $q(D)$, or there is a one-to-one mapping between the virtual cost and $q(W)-q(D)$. The general proof strategy is the following. Bayes' rule together with virtual costs yields a (non-linear) system of equations for $q(W)$ and $q(D)$. In each case, the system of equations has the form, $q(D) = \Lambda_A(q(W) - q(D))$ and $q(W) = \Lambda_B(q(W) - q(D))$. If we subtract these two equations, we obtain: $q(W) - q(D) = \Lambda_C(q(W) - q(D))$. The functions Λ_A, Λ_B, and Λ_C are determined by incumbents' types in the first and second contests. Denoting $x \equiv q(W) - q(D)$, the proof hinges on showing that there is a unique solution to the equation $x = \Lambda_C(x)$. Once we find x^*, the unique solution to this equation, we obtain the (unique) equilibrium beliefs from $q(D) = \Lambda_A(x^*)$ and $q(W) = \Lambda_B(x^*)$. We then prove the bounds on virtual costs stated in the proposition.

We proceed case by case, denoting by $j \to k$ the sequential contest where Incumbent-1's type is c_j and Incumbent-2's type is c_k.

1. Case $L \to H$. In a perfect Bayesian equilibrium, using Bayes' rule in (2) and the probability of winning/losing in the proof of Lemma 1, the belief system $(q(W), q(D))$ must satisfy

$$q(W) = \frac{(2-p)\,p}{(2-p)\,p + \frac{c_L}{\widetilde{c}_H}(1-p)^2} \quad \text{and} \quad q(D) = \frac{p^2}{p^2 + 2(1-p) - \frac{c_L}{\widetilde{c}_H}(1-p)^2} \tag{13}$$

The virtual cost $\widetilde{c}_H$, defined in Eq. (3), is a one-to-one function of $q(W)-q(D)$. Recall that $\gamma=\frac{c_L}{c_H}$. Replacing the definition of $\widetilde{c}_H$ in (13), subtracting $q(W)$ and $q(D)$ in (13), and defining $x=q(W)-q(D)$, any beliefs satisfying (13) must also solve

$$f(x)\equiv\frac{2p^2}{(1-\gamma)(1-p)^2(1-\gamma x)}+x(2-p)p+\gamma(1-p)^2(1+(1-\gamma x))-2p=0. \tag{14}$$

It remains to show that there exists a unique solution, $x^*\in[0,1]$ that solves $f(x^*)=0$. Standard computations show that $f(0)<0<f(1)$ and $f'(x)>0$, for all $x\in(0,1)$. Therefore, there is a unique solution $x^*\in[0,1]$ that solves Eq. (14). This solution uniquely pins down $q(W)$ and $q(D)$ from (13).

Bounds on virtual costs. We now show that $c_L<\widetilde{c}_H\le c_H$. Let $x=q(W)-q(D)$. Note that $1+x(1-\gamma)>0$ for all $x\in[1,-1]$. $c_L<\widetilde{c}_H$ if and only if $(1+x(1-\gamma))c_L\le c_H$. Since $c_L/c_H<1$ and $(2-\gamma)\gamma<1$ for $0\le\gamma<1$, this inequality always holds. For $\tilde{c}_H\le c_H$ we need that $x\ge 0$. Now, $x\ge 0$ is equivalent to $\Pr(W|c_1=c_L)\ge\Pr(W|c_1=\widetilde{c}_H)$. Since $c_L<\widetilde{c}_H$, using the results of the static contest we can show that $\Pr(W|c_1=c_L)\ge\Pr(W|c_1=\widetilde{c}_H)$ is equivalent to
$1-\frac{c_L}{2\widetilde{c}_H}\ge\frac{p}{2}\left(1-\frac{c_L}{\widetilde{c}_H}\right)$.

This condition always holds because $f(y)=1-\frac{y}{2}-\frac{p}{2}(1-y)$ is strictly decreasing in $[0,1]$ with $f(1)=1/2$.

2. Case $H\to L$. In a perfect Bayesian equilibrium, using Bayes' rule in (2) and the probability of winning/losing in the proof of Lemma 1, the belief system $(q(W),q(D))$ must satisfy

$$q(W)=\frac{\left(2-\frac{\widetilde{c}_L}{c_H}p\right)p}{2p+1-p^2-\frac{\widetilde{c}_L}{c_H}p(2-p)}\quad\text{and}\quad q(D)=\frac{\frac{\widetilde{c}_L}{c_H}p^2}{(1-p)^2+\frac{\widetilde{c}_L}{c_H}p(2-p)}. \tag{15}$$

The virtual cost $\widetilde{c}_L$, defined in Eq. (4), is a one-to-one function of $q(W)-q(D)$. We define $\gamma=\frac{c_L}{c_H}$. Replacing the definition of $\widetilde{c}_L$ in (15), subtracting $q(W)$ and $q(D)$ in (15), and defining $x=q(W)-q(D)$, any beliefs satisfying (15) must also solve

$$\begin{aligned}h(x)=&2\gamma p^2(1-xG)+x\,(A(1-xG)-B)\,((1-xG)(2-A)+B)\\&-2p((1-xG)(2-A)+B)(1-xG)=0,\end{aligned} \tag{16}$$

with $A=1+2p-p^2$, $B=\gamma p(2-p)$, and $G=1-\gamma$.

Standard computations show that $h(0)<0<h(1)$. Furthermore,

$$h''(x)=2G[3AG(2-A)x+2(B(1-A)-(A+Gp)(2-A))],$$

which is clearly increasing in x, so $h''(x)\le h''(1)$ for any $x\in[0,1]$. Standard computations show that $h''(1)<0$, so h is concave in $[0,1]$. Concavity plus the fact that $h(0)<0<h(1)$ implies that there is a unique x^* such that $h(x^*)=0$. This solution uniquely pins down $q(W)$ and $q(D)$ from (15).

Bounds on virtual costs. Let $x=q(W)-q(D)$. Note that $1-x(1-\gamma)>0$ for all $x\in[1,-1]$. $\tilde{c}_L<c_H$ if and only if $c_L<c_H(1-x(1-\gamma))$ or, equivalently, $x<1$, which always holds. For $c_L\le\tilde{c}_L$ we need that $x\ge 0$. Now, $x\ge 0$ is equivalent to $\Pr^L(W|C(p,\{\widetilde{c}_L,c_H\},c_H))\ge\Pr^H(W|C(p,\{\widetilde{c}_L,c_H\},c_H))$, which holds because from Lemma 1 it is
$1-\frac{\tilde{c}_Lp}{2c_H}\ge\frac{1+p}{2}-\frac{\tilde{c}_Lp}{c_H}$,

which holds when $\tilde{c}_L\le c_H$.

3. Case $c_H\to c_H$. The challenger faces two incumbents of type c_H. From Bayes' rule in (2) and the probability of winning/losing in the proof of Lemma 1, we have:

$$q(W)=\frac{(2-\gamma p)\,p}{2-\gamma\,(2-p)\,p-\frac{\tilde{c}_H}{c_H}(1-p)^2}\quad\text{and}\quad q(D)=\frac{\gamma p^2}{\gamma\,(2-p)\,p+\frac{\tilde{c}_H}{c_H}(1-p)^2}. \tag{17}$$

Using that the virtual costs are given by (3), the system in (17) must satisfy:

$$\frac{\tilde{c}_H}{c_H}=\left[1+\left[\frac{(2-\gamma p)\,p}{2-\gamma\,(2-p)\,p-\frac{\tilde{c}_H}{c_H}(1-p)^2}-\frac{\gamma p^2}{\gamma\,(2-p)\,p+\frac{\tilde{c}_H}{c_H}(1-p)^2}\right](1-\gamma)\right]^{-1}. \tag{18}$$

The left side is strictly increasing in $\tilde{c}_H$, and the right side is decreasing in $\tilde{c}_H$. Therefore the solution, $\tilde{c}^*_H$, which exists by Brower's fixed point theorem, is unique. Equilibrium beliefs are uniquely pindown by replacing $\tilde{c}^*_H$ in (17).

4. Case $c_L\to c_L$. The challenger faces two incumbents of type c_H. Using the probability of winning/losing in the proof of Lemma 1, we have:

$$\Pr{}^L(W|C)=\frac{c_L}{2p\tilde{c}_L}\cdot 1(c_L<p\tilde{c}_L)+\left(1-\frac{p\tilde{c}_L}{2c_L}\right)\cdot 1(c_L\ge p\tilde{c}_L) \tag{19}$$

$$\Pr{}^H(W|C)=0\cdot 1(c_L<p\tilde{c}_L)+\frac{\gamma}{2(1-p)}\left(1-\frac{p\tilde{c}_L}{c_L}\right)^2\cdot 1(c_L\ge p\tilde{c}_L) \tag{20}$$

Therefore, by Bayes' rule in (2), the second incumbent's beliefs after learning the result of the first contest are:

$$q(W)=\begin{cases}1 & \text{if } c_L<p\tilde{c}_L\\ \frac{p\left(1-\frac{p\tilde{c}_L}{2c_L}\right)}{p\left(1-\frac{p\tilde{c}_L}{2c_L}\right)+\frac{\gamma}{2}\left(1-\frac{p\tilde{c}_L}{c_L}\right)^2} & \text{if } c_L\ge p\tilde{c}_L\end{cases} \tag{21}$$

$$q(D)=\begin{cases}\frac{\left(1-\frac{c_L}{2p\tilde{c}_L}\right)p}{\left(1-\frac{c_L}{2p\tilde{c}_L}\right)p+(1-p)} & \text{if } c_L<p\tilde{c}_L\\ \frac{p\left(\frac{p\tilde{c}_L}{2c_L}\right)}{p\left(\frac{p\tilde{c}_L}{2c_L}\right)+(1-p)\left(1-\frac{\gamma}{2(1-p)}\left(1-\frac{p\tilde{c}_L}{c_L}\right)^2\right)} & \text{if } c_L\ge p\tilde{c}_L\end{cases} \tag{22}$$

Using that the virtual costs are given by (4)

$$\frac{c_L}{\tilde{c}_L}=(1-(q(W)-q(D))(1-\gamma)), \tag{23}$$

Using the expressions of $q(W)$ and $q(D)$ above, we can show that the right-hand side of this equation is increasing in $\tilde{c}_L$, whereas the left-hand side is strictly decreasing in $\tilde{c}_L$. Therefore, there exists a solution (by Brower's fixed point theorem), and this solution is unique. Equilibrium beliefs are uniquely pindown by replacing this unique solution in the expressions for $q(W)$ and $q(D)$ above. □

Proof of Proposition 3

Proof. We show how the challenger's payoff in the first contest is affected by the incentives created by the second contest. From Eq. (10), the total equilibrium payoff for the challenger of type j is

$$(1+\pi^j(C_W))\Pr(e^*_j,e^*_1)+\pi^j(C_D)(1-\Pr(e^*_j,e^*_1))-c_je^*_j.$$

We separate this payoff into two parts, the first- and second-contest payoffs:

$$\underbrace{\Pr(e^*_j,e^*_1)-c_je^*_j}_{\text{first contest payoff}}+\underbrace{\Pr(e^*_j,e^*_1)\pi^j(C_W)+\pi^j(C_D)(1-\Pr(e^*_j,e^*_1))}_{\text{second contest payoff}}.$$

We compare the challenger's payoff in the first contest by computing $\Pr(e^*_j,e^*_1)-c_je^*_j$, when (e^*_j,e^*_1) are the equilibrium efforts (distributions) $\widetilde{C}\equiv C(p,\{\tilde{c}_L,\tilde{c}_H\},c_1)$, which accounts for dynamic incentives created by sequential contests, and in $C\equiv C(p,\{c_L,c_H\},c_1)$, which is a static contest and is equivalent to the payoff of one contest in the simultaneous case.

Table 1
Probabilities of Winning and Expected Efforts under Different Combinations of First Incumbent and Challenger Strengths.

First incumbent	Challenger	Probability of winning	Expected effort
Weak	Strong	$\Pr^L(W\|\widetilde{C}) = 1 - \gamma\frac{(1-p)}{2}$	$\frac{2-p}{2c_H}$
Weak	Weak	$\Pr^H(W\|\widetilde{C}) = 1 - \frac{\tilde{c}_H}{c_H}(1-p) - \gamma p$	$\frac{1-p}{2c_H}$
Strong	Strong	$\Pr^L(W\|\widetilde{C}) = 1 - \frac{p}{2}$	$\frac{c_L}{\tilde{c}_H}$
Strong	Weak	$\Pr^H(W\|\widetilde{C}) = \frac{c_L}{2\tilde{c}_H}(1-p)$	$\frac{(1-p)c_L}{2\tilde{c}_H^2}$

We exploit that equilibrium objects in $C(p,\{c_L,c_H\},c_1)$ can be computed from the equilibrium objects in $C(p,\{\tilde{c}_L,\tilde{c}_H\},c_1)$ by replacing the virtual costs with the actual costs.

We divide the proof in two cases, based on the strength of the second incumbent.

Case 1, $c_2 = c_H$: We first consider the case of a weak incumbent in the second contest. Here, we have $\tilde{c}_L = c_L$ and from Proposition 1, $\tilde{c}_H < c_H$. Let $C \equiv C(p,\{c_L,c_H\},c_1)$ and $\widetilde{C} \equiv C(p,\{c_L,\tilde{c}_H\},c_1)$. From Lemma 1, we have the following equilibrium values for the challenger's probability of winning and expected effort, summarized in Table 1 below.

In Table 1 (first row), the probability of winning and expected effort are independent of $\tilde{c}_H$. Thus, the first contest payoff is the same in a sequential or simultaneous contest.

In Table 1 (second row), the probability of winning decreases with $\tilde{c}_H$, and the expected effort is independent of $\tilde{c}_H$. Thus, the first contest's payoff is higher in a sequential contest.

In Table 1 (third row), the probability of winning is independent of $\tilde{c}_H$ but the expected effort decreases with $\tilde{c}_H$. Thus, the first contest's payoff is lower in a sequential contest.

In Table 1 (fourth row), the probability of winning and expected effort both dependent on $\tilde{c}_H$. Thus, the first contest payoff is

$$\frac{c_L}{2\tilde{c}_H}(1-p) - c_H\frac{c_L}{2\tilde{c}_H^2}(1-p) = (\tilde{c}_H - c_H)\frac{c_L}{2\tilde{c}_H^2}(1-p)$$

Given that, in equilibrium, $\tilde{c}_H < c_H$, the first contest's payoff for the strong challenger is lower in a sequential contest than in a simultaneous one.

In summary, assuming a weak second incumbent, when the first incumbent is strong, both types of challengers receive a lower payoff in the first stage when the contests are simultaneous. Contrarily, when the first incumbent is weak, the weak challenger's first contest's payoff is higher with sequential contests, and the strong challenger's first contest's payoff is unaffected by whether the contests are simultaneous or sequential.

Case 2, $c_2 = c_L$: We next consider the case of a strong incumbent in the second contest.

In equilibrium, $\tilde{c}_H = c_H$, and from Proposition 1, $\tilde{c}_L > c_L$. Let $C \equiv C(p,\{c_L,c_H\},c_1)$ and $\widetilde{C} \equiv C(p,\{\tilde{c}_L,c_H\},c_1)$.

From Lemma 1, if $c_1 = c_H$,

$$\Pr^L(W|\widetilde{C}) = 1 - \gamma\frac{\tilde{c}_L p}{2c_L} \quad \text{and} \quad \Pr^H(W|\widetilde{C}) = \frac{1+p}{2} - \gamma\frac{\tilde{c}_L p}{c_L},$$

and if $c_1 = c_L$,

$$\Pr^L(W|\widetilde{C}) = \frac{c_L}{2p\tilde{c}_L}\cdot 1(c_L < p\tilde{c}_L) + \left(1 - \frac{p\tilde{c}_L}{2c_L}\right)\cdot 1(c_L \geq p\tilde{c}_L),$$

$$\Pr^H(W|\widetilde{C}) = 0\cdot 1(c_L < p\tilde{c}_L) + \frac{\gamma}{2(1-p)}\left(1 - \frac{p\tilde{c}_L}{c_L}\right)^2\cdot 1(c_L \geq p\tilde{c}_L).$$

Case 2.1: Weak challenger.

It is clear that $\Pr^H(W|C)$ decreases in $\tilde{c}_L$, which implies that the weak incumbent wins less often in a sequential contest.

If the first incumbent is weak, the expected effort of the weak challenger is $\frac{1-p}{c_H}$, which is the same in a simultaneous or sequential contest. Therefore, only the probability of winning matters when comparing the first contest's payoffs (i.e, $\Pr^H(W|\widetilde{C}) - c_H E[e_H^*(\widetilde{C})]$ versus $\Pr^H(W|C) - c_H E[e_H^*(C)]$), so the weak challenger is worse off in a sequential contest.

If the first incumbent is strong, then the weak challenger expected effort is $0\cdot 1(c_L < p\tilde{c}_L) + \frac{\gamma}{2(1-p)c_H}\left(1 - \frac{p\tilde{c}_L}{c_L}\right)^2\cdot 1(c_L \geq p\tilde{c}_L)$, and her expected payoff in the first stage is equal to zero for any value of $\tilde{c}_L$. In this case, it does not matter if the contest is simultaneous or sequential, the weak challenger gets and expected payoff of zero in the first contest.

Case 2.2: Strong challenger.

If the first incumbent is weak, from Lemma 1, the strong challenger's expected payoff in the first contest is $\Pr^L(W|\widetilde{C}) - c_L e^*(\widetilde{C})$ where $e^*(\widetilde{C}) = \frac{(2-p)}{2c_H}$, and therefore,

$$\Pr^L(W|\widetilde{C}) - c_L e^*(\widetilde{C}) = 1 - \gamma\frac{\tilde{c}_L p}{2c_L} - \frac{\gamma(2-p)}{2}.$$

This function is decreasing in $\tilde{c}_L$, which implies that the payoff is greater when $\tilde{c}_L = c_L$.

Finally, if the first incumbent is strong, from Lemma 1, we have two subcases. First, let $c_L < p\tilde{c}_L$. The strong challenger's expected effort is $e^*(\widetilde{C}) = \frac{c_L}{2p\tilde{c}_L^2}$, and therefore,

$$\Pr^L(W|\widetilde{C}) - c_L e^*(\widetilde{C}) = \frac{c_L}{2p\tilde{c}_L}\left(1 - \frac{c_L}{\tilde{c}_L}\right). \tag{24}$$

This function is non-monotone in $\tilde{c}_L$. Second, let $c_L \geq p\tilde{c}_L$. The strong challenger's expected effort is $e^*(\widetilde{C}) = \left(1 - \frac{p\tilde{c}_L}{c_L}\right)\frac{1}{c_H} + \frac{p}{2c_L}$, and therefore,

$$\Pr^L(W|\widetilde{C}) - c_L e^*(\widetilde{C}) = 1 - \gamma + \frac{p\tilde{c}_L}{c_L}\left(\gamma - \frac{1}{2}\right) - \frac{p}{2}. \tag{25}$$

Therefore, the strong challenger's payoff is non-monotone in $\tilde{c}_L$.

Next, we show that the strong challenger's payoff in the first stage of the dynamic game can be higher or lower than the payoff in the static contest $C(p,\{c_L,c_H\},c_L)$, which is $(1-\gamma)(1-p)$.

Suppose $p \leq \gamma$. Since $\tilde{c}_L < c_H$, $c_L/\tilde{c}_L \geq \gamma \geq p$, and we are always in the region where $c_L \geq p\tilde{c}_L$, and the strong challenger's payoff is given by (25). However, the challenger's payoff is increasing or decreasing in $\tilde{c}_L$ depending on how γ compares to $1/2$. For $\gamma > 1/2$, the strong challenger has a higher payoff with a second stage, but for $\gamma < 1/2$, the strong challenger's payoff is higher without the second stage.

In summary, assuming a strong second incumbent, when the first incumbent is weak, both types of challengers are worse with a second stage vs without it. When the first incumbent is strong, the weak challenger is unaffected by the second stage, but the effect on the strong challenger's payoff is ambiguous, its direction depends on the parameters. □

Proof of Proposition 4

Proof. We divide the proposition in two cases depending on whether incumbents are heterogeneous (one strong and one weak) or homogeneous (both strong or both weak).

1. Heterogeneous incumbents. Let $C_H \equiv C(p,\{c_L,c_H\},c_H)$ and $C_L \equiv C(p,\{c_L,c_H\},c_L)$. If the challenger of type c_j plays C_H and C_L simultaneously, his expected payoff is $\pi_S^j = \pi^j(C_H) + \pi^j(C_L)$. It is direct from Lemma 1 (see Fig. 1, left panel with $c_B = c_L$ and right panel with $c_B = c_H$) that

$$\pi^H(C_H) = p(1-\gamma),$$
$$\pi^H(C_L) = 0,$$
$$\pi^L(C_H) = 1-\gamma,$$
$$\pi^L(C_L) = (1-\gamma)(1-p).$$

Therefore,

$$\pi_S^H = p(1-\gamma) \quad \text{and} \quad \pi_S^L = (2-p)(1-\gamma).$$

For the sequential game, $\mathcal{S}(p,\{c_L,c_H\},c_j \to c_k)$, we define the equilibrium objects:

1. $q_{j\to k}(R)$, the second incumbent's belief after result R in the first contest.
2. $\pi^i_{j\to k}$, the expected payoff of challenger of type c_i when playing sequentially the order $j \to k$.

To compute the expected payoffs for each challenger, we use Proposition 1 to compute

$$\pi^i_{j\to k} = (1 + \pi^i(C_W) - \pi^i(C_D))(\Pr(e^*, e_1^*) - \tilde{c}_i e^*) + \pi^i(C_D). \tag{26}$$

We use Proposition 1 and follow these steps:

1. Compute the equilibrium payoff of static game where the challenger's types are $\tilde{c}_i$ instead of c_i, exploiting the characterization in (11), to compute $\Pr(e^*, e_1^*) - \tilde{c}_i e^*$.
2. Compute the equilibrium payoffs of the second contest depending on the result of the first contest, which correspond to the static payoffs above but replacing p by $q_{j\to k}(R)$, since the result R determines the beliefs in the second contest.
3. Multiply the payoff in step 1. by $1 + \pi^i(C_W) - \pi^i(C_D)$ to obtain the first term in (26) and add the expected payoff of the second contest, assuming the result of the first contest was D from step 2. to obtain the term $\pi^j(C_D)$ in (26).

When the sequence of incumbents follows the order $L \to H$, the first contest is $C(p, \{c_L, \tilde{c}_H\}, c_L)$. The payoff for the strong challenger in this contest with virtual costs is $(\tilde{c}_H - c_L)(1-p)/\tilde{c}_H$. The payoff of the weak challenger is 0.

In the second contest, for the weak challenger $1+\pi^H(C_W)-\pi^H(C_D) = 1 + (1-\gamma)(q_{L\to H}(W) - q_{L\to H}(D))$ and his payoff in case of defeat in the first contest is $q_{L\to H}(D)(1-\gamma)$. Then,

$$\begin{aligned}\pi^H_{L\to H} &= 0 \times (1 + (1-\gamma)(q_{L\to H}(W) - q_{L\to H}(D))) + q_{L\to H}(D)(1-\gamma)\\ &= q_{L\to H}(D)(1-\gamma).\end{aligned}$$

In the second contest, for the strong challenger $1+\pi^L(C_W)-\pi^L(C_D) = 1$ and his payoff in case of defeat in the first contest is $(1-\gamma)$. Then,

$$\pi^H_{L\to H} = 1 \times (\tilde{c}_H - c_L)(1-p)/\tilde{c}_H + (1-\gamma),$$

with $\tilde{c}_H = c_H/(1 + (1-\gamma)(q_{L\to H}(W) - q_{L\to H}(D)))$.

Analogous computations can be performed when the sequence of incumbents follows the order $L \to H$, taking into account that we have $C(p, \{\tilde{c}_L, c_H\}, c_L)$ on the first stage. We have:

1. The challenger's expected payoffs in $\mathcal{S}(p, \{c_L, c_H\}, c_L \to c_H)$ are
$$\begin{aligned}\pi^H_{L\to H} &= q_{L\to H}(D)(1-\gamma),\\ \pi^L_{L\to H} &= \left(2 - p - (1-p)\gamma(q_{L\to H}(W) - q_{L\to H}(D))\right)(1-\gamma).\end{aligned}$$
2. The challenger's expected payoffs in $\mathcal{S}(p, \{c_L, c_H\}, c_H \to c_L)$ are
$$\begin{aligned}\pi^H_{H\to L} &= p\left(1 - \frac{\gamma(q_{H\to L}(W) - q_{H\to L}(D))}{(1 - (q_{H\to L}(W) - q_{H\to L}(D))(1-\gamma))}\right)(1-\gamma),\\ \pi^L_{H\to L} &= \left(2 - q_{H\to L}(W)\right)(1-\gamma).\end{aligned}$$

It is direct from Proposition 2 and the fact that $q(D) < p < q(W)$ that the expected payoff of each challenger's type is higher when he faces the incumbents simultaneously than sequentially, regardless of the order of the incumbents.

2. Homogeneous incumbents. From Proposition 3, when facing homogeneous incumbents, the effect of playing sequentially is ambiguous. We show how this changes for different values of p and γ. Fig. 7 shows combinations of parameters for which the challenger prefers to face the rivals sequentially rather than simultaneously. □

Proof of Proposition 5

Proof. We first show that the strong challenger's expected payoff is increasing in p, regardless of the order of opponents, and the opposing happens for the weak challenger. We divide the proof in two cases:

Case 1: Incumbents of Decreasing Strength ($L \to H$)

Define $A(x,p) = (2-p)p + (1 + xG)\gamma(1-p)^2$, where $G = 1-\gamma$. Note that $A \in [p, 1]$ since $(2-p) > 1$ and $\gamma(1+xG) < 1$. For given beliefs $q(W)$ and $q(D)$, let $x = q(W) - q(D)$. From Bayes's rule in Eq. (2) and the definition of virtual costs in (3) we can write:

$$q(W) = \frac{(2-p)p}{A} \text{ and } q(D) = \frac{p^2}{2-A}.$$

Subtracting these two equations, for any $p \in (0,1)$, $x_{L\to H}$ is the solution to $x = \frac{(2-p)p}{A(x,p)} - \frac{p^2}{2-A(x,p)}$, and therefore

$$\frac{dx_{L\to H}}{dp} = \frac{\frac{2(2(1-p)-A)}{A(2-A)} - \frac{\partial A}{\partial p}\cdot D}{1 + \frac{\partial A}{\partial x}\cdot D}, \tag{27}$$

where $D = \frac{(2-p)p}{A^2} + \frac{p^2}{(2-A)^2}$.

We show first that $\pi^H_{L\to H}$ increases with p (case 1.1) and then that $\pi^L_{L\to H}$ decreases with p (case 1.2).

Case 1.1: Since $\pi^H_{L\to H} = q_{L\to H}(D)(1-\gamma)$ we only need to show that $q_{L\to H}(D)$ is increasing in p. Taking derivative of $q_{L\to H}(D)$ with respect to p, we obtain:

$$\frac{dq_{L\to H}(D)}{dp} = \frac{2p(2-A) + p^2\frac{dA}{dp}}{(2-A)^2}.$$

and therefore the sign of $\frac{dq_{L\to H}(D)}{dp}$ is the same as the sign of the numerator, which is positive if $\frac{dA}{dp} \geq 0$, since $A < 1$.

It remains to show that $\frac{dA}{dp} \geq 0$. From (27), we get

$$\frac{dA}{dp} = \frac{1}{1 + \frac{\partial A}{\partial x}\cdot D}\left[\frac{\partial A}{\partial p} + \frac{\partial A}{\partial x}\left(\frac{2(2(1-p)-A)}{A(2-A)}\right)\right].$$

Because $\frac{\partial A}{\partial x} > 0$, the sign of $\frac{dA}{dp}$ is the same as the sign of $Z \equiv \frac{\partial A}{\partial p} + \frac{\partial A}{\partial x}\left(\frac{2(2(1-p)-A)}{A(2-A)}\right)$.

Furthermore, because $\frac{\partial A}{\partial x} \geq 0$ and $\frac{\partial A}{\partial p} \geq 0$, $\frac{dA}{dp} \geq 0$ when $2(2(1-p) - A) \geq 0$. Therefore, the we need to prove that $Z > 0$ when $2(1-p) < A$. Using previous definitions, since $A \in [p,1]$, and $2(1-p) < A$, it can be shown that $Z \geq 2(1-p)G\gamma\left[2(1-p) - x_{L\to H}\right]$

Lastly, using that $x_{L\to H}$ is decreasing in γ, and $x \leq x_{L\to H}(0,p) = \frac{2(1-p)}{1+(1-p)^2} \leq 2(1-p)$, we conclude that $Z \geq 0$, when $2(1-p) < A$. Since $Z \geq 0$ when $2(1-p) \geq A$, we conclude that $Z \geq 0$ always. This implies that $\frac{dA}{dp} \geq 0$.

Case 1.2: We now show that $\pi^L_{L\to H} = \left(2 - p - (1-p)\gamma x_{L\to H}(W)\right)(1-\gamma)$ decreases with p. This is equivalent to

$$\gamma\left(x_{L\to H} - (1-p)\frac{dx_{L\to H}}{dp}\right) \leq 1. \tag{28}$$

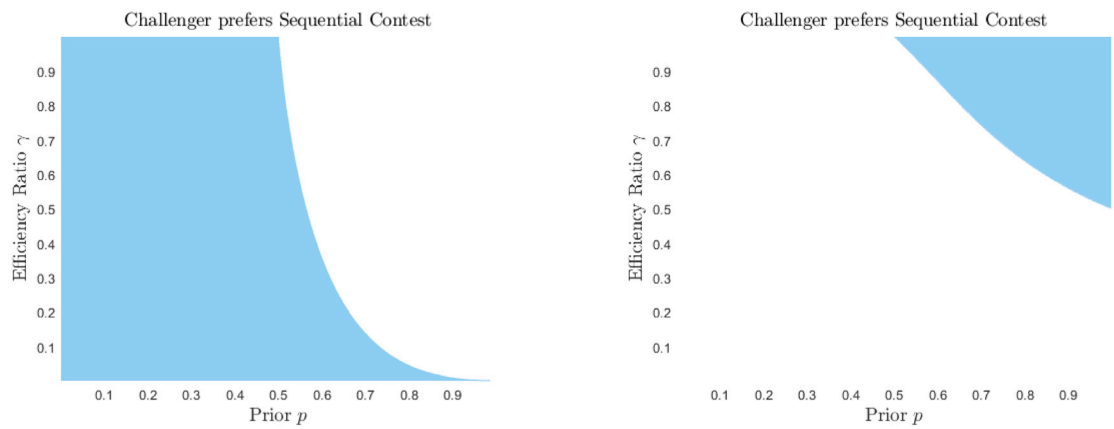
To show that this inequality holds, we study two cases. First, if $\frac{dx_{L\to H}}{dp} \geq 0$ it holds because $\gamma x_{L\to H} \leq 1$. Second, when $\frac{dx_{L\to H}}{dp} < 0$, showing that inequality (28) holds is more involved. We show the following upper bound on the left-hand side of the inequality:

$$\gamma(x_{L\to H} - (1-p)\frac{dx_{L\to H}}{dp}) \leq 2(1-p)\gamma\left(1 + 2p - p^2 - (1-p^2)\gamma\right).$$

To prove that this upper bound holds when $\frac{dx_{L\to H}}{dp} < 0$, note that we can write $D = \frac{q(W)}{A} + \frac{q(D)}{2-A}$. Using that $q(W) \leq 1$ and $q(D) \leq p$ we get $D \leq \frac{1}{A} + \frac{p}{2-A}$.

Second, $\frac{\partial A}{\partial p} \leq 2(1-p)(1-\gamma)$ and since $\frac{\partial A}{\partial x} \geq 0$ and $A \leq 1$, for any (p,γ) such that $\frac{dx_{L\to H}}{dp} < 0$ we have $\frac{dx_{L\to H}}{dp} \geq 2(1-p)\gamma - 2p(1 + (1-p)(1-\gamma))$

Also, we know that $x_{L\to H}(\gamma, p)$ is decreasing in γ, then $x \leq x(0,p) = \frac{2(1-p)}{1+(1-p)^2} \leq 2(1-p)$. Combining all these results we obtain the desired upper bound.



(a) $\mathcal{S}(p, \{c_L, c_H\}, H \to H)$ **(b)** $\mathcal{S}(p, \{c_L, c_H\}, L \to L)$

Fig. 7. In the shaded region, the challenger prefers to face the rivals sequentially.

To conclude the proof, we show that the upper bound is less than one by showing that

$$\max_{p\in[0,1]} \max_{\gamma\in[0,1]} 2(1-p)\gamma \left(1+2p-p^2-(1-p^2)\gamma\right) \leq 1,$$

The solution of the inner maximization problem is $\gamma^*(p) = \max\left\{0, \min\left\{1, \frac{1+2p-p^2}{2(1-p^2)}\right\}\right\}$.

First, if $\gamma^*(p)$ is a corner solution, then the outer maximization is over a function that is less than one for any p. When $\gamma^*(p)=1$, the objective function of the outer maximization is $2(1-p)\left(1+2p-p^2-(1-p^2)\right) = 4(1-p)p \leq 1$. If p is such that $\gamma^*(p)=0$, then the objective function is zero.

Second, if $\gamma^*(p) \in (0,1)$, then the objective function becomes $\frac{(1+2p-p^2)^2}{2(1+p)}$. Notice that in this case $\gamma^*(p)$ is strictly increasing when $p\in[0,1]$, and takes the value $1/2$ when $p=0$, and 1 for some $\bar{p}<1$. This means that the interior solution will be achieved only for $p \in (0,\bar{p})$.

The objective function of the outer maximization problem is increasing in p iff $\gamma^*(p) \leq 1$. Then, conditional on $p\in[0,\bar{p}]$, the objective function is maximized for $p^* = \bar{p}$, but in that case we get $4(1-\bar{p})\bar{p}$ which is always less than 1.

Therefore, inequality (28) holds, implying that $\pi^L_{L\to H}$ is decreasing in p.

Case 2: Incumbents of Increasing Strength ($H \to L$)

First, from the proof of Proposition 2, Eq. (16), in equilibrium, $x = q(W) - q(D)$ satisfies

$$\begin{aligned} 0 =& 2\gamma p^2(1-xG) + x\left(A(1-xG)-B\right)\left((1-xG)(2-A)+B\right) \\ &- 2p((1-xG)(2-A)+B)(1-xG), \end{aligned}$$

with $A = 1+2p-p^2$, $B=\gamma p(2-p)$, and $G = 1-\gamma$. Moreover, x is the only solution for that equation in $[0,1]$. Such root can be computed explicitly in terms of γ and p.

We now proceed in two steps, showing first that $\pi^L_{H\to L}$ decreases with p (case 2.1) and then showing that $\pi^H_{H\to L}$ increases with p (case 2.2).

Case 2.1: We first show that $\pi^L_{H\to L} = \left(2-q_{H\to L}(W)\right)(1-\gamma)$ is decreasing in p, which is the same as showing that $q_{H\to L}(W)$ is increasing in p.

From Bayes' rule (Eq. (2)), we have:

$$q(W) = \frac{\left(1-\frac{\tilde{c}_L p}{2c_H}\right)p}{\left(1-\frac{\tilde{c}_L p}{2c_H}\right)p + \left(\frac{1+p}{2} - \frac{\tilde{c}_L p}{c_H}\right)(1-p)} \quad \text{and}$$

$$q(D) = \frac{\left(\frac{\tilde{c}_L p}{2c_H}\right)p}{\left(\frac{\tilde{c}_L p}{2c_H}\right)p + \left(\frac{1-p}{2} + \frac{\tilde{c}_L p}{c_H}\right)(1-p)}.$$

Using the definition of the virtual costs in (4), we define $m = \frac{\tilde{c}_L}{c_H} = \frac{\gamma}{1-xG}$ and note that $m \in [\gamma, 1]$. We can rewrite the equations above as:

$$q(W) = \frac{1}{2}\cdot\frac{(2-mp)\,p}{A} \quad \text{and} \quad q(D) = \frac{1}{2}\cdot\frac{mp^2}{1-A}.$$

where $A = \frac{1}{2}[1+2p-p^2-p(2-p)m]$. Taking *partial derivatives* with respect to p we obtain:

$$\frac{\partial q(W)}{\partial p} = \frac{1-mp}{A} - (1-p)(1-m)\frac{(1-0.5\,mp)p}{A^2},$$
$$\frac{\partial q(D)}{\partial p} = \frac{mp}{1-A} + (1-p)(1-m)\frac{0.5\,mp^2}{(1-A)^2}.$$

Now, taking *partial derivatives* with respect to x we obtain:

$$\frac{\partial q(W)}{\partial x} = \left[-\frac{0.5p^2}{A} + \frac{(1-0.5mp)p}{A^2}\frac{(2-p)p}{2}\right]\frac{mG}{1-xG},$$
$$\frac{\partial q(L)}{\partial x} = \left[\frac{0.5p^2}{1-A} - \frac{0.5mp^2}{(1-A)^2}\frac{(2-p)p}{2}\right]\frac{mG}{1-xG}.$$

Using that $q(W) > p$, $p\in[0,1]$, and $m\in[\gamma,1]$ we get:

$$\frac{\partial q(W)}{\partial p} = \frac{1}{A}\left[(1-mp)-(1-p)(1-m)q(W)\right] > 0$$
$$\frac{\partial q(D)}{\partial x} = \left[-p+p_b(W)(2-p)\right]\frac{p}{2A}\frac{mG}{1-xG} > 0$$

Taking the total derivative with respect to p, we get

$$\frac{dq(R)}{dp} = \frac{\partial q(R)}{\partial x}\frac{dx}{dp} + \frac{\partial q(R)}{\partial p},$$

for $R\in\{L,D\}$. This implies that $\frac{dq(W)}{dp} > 0$ if and only if $\frac{\partial q(W)}{\partial p} + \frac{\partial q(W)}{\partial x}\frac{dx}{dp} > 0$. Since the two partial derivatives of $q(W)$ with respect to p and x are positive, the inequality will hold whenever $dx/dp \geq 0$. Thus, we are only concerned that the condition could be violated when $dx/dp < 0$. Using that $x = q(W)-q(D)$ we have

$$\frac{dx}{dp} = \frac{\frac{\partial q(W)}{\partial p} - \frac{\partial q(D)}{\partial p}}{1 - \frac{\partial q(W)}{\partial x} + \frac{\partial q(D)}{\partial x}}$$

It can be shown that the denominator is strictly positive. Therefore, $\frac{dq(W)}{dp} > 0$ holds whenever

$$\frac{\partial q(W)}{\partial p}\left(1-\frac{\partial q(W)}{\partial x}+\frac{\partial q(D)}{\partial x}\right) + \frac{\partial q(W)}{\partial x}\left(\frac{\partial q(W)}{\partial p}-\frac{\partial q(D)}{\partial p}\right) > 0,$$

which is equivalent to show that

$$Q = \frac{\partial q(W)}{\partial p} + \frac{\partial q(W)}{\partial p}\frac{\partial q(D)}{\partial x} - \frac{\partial q(W)}{\partial x}\frac{\partial q(D)}{\partial p} > 0$$

Using the expressions for the partial derivatives we get

$$Q = (1-A)(1-mp-(1-p)(1-m)q(W)) + \frac{m^2Gp}{2\gamma}\left[p - q(L)(2-p) - xp(1-p) - xmp\right].$$

Next, using the definition of $q(W)$, $q(L)$ and the relationship $x = \frac{m-\gamma}{mG}$, we get

$$Q = (1-A)\left(1-mp-(1-p)(1-m)\frac{(2-mp)\,p}{2A}\right) + \frac{m^2Gp}{2\gamma}\left[p - \frac{mp^2}{2(1-A)}(2-p) - \frac{(m-\gamma)p}{mG}(1-p+m)\right]$$

This term is positive using the exact formula for x as a function of p and γ, obtained as the only root in $[0,1]$ of a polynomial of degree 3.

Case 2.2. We want to show that $\pi^H_{H\to L} = p\left(1 - \frac{\gamma x_{H\to L}}{(1-x_{H\to L}(1-\gamma))}\right)(1-\gamma)$ is increasing in p. Following the notation of Case 2.1., we can write $\pi^H_{H\to L} = p(G - mxG)$.

Note that $m = \frac{\gamma}{1-xG} \Rightarrow 1 - \frac{\gamma}{m} = xG$, so $p(G-mxG) = p(G-m+\gamma) = p(1-m)$.

Thus, we want to show that $p(1-m)$ is increasing in p. First, when m decreases in p this is true. Therefore, it remains to show that $1-m-pm' > 0$ whenever $m' > 0$. Using the definition of m, the inequality is equivalent to

$$(1-Gx)^2 - \gamma(1-Gx) + p\gamma G\frac{dx}{dp} > 0.$$

Using the previous part, we can substitute for $\frac{dx}{dp}$, and since $1 - \frac{\partial q(W)}{\partial x} + \frac{\partial q(D)}{\partial x} > 0$, what we need to show is

$$(1-Gx)G(1-x)\left(1 - \frac{\partial q(W)}{\partial x} + \frac{\partial q(D)}{\partial x}\right) + p\gamma G\left(\frac{\partial q(W)}{\partial p} - \frac{\partial q(D)}{\partial p}\right) > 0.$$

Let us call

$$R = (1-Gx)G(1-x)\left(1 - \frac{\partial q(W)}{\partial x} + \frac{\partial q(D)}{\partial x}\right) + p\gamma G\left(\frac{\partial q(W)}{\partial p} - \frac{\partial q(D)}{\partial p}\right).$$

As before, using the definitions of $q(W)$, $q(D)$ and m, we can write R as a function of p, γ and x, and since x also has an exact formula, the quantity can be written only in terms of p and γ. The resulting expression is always positive for p, γ in $[0,1]$, so $p(1-m)$ is increasing in p. □

Proof of Proposition 6

Proof. First, we show that there is no separating equilibrium satisfying Definition 5. Suppose the challenger of type c_L chooses order $L \to H$ and the challenger of type c_H chooses order $H \to L$. Then, off-path beliefs are $\mu_{L\to H} = 1$ and $\mu_{H\to L} = 0$. But the payoff of the challenger of type c_L is (strictly) decreasing and continuous, and $\pi^L_{L\to H}(0) = \pi^L_{H\to L}(0)$ (Proposition 5). Therefore, $\pi^L_{L\to H}(1) < \pi^L_{L\to H}(0) = \pi^L_{H\to L}(0)$. This implies that type c_L would benefit from deviating to choosing the order $H \to L$. The same argument applies if the separation occurs by challenger's types choosing the opposite orders.

We next present an algorithmic construction of all the equilibria of the game specified in Definition 5. Our construction delivers regions in the parameter space (p,γ) where equilibrium is a pooling or a semi-pooling equilibria.

Let $\mathcal{P} \subset [0,1]$ such that for any $p \in \mathcal{P}$, $\hat{p}^L_{j\to k}(p) = \hat{p}^H_{j\to k}(p)$. To simplify the proof, we will assume that $\mathcal{P}$ is finite, although this is not required. $\mathcal{P}$ is non-empty because $\{0,1\} \subseteq \mathcal{P}$. We index the elements in $\mathcal{P}$ by $p_0 < p_1 < \cdots < p_{N+1} = 1$. Then,

$$[0,1] = \bigcup_{n=0}^{N}[p_n, p_{n+1}]$$

Fig. 8 illustrates the case of $\gamma = 0.6$, where $\mathcal{P} = \{p_0, p_1, p_2\}$, with $r_0 = p_0 = 0$, $p_1 = 0.59$, $r_1 = 0.65$, $r_2 = p_2 = 1$.

Consider the interval $I_n = (p_n, p_{n+1})$.

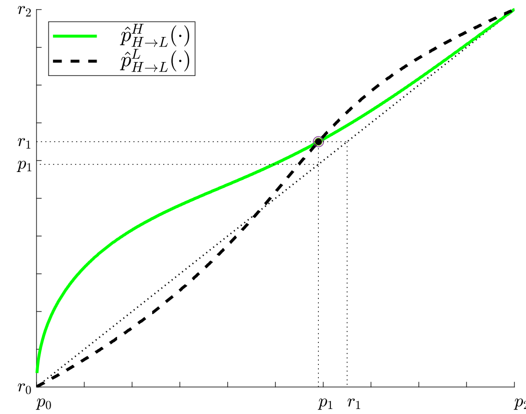


Fig. 8. Functions $\hat{p}^i_{H\to L}(p)$, for $\gamma = 0.6$.

1. If $\hat{p}^H_{j\to k}(p) \geq \hat{p}^L_{j\to k}(p)$ for all $p \in I_n$, then we can construct a pooling equilibrium where both types of the challenger choose order $j \to k$. Any off-path belief in $[\hat{p}^L_{j\to k}(p), \hat{p}^H_{j\to k}(p)]$ deters deviations.
2. If $\hat{p}^H_{j\to k}(p) < \hat{p}^L_{j\to k}(p)$ for all $p \in I_n$. Let $r_n = \hat{p}^L_{j\to k}(p_n) = \hat{p}^H_{j\to k}(p_n)$ and define the interval $R_n = (r_n, r_{n+1})$. We have two subcases:
   (a) For $r \in R_n$ we have $[\hat{p}^H_{j\to k}]^{-1}(r) > [\hat{p}^L_{j\to k}]^{-1}(r)$. If not, then there exists $p \in I_n$ such that $r = \hat{p}^H_{j\to k}(p)$, but then $\hat{p}^H_{j\to k}(p) > \hat{p}^L_{j\to k}(p)$, which is a contradiction.
   Using the relationship with the inverse we get $\hat{p}^H_{k\to j}(r) > \hat{p}^L_{k\to j}(r)$ for all $r \in R_n$. Then, for all $p \in R_n$ we can construct a pooling equilibrium where both types of the challenger choose order $k \to j$. Any off-path belief in $[\hat{p}^H_{k\to j}(r), \hat{p}^H_{k\to j}(r)]$ deters deviations.
   (b) For $p \notin R_n$, we also have $\hat{p}^H_{k\to j}(p) < \hat{p}^L_{k\to j}(p)$. Then, there is no pooling at any order for this p.
   Claim: There exists n such that p is "between" p_n and r_n. If $p \notin R_n$, there are two cases: $\min\{\hat{p}^L_{k\to j}(p_n), \hat{p}^L_{k\to j}(p_{n+1})\} > p$ or $\max\{\hat{p}^L_{k\to j}(p_n), \hat{p}^L_{k\to j}(p_{n+1})\} < p$ (if not, then $p \in R_n$). In the first case, $p \in [\hat{p}^L_{k\to j}(p_{n+1}), p_{n+1}]$ and in the second case $p \in [p_n, \hat{p}^L_{k\to j}(p_n)]$. Thus, such a p is always a convex combination of p_n and r_n, for some n. If that is the case, we can construct a partial pooling, mixing between p_n and r_n. □

Appendix C. Supplementary data

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.jmateco.2024.103080.

Data availability

No data was used for the research described in the article.

References

Amann, Erwin, Leininger, Wolfgang, 1996. Asymmetric all-pay auctions with incomplete information: the two-player case. Games Econom. Behav. 14 (1), 1–18.
Antsygina, Anastasia, 2022. Optimal allocation of multi-dimensional prizes in contests with heterogeneous agents. Available at SSRN 3929508.
Barbieri, Stefano, Serena, Marco, 2022. Repeated contests with toughness!
Baye, Michael R., Kovenock, Dan, De Vries, Casper G., 2005. Comparative analysis of litigation systems: An auction-theoretic approach. Econ. J. 115 (505), 583–601.
Beccuti, Juan, Möller, Marc, 2020. Fighting for Lemons: The Encouragement Effect in Dynamic Contests with Private Information. Technical Report, Discussion Papers.

Bernhardt, Dan, Lee, Frances Zhiyun X., 2015. Trial incentives in sequential litigation. Am. Law Econ. Rev. 17 (1), 214–244.
Beviá, Carmen, Corchón, Luis C., 2013. Endogenous strength in conflicts. Int. J. Ind. Organ. 31 (3), 297–306.
Chen, Bo, Chen, Bo, 2024. Optimal disclosure in all-pay auctions with interdependent valuations. Games Econom. Behav. 143, 204–222.
Chen, Bo, Serena, Marco, 2023. Disclosure policies in all-pay auctions with bid caps. J. Econ. Behav. Organ. 209, 141–160.
Corchón, Luis C, Yıldızparlak, Anıl, 2013. Give peace a chance: the effect of ownership and asymmetric information on peace. J. Econ. Behav. Organ. 92, 116–126.
Denter, Philipp, Morgan, John, Sisak, Dana, 2021. Showing off or laying low? The economics of psych-outs. Am. Econ. J. Microecon.
Farmer, Amy, Pecorino, Paul, 2013. Discovery and disclosure with asymmetric information and endogenous expenditure at trial. J. Leg. Stud. 42 (1), 223–247.
Fu, Qiang, Gürtler, Oliver, Münster, Johannes, 2013. Communication and commitment in contests. J. Econ. Behav. Organ. 95, 1–19.
Groh, Christian, Moldovanu, Benny, Sela, Aner, Sunde, Uwe, 2012. Optimal seedings in elimination tournaments. Econom. Theory 49 (1), 59–80.
Hillman, Arye L., Riley, John G., 1989. Politically contestable rents and transfers. Econ. Politics 1 (1), 17–39.
Ifergane, Tomer, Sela, Aner, et al., 2017. Signaling in Contests. Technical Report.
Konrad, Kai A., Kovenock, Dan, 2009. Multi-battle contests. Games Econom. Behav. 66 (1), 256–274.
Kräkel, Matthias, 2014. Optimal seedings in elimination tournaments revisited. Econ. Theory Bull. 2 (1), 77–91.
Krumer, Alex, Megidish, Reut, Sela, Aner, 2017. Round-robin tournaments with a dominant player. Scand. J. Econ. 119 (4), 1167–1200.
Kubitz, Greg, 2023. Two-stage contests with private information. Am. Econ. J. Microecon. 15 (1), 239–287.
Lemus, Jorge, Temnyalov, Emil, 2023. Diversification and information in contests. Econom. Theory 1–32.
Moldovanu, Benny, Sela, Aner, 2001. The optimal allocation of prizes in contests. Amer. Econ. Rev. 91 (3), 542–558.
Münster, Johannes, 2009. Repeated contests with asymmetric information. J. Public Econ. Theory 11 (1), 89–118.
Siegel, Ron, 2014. Asymmetric all-pay auctions with interdependent valuations. J. Econom. Theory 153, 684–702.
Spence, Michael, 1973. Job market signaling. Q. J. Econ. 87 (3), 355–374.