

An ascending auction with multi-dimensional signals [☆]

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Abstract

This paper examines a single-unit ascending auction where agents observe two-dimensional Gaussian signals. The model combines the pure private-values model with the pure common-values model. The challenge is to characterize how the multi-dimensional signals observed by an agent are aggregated onto that agent's one-dimensional bid. The challenge is solved by projecting an agent's private signals onto a one-dimensional *equilibrium statistic*; the equilibrium bidding strategies are constructed as if each agent observed only his own equilibrium statistic. An agent's equilibrium statistic aggregates this agent's private signals while taking into account the additional information deduced from the other agents' bids. In contrast to one-dimensional environments, an ascending auction may have multiple symmetric equilibria that yield different social surpluses.

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1. Introduction

Motivation. Auctions have been extensively studied in economics. It is an empirically relevant and a theoretically rich literature: auctions are commonly used to allocate goods across agents, and there is a rich class of models that allows the study of bidding in auctions. A critical assumption in most auction models is that agents observe one-dimensional signals. In this paper, the main objective is to characterize the equilibrium of an auction in which agents observe multi-dimensional signals. As a by-product, we provide predictions of auctions that arise only when agents observe multi-dimensional signals.

Our paper is motivated by the observation that, in many environments, agents' information is naturally a multi-dimensional object. Consider, for example, the auction of an oil field, and suppose that an agent's valuation of the oil field is determined by its size and by the agent-specific cost of extracting oil. Furthermore, assume that each agent privately observes her own cost and all agents observe conditionally independent noisy signals about the oil field's size. This environment is therefore one in which agents observe two-dimensional signals. In most auction environments, agents similarly observe multi-dimensional signals about their valuation of the good (e.g., timber, highway construction procurements, art, real estate).¹

There is an important conceptual difference between bidding in an auction with one-dimensional signals and in one with multi-dimensional signals. If agents observe one-dimensional signals, then observing agent n 's bid is informationally equivalent to observing agent n 's signal. Yet, in environments with multi-dimensional signals, observing agent n 's bid is *not* informationally equivalent to observing all the signals observed by agent n . It follows that a bid reflects only an aggregate statistic of an agent's signals. In the oil field example, agent m cannot tell whether agent n 's low bid is due to a high cost of extracting oil or to agent n 's belief that the oil reservoir is small. The extent to which agent n 's bidding is driven by his private costs or his beliefs about the oil reservoir's size is critical for agent m to determine her own bidding strategy.² After all, agent m 's valuation of the oil field is independent of agent n 's costs but is not independent of agent n 's signal about the size of the oil reservoir. This distinction leads to a crucial difference in the equilibrium bidding. The conceptual challenge is to account for the feedback between agents' bids: the way agent n 's signals are aggregated onto his bid depends on how agent m 's signals are aggregated onto her bid.

Model. Our model consists of N agents bidding for an indivisible good in an ascending auction. The utility of an agent who wins the object is determined by a common shock and an idiosyncratic shock. Each agent privately observes his own idiosyncratic shock and, additionally, each agent observes a (noisy) signal about the common shock. The signals and the payoff shocks are normally distributed. We focus on symmetric environments and symmetric equilibria.

The two-dimensional signals contain elements of a pure-common values environment and a pure-private values environment; our only departure from classic models in the auction literature

¹ In timber auctions, agents may differ in their harvesting cost and in their estimate of harvest quality (Haile (2001) or Athey and Levin (2001)). In highway construction procurements, the winning bidder faces idiosyncratic cost shocks and common cost shocks (Somaini (2011) or Hong and Shum (2002)). In art and real estate auctions, an agent privately observes his own taste shock and a noisy signal of an unknown common shock, where the latter can represent the good's quality or its future resale value.

² To facilitate the exposition, we shall often refer to agents n and m via (respectively) masculine and feminine pronouns.

is the information structure's multi-dimensionality.³ This allows us to identify those elements of bidding strategies that arise only due to the multi-dimensional signals.

The focus on an ascending auction and on Gaussian signals is useful in fully characterizing a class of equilibria. The ascending auction is a frequently used auction format and the assumption of Gaussian signals has been used in the empirical auction literature (see, for example, Hong and Shum (2002)). Therefore, this is a natural model for the study of auctions with multi-dimensional signals.

Equilibrium construction. This paper's main result is to construct a class of equilibria in the ascending auction. In the class of equilibria we construct the drop-out time of an agent is determined by a linear combination of the signals he observes; we refer to this linear combination of signals as an *equilibrium statistic*. The equilibrium bidding strategies are the same as if each agent observed only his own equilibrium statistic. An agent's equilibrium statistic is a sufficient statistic — for this agent's two private signals— to compute his valuation at the end of the auction (i.e. after all drop-out times are observed); however, an agent's equilibrium statistic does not satisfy this sufficiency property before the auction ends.

The equilibrium characterization is tractable because the drop-out time of an agent is determined by a linear combination of the signals he observes: the equilibrium statistic. The linearity arises because expectations with Gaussian signals are linear. The ascending auction allows us to keep the Bayesian updating within the Gaussian family when we evaluate the equilibrium conditions. In the equilibria we characterize, an agent's drop-out time remains optimal even after observing the drop-out time of all other agents.⁴ Consequently, we evaluate the best response conditions using the *realized* drop-out time of each agent (and not a lower bound for those times). This property of the equilibria in an ascending auction, in conjunction with the assumed Gaussian nature of signals, renders the problem tractable. For example, a first-price auction with Gaussian signals does not preserve the same tractability because in such an auction it is not possible to evaluate an agent's best-response conditions using the realized bids of other agents.

We show that, in contrast to one-dimensional environments, multi-dimensional environments may feature multiple symmetric equilibria of an ascending auction.⁵ The various equilibria yield distinct levels of revenue as well as a different social surplus.⁶ The multiplicity of equilibria is due to a strategic complementarity in the weight that agents place—in their bidding strategy—on their own idiosyncratic shock. This complementarity arises because signals must be aggregated onto an agent's bid.

Literature review. There is an extensive literature on auctions with one-dimensional signals. Much of this research is based on the seminal contribution of Milgrom and Weber (1982), who eloquently describe the assumption as follows:

“To represent a bidder's information by a single real-valued signal is to make two substantive assumptions. Not only must his signal be a sufficient statistic for all of the information he

³ If agents observed only their idiosyncratic shock then this would be a classic pure-private values environment; if agents observed only the signal of the common shock, this would be a classic pure-common values environment.

⁴ Formally, the set of equilibria we characterize constitute a posterior equilibrium. This is stronger notion of equilibrium is due to Green and Laffont (1987).

⁵ Bikhchandani et al. (2002) show that there is a continuum of symmetric equilibria. Nevertheless, both the allocation and the equilibrium price are the same across equilibria. See Krishna (2009) for a textbook discussion.

⁶ By “revenue” we mean ex ante expected revenue, and by “social surplus” we mean ex ante expected surplus.

possesses concerning the value of the object to him, it must also adequately summarize his information concerning the signals received by the other bidders.” (p. 1097)

These remarks offer a clear explanation of what is entailed by assuming one-dimensional signals. They also illustrate the difficulty of characterizing an equilibrium when agents observe multi-dimensional signals: in general, an agent’s bid is not determined simply by her interim expected valuation.⁷

The literature on auctions with multi-dimensional signals has made progress in two ways. The first way is to make the appropriate assumptions about the distribution of signals such that an agent’s bid is determined only by her interim expected valuation. The second way is to provide properties of an auction without having to characterize the equilibrium bids. The distinguishing features of our paper are that we do not impose any assumptions on how signals are correlated across agents and we fully characterize a class of equilibria. This allows us to study how signals are aggregated onto an agent’s bid while taking into account the feedback effects between bids of different agents, which ultimately delivers new predictions about ascending auctions. We now discuss the literature on auctions with multi-dimensional signals and interdependent valuations.⁸

Wilson (1998) studies an ascending auction with two-dimensional signals. Wilson (1998) assumes that the random variables are log-normally distributed and also drawn from a diffuse prior. Because of its tractability (which our model shares to a great extent), the model studied by Wilson (1998) is often used in empirical work.⁹ In Section 4.4, we explain how the model in Wilson (1998) can be obtained as a particular limit of our model.

Dasgupta and Maskin (2000) studies a generalized VCG mechanism. They show that, if agents’ signals are independently distributed across agents, then an agent’s interim expected valuation delivers a one-dimensional statistic that can be used to characterize the mechanism’s Nash equilibria.¹⁰ The intuition is that an agent has no information about the signals observed by other agents and so the impact of the signals on the agent’s payoff is a sufficient statistic of all the information this agent observes. Provided the signals are independently distributed, this approach can also be used in many other mechanisms—including a first-price auction and an ascending auction (see Goeree and Offerman (2003) or Levin et al. (2007)). We discuss the assumption of independent signals after we provide the information structure in our model (in Footnote 12).

Jackson (2009) provides an example of an ascending auction for which no equilibrium exists. The model he describes is similar to ours in that it features both a private and a common signal; however, the distribution of signals and payoff shocks in that model have finite support (so they are non-Gaussian). Jackson (2009) establishes that the existence of an equilibrium is not guaranteed in an auction model with multi-dimensional signals. The extent to which it is possible to construct equilibria with multi-dimensional non-Gaussian information structures is still an open question.

⁷ An agent’s interim expected valuation is his expected valuation conditional only on his private signals.

⁸ There is a literature that studies multi-dimensional signals in private values environments (see, for example, Fang and Morris (2006) or Jackson and Swinkels (2005)). These works are primarily based on first-price auctions, and they seek to explain how multi-dimensional signals change the “bid shading” in such auctions. Multi-dimensional signals play a different role in this literature than in our model. In fact, an ascending auction has an equilibrium in dominant strategies when agents have private values.

⁹ See Hong and Shum (2003) for further discussion on the use of normal distributions in empirical analysis.

¹⁰ McLean and Postlewaite (2004) studies an interesting variation of the VCG mechanism for environments in which agents observe multi-dimensional signals that are *not* independently distributed.

Pesendorfer and Swinkels (2000) study a sealed-bid uniform price auction in which there are k goods for sale, each agent has unit demand, and each agent observes two-dimensional signals. They study the limit in which the number of agents grows to infinity. Pesendorfer and Swinkels (2000) are able to provide asymptotic properties of any equilibrium without the need to characterize such an equilibrium or even prove its existence.

Finally, there is a literature that studies mechanism design with multi-dimensional signals. Jehiel and Moldovanu (2001) show that when agents observe multi-dimensional independently distributed signals there is no efficient mechanism. The impossibility result in Jehiel and Moldovanu (2001) does not apply to our model because in our model signals are not independently distributed (this is precisely what makes the construction of an equilibrium challenging); however, the Nash equilibrium in our model is still inefficient. Jehiel et al. (2006) show that when agents observe multi-dimensional signals then it is not possible to construct a mechanism that has an ex post equilibrium. The equilibria of the ascending auction when agents observe multi-dimensional signals is not an ex post equilibrium.

The rest of our paper proceeds as follows. Section 2 describes the model and Section 3 studies one-dimensional signals. Section 4 characterizes the equilibrium with two-dimensional signals. Section 5 illustrates the presence of multiple equilibria and Section 6 examines the effect of a public signal. We conclude in Section 7 with a discussion about our paper's main assumptions and possible extensions. Appendix I gives the proofs that are not included in the main text.

2. Model

2.1. Payoffs and information

We study N agents bidding for an indivisible good in an ascending auction. The utility of an agent $n \in N$ who wins the object at price p is given by:

$$u(i_n + c) - p, \quad (1)$$

where $u(\cdot)$ is a strictly increasing function, $i_n \in \mathbb{R}$ is an idiosyncratic shock, and c is a common shock. The utility of an agent who does *not* win the good is zero. To make the notation more compact, we define

$$v_n \triangleq i_n + c. \quad (2)$$

The payoff shock v_n summarizes the valuation of agent n .

The idiosyncratic shocks and the common shock are jointly normally distributed with mean 0 and with respective variance σ_i^2 and σ_c^2 . Assuming that the idiosyncratic and common shock(s) have zero mean reduces the notation but plays no role in the analysis. The idiosyncratic shocks have correlation $\rho_i \in (-1/(N-1), 1)$ across agents and are distributed independently of the common shock.¹¹

Agent n observes two signals, of which the first is a perfectly informative signal about her own idiosyncratic shock i_n . The second is a noisy signal about the common shock:

$$s_n \triangleq c + \varepsilon_n; \quad (3)$$

¹¹ The minimum statistically feasible correlation is $-1/(N-1)$. Hence we impose no restrictions on the set of feasible correlations beyond the requirement that it is an interior correlation.

here ε_n is a noise term that is independent across agents, independent of all other random variables in the model, and normally distributed with variance σ_ε^2 . The private information of agent n is summarized by the pair of random variables (i_n, s_n) .¹² If every agent n observed only signal i_n , then this would be a pure-private values model; if each agent n observed only signal s_n , it would be a pure-common values model.

Interpretation of the utility function. We now interpret the agents' utility function in two applications. Observe that when $u(\cdot) = \exp(\cdot)$ agent n 's utility can be written as the product of two log-normal random variables (i.e., $u(i_n + c) = \exp(i_n) \cdot \exp(c)$).

For the oil field example, $\exp(c)$ can be interpreted as the oil field's size and $\exp(i_n)$ as the technology of firm n . The total amount of oil that can be extracted from an oil reserve of size $\exp(c)$ by firm n using technology $\exp(i_n)$ is equal to $\exp(i_n) \cdot \exp(c)$. Li et al. (2000) use log-additive payoffs (as in (1)) to study Outer Continental Shelf wildcat auctions.

If we multiply the utility function by -1 then the model can be interpreted as the "procurement" of a project, where $\exp(i_n) \cdot \exp(c)$ is the cost of delivering that project. Here $\exp(i_n)$ can be interpreted as the total amount of inputs that bidder n needs to complete the project and $\exp(c)$ as a price index of those inputs. Hong and Shum (2002) use log-additive payoffs (as in (1)) to study procurement auctions held by the state of New Jersey.

2.2. Ascending auction

We study an ascending auction.¹³ An auctioneer raises the price continuously. At each moment in time, an agent can drop out of the auction; in that event, the agent neither pays anything nor obtains the good. The object is won by the last agent to drop out of the auction,¹⁴ who pays the price at which the second-to-last agent dropped out of the auction.¹⁵ Because each drop-out time is associated with a unique price, we shall use the terms "price" and "drop-out time" interchangeably.

The outcome of the ascending auction is described by the order in which each agent drops out and the respective prices at which they do so. The number of agents left in the auction when agent n drops out is denoted by a permutation π .¹⁶ For example, the identity of the last agent to drop out of the auction is given by $\pi^{-1}(1)$. The price at which agents drop out is denoted $p_1 > \dots > p_N$. So for any strategy profile, the expected utility of agent n is

$$\mathbb{E}[\mathbb{1}\{\pi^{-1}(1) = n\}(u(v_n) - p_2)],$$

where $\mathbb{1}\{\cdot\}$ is the indicator function. We study the auction's symmetric Nash equilibria.

¹² A model of independently distributed signals (as in Dasgupta and Maskin (2000) or Goeree and Offerman (2003)) is recovered by assuming that $\rho_i = 0$ and assuming the noise terms $\{\varepsilon_n\}_{n \in N}$ are negatively correlated across agents. In particular, the variance of the noise terms must be large enough and sufficiently negatively correlated so that $\text{cov}(s_n, s_m) = \sigma_v^2 + \sigma_\varepsilon^2 \cdot \text{corr}(\varepsilon_m, \varepsilon_n) = 0$.

¹³ We follow Krishna (2009) in the formal description of the ascending auction.

¹⁴ We assume that the auction continues until all agents have dropped out. The price at which the last agent drops out is clearly payoff irrelevant because he pays only the price at which the second-to-last agent dropped out. This fact allows us to simplify the notation in some parts of the paper because there is always one drop-out time for each agent.

¹⁵ In case of a tie, the good is sold with equal probability to one of the agents who was the last to drop out. In equilibrium, there will be no ties and an agent cannot tie with another agent by unilaterally changing his strategy. Hence, as is standard in an ascending auction, the tie-breaking rule does not matter.

¹⁶ A permutation is a bijective function $\pi : N \rightarrow N$.

A strategy of agent n is a set of functions $\{P_k\}_{k \in N}$ with

$$P_k : \mathbb{R}^2 \times \mathbb{R}^{N-k} \rightarrow \mathbb{R}_+. \quad (4)$$

The function $P_k(i, s_n, p_{k+1}, \dots, p_N)$ is the drop-out time of agent n when k agents are left in the auction and the observed drop-out times are $p_N < \dots < p_{k+1}$. The function $P_k(i, s_n, p_{k+1}, \dots, p_N)$ must satisfy the following inequality:

$$P_k(i, s_n, p_{k+1}, \dots, p_N) \geq p_{k+1}.$$

That is, agent n cannot drop out of the auction at a price *lower* than the price at which another agent has already dropped out. Note that we restrict attention to symmetric equilibria in symmetric environments. It is therefore sufficient to specify the price at which an agent dropped of the auction yet the agent's identity is irrelevant.

3. Benchmark: one-dimensional signals

We first study one-dimensional signals; the analysis of this case will be helpful for understanding the analysis of two-dimensional environments. All results in this section are direct corollaries or simple extensions of results that are already well known in the literature.

3.1. Information structure

We assume agent n observes a one-dimensional signal

$$s'_n = i_n + b \cdot (c + \varepsilon_n), \quad (5)$$

where $b \in \mathbb{R}_+$ is an exogenous parameter. In other words, agent n observes only a linear combination of the two-dimensional signal (i_n, s_n) . The one-dimensional signal (5) provides a parameterized class of information structures that include pure private values and pure common values as limits. If $b = 0$, then the model is that of a pure-private values auction; if $b \rightarrow \infty$, then we are modeling a pure-common values auction.

The specific form of the signal in (5) illuminates the connections to the model in which an agent observes both signals separately. This class of one-dimensional signals is, in essence, a special case of the model studied by Milgrom and Weber (1982).¹⁷ Although we believe that (5) specifies a natural class of one-dimensional information structures, we are not aware of any other paper that examines this class of signals *except* for the case $b = 1$.¹⁸

¹⁷ More precisely: if $b \leq 1$ then this environment is a particular case of the model studied by Milgrom and Weber (1982), but if $b > 1$ then this environment may fail to satisfy all the assumptions in that paper; even so, their analysis goes through without important changes. For example, an information structure with $b > 1$ and $\sigma_\varepsilon^2 = 0$ would not satisfy the monotonicity assumption in Milgrom and Weber (1982) because, in that case, agent n 's utility would be decreasing in the realization of agent m 's signal. Yet, the failure of this monotonicity condition is sufficiently "mild" that the analysis in Milgrom and Weber (1982) goes through unchanged.

¹⁸ Hong and Shum (2002) study a model where the payoff environment is as in (1) and where agents observe one-dimensional signals—as in (5) with $b = 1$ (see also Hong and Shum (2003)). The model studied by Wilson (1998) also reduces to a one-dimensional signal as in (5) with $b = 1$ (we discuss this in more detail later).

3.2. Characterization of equilibrium with one-dimensional signals

To characterize the equilibrium we relabel agents such that the realization of signals satisfy

$$s'_1 > \dots > s'_N.$$

Since signals are noisy, it follows that the order over valuations may not be preserved. For example, it may be that $v_{n+1} > v_n$ even though, by construction, $s'_{n+1} \leq s'_n$.

If the signals $(s'_1, \dots, s'_{n-1})$ are equal to s'_n (i.e., all signals higher than s'_n are equal to s'_n), then we write the expectation of v_n as

$$\mathbb{E}[v_n \mid s'_n, \dots, s'_n, s'_{n+1}, \dots, s'_N]. \quad (6)$$

For example, if $N = 3$ then $\mathbb{E}[v_2 \mid s'_2, s'_2, s'_3]$ denotes the expected valuation of the agent with the second-highest signal—conditional on (i) the realization of her own signal, (ii) the realization of agent 3's signal, and (iii) the realization of agent 1's signal being equal to s'_2 .

Proposition 1 (*Equilibrium of ascending auction*). *The ascending auction with one-dimensional signals as in (5) has a Nash equilibrium in which agent n 's drop-out time is given by*

$$p_n = \mathbb{E}[u(v_n) \mid s'_n, \dots, s'_n, s'_{n+1}, \dots, s'_N]. \quad (7)$$

In equilibrium, agent 1 wins the good and pays $p_2 = \mathbb{E}[u(v_2) \mid s'_2, s'_2, s'_3, \dots, s'_N]$.

Proposition 1 provides the classic equilibrium characterization found in Milgrom and Weber (1982), which is essentially the unique symmetric equilibrium (see Note 5). In equilibrium, the agent with the n th-highest signal drops out of the auction at her expected valuation conditional on the signals observed by agents who have already dropped out (i.e., agents $m > n$) and assuming that the $n - 1$ signals that are higher than s'_n are all equal to s'_n .

The equilibrium strategies (see (7)) satisfy the following two conditions: (i) agent 1 does not regret winning the good at price p_2 , and (ii) no agent $m > 1$ regrets dropping out of the auction instead of waiting until agent 1 drops out. These conditions are formally expressed as follows¹⁹:

$$\mathbb{E}[u(v_1) \mid s'_1, \dots, s'_N] - \mathbb{E}[u(v_2) \mid s'_2, s'_2, \dots, s'_N] \geq 0; \quad (8)$$

$$\forall m > 1, \quad \mathbb{E}[u(v_m) \mid s'_1, \dots, s'_N] - \mathbb{E}[u(v_1) \mid s'_1, s'_1, s'_2, \dots, s'_{m-1}, s'_{m+1}, \dots, s'_N] \leq 0. \quad (9)$$

Condition (8) states that the expected valuation of agent 1 conditional on all the signals is greater than the price at which agent 2 drops out of the auction. Therefore, agent 1 does not regret winning the good. Condition (9) states that the expected valuation of agent m conditional on all the signals is less than the price at which agent 1 would drop out of the auction if agent m waited until agent 1 dropped out. Hence agent $m > 1$ does not regret dropping out of the auction even if she observed the realization of all the signals. These considerations lead to an important property: the strategy profile (see (7)) remains a Nash equilibrium even if every agent observes the signals of all other agents.

¹⁹ If agent $m > 1$ waited until all other agents dropped out of the auction, then she would win the good at price $\tilde{p}_2 = \mathbb{E}[u(v_1) \mid s'_1, s'_1, s'_2, \dots, s'_{m-1}, s'_{m+1}, \dots, s'_N]$. This is the expected valuation of agent 1 conditional on the signals of all agents other than agent m and assuming that agent m observed a signal equal to that observed by agent 1.

4. Equilibria with multi-dimensional signals

We now characterize a class of equilibria when agents observe two-dimensional signals (i_n, s_n) . The first step of the equilibrium characterization is to project the signals onto a one-dimensional object that we call an equilibrium statistic. Then we show that there exists a class of equilibria where each agent behaves as if she observes only her own equilibrium statistic. We then show that our equilibrium construction characterizes all equilibria in linear strategies.

4.1. Equilibrium statistic

The fundamental object that enables us to characterize an equilibrium is the *equilibrium statistic*, which amounts to the projection of signals that determine agents' respective drop-out times.

Definition 1 (*Equilibrium statistic*). The random variables $\{t_n\}_{n \in N}$ constitute an equilibrium statistic if there exists a $\beta \in \mathbb{R}$ such that, for all $n \in N$:

$$t_n = i_n + \beta \cdot s_n; \quad (10)$$

$$\mathbb{E}[v_n \mid i_n, s_n, t_1, \dots, t_N] = \mathbb{E}[v_n \mid t_1, \dots, t_N]. \quad (11)$$

An equilibrium statistic is a linear combination of signals that satisfy the statistical condition (11). The expected value of v_n conditional on all equilibrium statistics $\{t_n\}_{n \in N}$ is equal to the expected value of v_n conditional on (i) all equilibrium statistics $\{t_n\}_{n \in N}$ and (ii) both signals (i_n, s_n) . In other words, if agent n knows the equilibrium statistic of other agents, then her equilibrium statistic is a sufficient statistic—for both signals observed by agent n —to compute the expectation of v_n . Note that the weight β is the same for all agents. The reason is that we focus on symmetric equilibria, and hence, all agents use the same weight. Throughout the paper, t_n will denote an equilibrium statistic.

Our next proposition characterizes the set of equilibrium statistics.

Proposition 2 (*Equilibrium statistic*). A linear combination of signals $t_n = i_n + \beta \cdot s_n$ is an equilibrium statistic if and only if β is a root of the cubic polynomial $x_3 \cdot \beta^3 + x_2 \cdot \beta^2 + x_1 \cdot \beta + x_0$, where:

$$\begin{aligned} x_3 &= \frac{1}{(1 - \rho_i)(1 + (N - 1)\rho_i)} \frac{(\sigma_\varepsilon^2 + N \cdot \sigma_c^2)}{\sigma_i^2 \sigma_c^2}; \quad x_2 = \frac{-1}{(1 - \rho_i)\sigma_i^2}; \\ x_1 &= \frac{\sigma_\varepsilon^2 + \sigma_c^2}{\sigma_\varepsilon^2 \sigma_c^2}; \quad x_0 = \frac{-1}{\sigma_\varepsilon^2}. \end{aligned} \quad (12)$$

Moreover, all real roots of the polynomial are between 0 and 1.

According to this proposition, the set of equilibrium statistics is determined by a cubic equation that always has at least one real root.

4.2. Equilibrium construction

We show that, for every equilibrium statistic, there exists a Nash equilibrium in which each agent n behaves as if she observed *only* her equilibrium statistic t_n . The characterization of the

equilibrium strategies is analogous to the treatment in Section 3, but now we use the equilibrium statistic (instead of (5)). One must bear in mind that the equilibrium statistic is simply an auxiliary element that helps characterize a class of equilibria, but it is not (in general) a sufficient statistic for an agent's private signals.

Much as in the previous analysis of one-dimensional signals, for the two-dimensional case we assume that agents are ordered as follows:

$$t_1 > \dots > t_N. \quad (13)$$

If there are multiple equilibrium statistics, then there will be a Nash equilibrium for each one. Different equilibrium statistics induce a different order (as in (13)), so the Nash equilibrium is described in terms of the order induced by each equilibrium statistic.

Theorem 1 (*Symmetric equilibrium with multi-dimensional signals*). *For every equilibrium statistic, there exists a Nash equilibrium in which agent n 's drop-out time is given by*

$$p_n = \mathbb{E}[u(v_n) \mid t_n, \dots, t_n, t_{n+1}, \dots, t_N]. \quad (14)$$

In equilibrium, agent 1 wins the object and pays $p_2 = \mathbb{E}[u(v_2) \mid t_2, t_2, \dots, t_N]$.

Theorem 1 establishes that there exists a class of equilibria in which agents project their signals onto a one-dimensional statistic via the equilibrium statistic $t_n = i_n + \beta \cdot s_n$. In equilibrium, every agent n behaves as if he observed *only* t_n , which is a one-dimensional object.

We prove Theorem 1 in two steps: we first provide the equilibrium conditions and then show that these conditions are satisfied. The equilibrium conditions are similar to (8) and (9): an agent's drop-out time remains optimal even after observing the realized drop-out times of all agents in the auction.²⁰ An important aspect of this setup is that agent n can learn the equilibrium statistic of agent m by observing the latter's drop-out time; but agent n does not learn both of agent m 's signals separately. Hence the optimality condition for agent n 's drop-out time accounts for (i) both of agent n 's observed signals and (ii) the equilibrium statistic $\{t_m\}_{m \neq n}$ of other agents. The equilibrium statistic's properties can be used to show that the optimality conditions are satisfied. Toward that end, we reduce the equilibrium conditions in the two-dimensional environment to the corresponding equilibrium conditions that arise in a one-dimensional environment.

Proof of Theorem 1. We check the following two conditions: (i) agent 1 never regrets winning the object at price p_2 after all agents $m > 1$ drop out of the auction; and (ii) no agent $m > 1$ regrets dropping out of the auction instead of waiting until all other agents (including agent 1) drop out. Formally, the conditions that must be satisfied are as follows:

$$\mathbb{E}[u(v_1) \mid i_1, s_1, t_1, \dots, t_N] - \mathbb{E}[u(v_2) \mid t_2, t_2, \dots, t_N] \geq 0; \quad (15)$$

$$\forall m > 1, \quad \mathbb{E}[u(v_m) \mid i_m, s_m, t_1, \dots, t_N] - \mathbb{E}[u(v_1) \mid t_1, t_1, t_2, \dots, t_{m-1}, t_{m+1}, \dots, t_N] \leq 0. \quad (16)$$

Condition (15) states that the expected valuation of agent 1—conditional on the two signals she observes and on the information she infers from the drop-out time of other agents—is greater than the price at which agent 2 drops out of the auction. It follows that agent 1 does not regret

²⁰ Formally, the Nash equilibrium we characterize is also a posterior equilibrium (cf. Green and Laffont (1987)).

winning the good. Condition (16) states that the expected valuation of agent m —conditional on the two signals he observes and on the information he infers from the drop-out time of other agents—is less than the price at which agent 1 would drop out of the auction if agent m waits until agent 1 drops out. Therefore, agent $m > 1$ does not regret dropping out of the auction before agent 1.

We now prove that (15) and (16) are satisfied. We can use (11) to write

$$\forall n, \quad \mathbb{E}[u(v_n) \mid i_n, s_n, t_1, \dots, t_N] = \mathbb{E}[u(v_n) \mid t_1, \dots, t_N].$$

Note that the expectations in (11) were taken *without* the exponential function. But since all random variables are (assumed to be) Gaussian, it follows that the distribution of v_n conditional on $(i_n, s_n, t_1, \dots, t_N)$ is the same as the distribution of v_n conditional on $(t_1, \dots, t_N)$.²¹ So if (11) is satisfied, then it is satisfied also for any function of v_n . Hence (15) and (16) are satisfied if and only if

$$\mathbb{E}[u(v_1) \mid t_1, \dots, t_N] - \mathbb{E}[u(v_2) \mid t_2, t_2, \dots, t_N] \geq 0; \quad (17)$$

$$\forall m > 1, \quad \mathbb{E}[u(v_m) \mid t_1, \dots, t_N] - \mathbb{E}[u(v_1) \mid t_1, t_1, t_2, \dots, t_{m-1}, t_{m+1}, \dots, t_N] \leq 0. \quad (18)$$

We remark that checking (17) and (18) is equivalent to checking their counterparts, (8) and (9), in one-dimensional environments. In other words: since we proved (in Section 3) that (8) and (9) are satisfied, it follows—after simply replacing b with β —that (17) and (18) are also satisfied. $\square$

For the class of equilibria characterized by Theorem 1, the analysis in Section 3 can be applied once we replace s'_n with t_n (or, equivalently, b with β). The key element of the characterization that determines an equilibrium's qualitative properties is β —that is, the weight placed by the equilibrium statistic on signals about the common shock. If $\beta \approx 0$, then the auction's outcome will be efficient and the outcome will resemble the one that obtains in a pure-private values environment. As β increases, social surplus is reduced and the model resembles more of an “interdependent values” environment. Note that all equilibrium statistics satisfy $\beta \leq 1$. Yet, if $\beta \approx 1$ and if the variance of the idiosyncratic shock is small enough relative to the variance of the common shock and the noise term, then the model will resemble a pure-common values model.

4.3. Linear equilibria in ascending auctions

Theorem 1 constructs a class of equilibria in our ascending auction. We have not been able to prove or disprove that other equilibria exist. We now show that our equilibrium construction characterizes all equilibria in *linear* strategies, which we now formally define.

We recall that an agent's strategy is given by functions $\{P_k\}_{k \in N}$. We say a strategy is *linear* if there exists $b \in \mathbb{R} \cup \{\infty\}$ and functions $\{\hat{P}_k\}_{k \in N}$ such that²²:

$$\forall k \in N, \quad P_k(i, s_n, p_{k+1}, \dots, p_N) = \hat{P}_k(i_n + b \cdot s_n, p_{k+1}, \dots, p_N).$$

²¹ For any jointly distributed (x, y) , we have $x|y \sim \mathcal{N}(\mathbb{E}[x|y], \sigma_x^2 - \text{var}(\mathbb{E}[x|y]))$. Because $\mathbb{E}[v_n \mid i_n, s_n, t_1, \dots, t_N] = \mathbb{E}[v_n \mid t_1, \dots, t_N]$, we also have that $\text{var}(\mathbb{E}[v_n \mid i_n, s_n, t_1, \dots, t_N]) = \text{var}(\mathbb{E}[v_n \mid t_1, \dots, t_N])$. As a result, $v_n \mid (i_n, s_n, t_1, \dots, t_N) \sim v_n \mid (t_1, \dots, t_N)$.

²² If $b = \infty$, then we write $\hat{P}_k(s_n, p_{k+1}, \dots, p_N)$ (that is, $\hat{P}_k$ does not depend on i_n).

In other words, a strategy is linear if at each stage of the auction, an agent's drop-out time depends only on a linear combination of (i_n, s_n) . We say a linear strategy is strictly monotonic if $\hat{P}_k$ is strictly monotonic in its first argument for every k .

Proposition 3 (*Uniqueness within symmetric linear equilibria*). *In every symmetric equilibrium in strictly monotonic linear strategies, the winner of the auction and the price at which the good is sold is the same as in Theorem 1.*

Proposition 3 states that the equilibria constructed in the previous section characterize the winner of the auction and the price at which the good is sold in every equilibrium in strictly monotonic linear strategies. Bidders that drop out before the second-to-last bid may drop out at a price different than that prescribed by Theorem 1. We remark that this source of equilibrium multiplicity is also present in ascending auctions with one-dimensional signals (as shown by Bikhchandani et al. (2002)).

We observe that in a linear strategy, an agent uses the same linear combination of signals in every step of the auction. In other words, weight b is not indexed by k . This guarantees that ex post an agent's expected valuation can be computed using only normal random variables.²³

4.4. Diffuse prior

Wilson (1998) studies an ascending auction, in which agents observe two-dimensional signals (as in our model) and the utility function is equal to the exponential function (i.e., $u(\cdot) = \exp(\cdot)$). However, in that paper, the shocks are drawn from a diffuse prior. More specifically, he assumes that the idiosyncratic shocks are normally distributed $i_n \sim \mathcal{N}(\mu_i, \sigma_\eta^2)$, and μ_i is itself normally distributed with a diffuse prior (i.e., $\text{var}(\mu_i) = \infty$). This modeling device implies that $\text{var}(i_n) = \infty$ but $\text{var}(i_n - i_m) = \sigma_\eta^2$; that is, the variance of the idiosyncratic shock is infinite but the variance of the difference between two idiosyncratic shocks is finite. Finally, it is also assumed that the common shock c is drawn according to a diffuse prior (i.e., $\text{var}(c) = \infty$).

By taking the limits $\sigma_c^2 \rightarrow \infty$, $\sigma_i^2 \rightarrow \infty$, and $\rho_i \rightarrow 1$ we can obtain the same assumptions on the distribution of signals and payoff shocks as Wilson (1998). The limits are taken at rates such that $(1 - \rho_i)\sigma_i^2 = \sigma_\eta^2$; in this limit the variance of the idiosyncratic shock is infinite but the variance of the difference between two idiosyncratic shocks is finite. In this limit, the polynomial (12) can be written as follows²⁴:

$$x^2 \frac{1}{(1 - \rho_i)\sigma_i^2} (x - 1) + \frac{1}{\sigma_\varepsilon^2} (x - 1).$$

Clearly, the only (real) root of this polynomial is $x = 1$. Thus, the equilibrium in Wilson (1998) corresponds to an equilibrium as described in Theorem 1, where the equilibrium statistic is given by $\beta = 1$.

²³ For example, if at steps k, k' the equilibrium strategies used different weights $b_k \neq b_{k'}$, then at the end of the auction agent n would be able to learn that the signals of the auction's winner, say agent m , satisfy: $i_m + b_k s_m \geq t$ and $i_n + b_{k'} s_n = t'$ for some t, t' . Thus, ex post agent n will only be able to infer a lower bound on $i_m + b_k s_n$, which is not a normal random variable.

²⁴ The terms of the cubic polynomial (12) converge to: $x_3 \rightarrow \frac{1}{(1 - \rho_i)\sigma_i^2}$; $x_2 \rightarrow \frac{-1}{(1 - \rho_i)\sigma_i^2}$; $x_1 \rightarrow \frac{1}{\sigma_\varepsilon^2}$; $x_0 \rightarrow \frac{-1}{\sigma_\varepsilon^2}$.

The analysis with diffuse priors is conceptually different from the analysis in our model. With diffuse priors, an agent's bidding strategy is measurable with respect to his own interim expected payoff shock. In other words, in the limit previously described:

$$\mathbb{E}[v_n | i_n, s_n] = i_n + s_n.$$

In this limit, the equilibrium statistic is also $t_n = i_n + s_n$. Therefore the equilibrium statistic is equal to an agent's expected payoff shock conditional only on his private information. In an auction, agents learn from the other agents' drop-out time, but when the priors are diffuse, this additional information does not change the weight that they place on their signals. In the equilibria we constructed, agents continue to bid as if they observed one-dimensional signals (that is, the equilibrium statistic). However, in general, agents' must adjust their bidding strategies so that this is not determined by their expected valuation conditional only on their private signals.

5. Equilibrium multiplicity

The cubic polynomial that determines the set of equilibrium statistics (see (12)) may have multiple roots. The implication is that an ascending auction with multi-dimensional signals may have multiple symmetric equilibria—that is, a different equilibrium for every root. We illustrate the multiplicity of equilibria with a parameterized example. Here we describe the multiple equilibria based on a numerical example; analytical results that give conditions for multiple equilibria are given in the Online Appendix. After we have illustrated the multiplicity of equilibria, we argue that the multiplicity of equilibria can be explained by the presence of a strategic complementarity.

5.1. Illustration of multiplicity

In Fig. 1(a) we plot the set of equilibrium statistics for different values of the variance in the noise. The graph's different colors correspond to the different roots of the cubic polynomial that determines the set of equilibrium statistics (see (12)). We can see that there are values of the noise term (e.g., $\sigma_\varepsilon = 50$) for which multiple equilibria exist.

In Fig. 1(b) we plot the expected social surplus generated in the auction corresponding to the equilibrium statistic shown in part (a) of the figure. There is one equilibrium in which β is small (the blue line). This equilibrium will resemble a private values environment: the social surplus generated will be high and the winner's curse will be low. There is also one equilibrium in which β is large (the red line). This equilibrium will resemble a common values environment: the social surplus generated will be low and the winner's curse will be high. Neither the auction's generated revenue nor the buyers' rents are plotted in this graph, but both are qualitatively similar to our plot of the social surplus generated in the auction.

The social surplus generated by the auction is non-monotonic in the size of the noise term (σ_ε^2) because there two countervailing effects. First, if β is fixed then, as σ_ε^2 increases, the correlation between an agent's drop-out time and the noise term ε_n also increases; the result is a reduction in social surplus. Second, as σ_ε^2 increases, the weight given to s_n decreases and so the weight placed on the noise term ε_n also decreases. The latter effect, in turn, reduces the correlation between an agent's drop-out time and the noise term; the result is an increase in social surplus.

If the noise term is extremely small or large (i.e., as $\sigma_\varepsilon^2 \rightarrow 0$ or $\sigma_\varepsilon^2 \rightarrow \infty$), then there is a unique equilibrium. This follows because, in both limits, the model approaches a private values model. In the former case, agents know almost perfectly the realization of c simply by virtue of

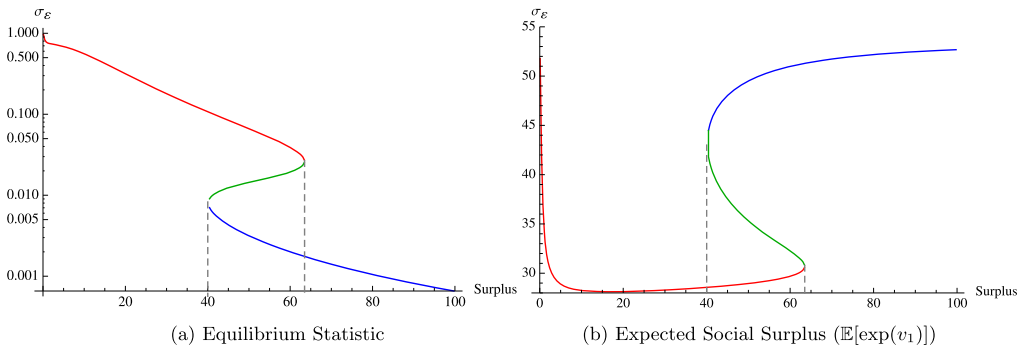


Fig. 1. Outcome of ascending auction for $u(\cdot) = \exp(\cdot)$, $\sigma_c = 5/2$, $\sigma_i = 0.6$, $\rho_i = 3/4$, and $N = 50$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

their private information; in the latter case, agents ignore s_n and so the model is again a private values model. Note that in both limits, the equilibrium is efficient.

5.2. Strategic complementarity

The existence of multiple equilibrium statistics is driven by a complementarity. To give an intuitive account for why there is a complementarity, we provide some intuition regarding how β is determined. Analogously to how the Nash equilibrium of any game can be understood by analyzing agents' best-response functions, we can explain how the equilibrium statistic is determined by analyzing how the expectations are determined “out of equilibrium”. We fix an exogenous one-dimensional signal,²⁵

$$s'_m = s_m + \frac{1}{b} i_m, \quad (19)$$

and define the terms $\gamma_i, \gamma_s, \gamma' \in \mathbb{R}$ implicitly as follows²⁶:

$$\mathbb{E}[v_n \mid i_n, s_n, \{s'_m\}_{m \neq n}] = \gamma_i \cdot i_n + \gamma_s \cdot s_n + \frac{\gamma'}{N-1} \cdot \sum_{m \neq n} s'_m. \quad (20)$$

The weight b is an equilibrium statistic if and only if it satisfies

$$b = \frac{\gamma_s}{\gamma_i}.$$

It is possible to show that, when $\rho_i > 0$, γ_i is decreasing in b (at least within a certain range of b), from which it follows that the weights placed by agents on their respective idiosyncratic shocks exhibit strategic complementarity. In particular, if agent m increases the weight he places on i_m then agent n will likewise increase the weight she places on i_n .

The intuition for this complementarity is as follows. From agent n 's perspective, i_m is a noise term in s'_m . That is, agent n would prefer simply to observe s_m . If $\rho_i = 0$, then γ_i is constant in b and is also equal to 1. This state of affairs is natural given that an agent knows her own

²⁵ The signal is the same as in Section 3, but we have divided it by $1/b$. The only effect of this maneuver is that it makes some of the subsequent comparisons more transparent.

²⁶ By symmetry, all signals $\{s'_m\}_{m \neq n}$ are given the same weight (denoted by γ').

idiosyncratic shock, which is independent of the noise in s'_m ; hence agent n assigns this signal a weight of 1. The conceptual difference between $\rho_i = 0$ and $\rho_i \neq 0$ is that, in the latter case, i_n influences agent n 's beliefs about c . When i_n is correlated with i_m , agent n can use i_n to filter out the noise in s'_m . The non-monotonicity of γ_i reflects the use of i_n to filter noise. Because agent n would prefer simply to observe s_m , if $b \rightarrow \infty$ then n can observe s_m directly; in this case, she need not use i_n to filter out the noise s'_m and so $\gamma_i = 1$. If $b \rightarrow 0$ then the signal s'_m provides no information about c and so agent n does not use s'_m to predict c at all; here, too, $\gamma_i = 1$. Thus it is only for intermediate values of b that agent n uses i_n to predict c .

We can now relate our results to those that have appeared in the literature studying demand function competition and rational expectation equilibrium. The existence of multiple equilibria in our model is closely related to the multiplicity of equilibria found by Ganguli and Yang (2009) and Manzano and Vives (2011). Even though these papers study a model where agents have CARA utility, the source of the multiplicity is also a complementarity in the information aggregation (much like in our model).²⁷

6. Extension: effect of public signals

In this section, we study how the precision of a public signal affects the social surplus and revenue generated by an auction. Our analysis reveals that the comparative statics in two-dimensional environments may be different from that in one-dimensional environments. This difference arises because the equilibrium statistic – which ultimately determines an agent's bidding strategy – is endogenous. This implies that changes in the equilibrium statistic partially determine the comparative statics. Here we only give a qualitative description based on a numerical example; analytical results are relegated to the Online Appendix.

We assume that agents have access to a public signal (i.e., in addition to (i_n, s_n)) about the average idiosyncratic shock:

$$\bar{s} = \frac{1}{N} \sum_{n \in N} i_n + \epsilon; \quad (21)$$

here ϵ is independent of all random variables defined so far and is normally distributed with variance σ_ϵ^2 . The signal $\bar{s}$ can be interpreted as providing more information about bidders' characteristics (e.g., more information about the industry's average cost of extracting oil). The analysis in Section 4 can be extended in a straightforward manner to accommodate public signals. The only modification required is that the public signals be added as "conditioning variables" in the expectations.²⁸

In Fig. 2(a) we plot the expected social surplus induced by the equilibrium outcome as a function of the public signal's precision. We can see that social surplus is increasing in the public

²⁷ Goldstein and Yang (2015) also shows that there may be a complementarity in trading in a model with multiple common shocks and noise traders.

²⁸ That is, our definition of an equilibrium statistic (see (11)) is modified as follows:

$$\mathbb{E}[v_n | i_n, s_n, t_1, \dots, t_N, \bar{s}] = \mathbb{E}[v_n | t_1, \dots, t_N, \bar{s}].$$

In addition, the strategy of agents (see (14)) is now written as

$$p_n = \mathbb{E}[\exp(v_n) | t_n, \dots, t_n, t_{n+1}, \dots, t_N, \bar{s}].$$

It should be clear that, under these two modifications, all of the analysis in Section 4 is unchanged.

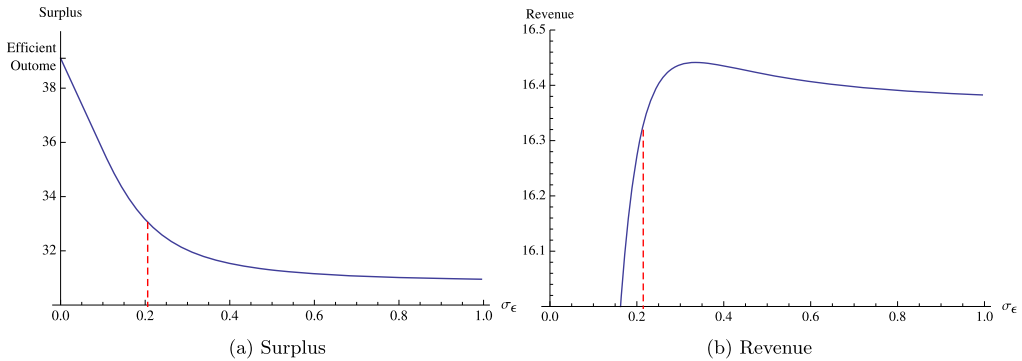


Fig. 2. Outcome of ascending auction for $u(\cdot) = \exp(\cdot)$, $\sigma_c = 5/2$, $\sigma_i = 0.7$, $\rho_i = 3/4$, and $N = 3$.

signal's precision. In Fig. 2(b) we plot the expected revenue induced by the equilibrium outcome. The revenue is non-monotonic in the precision of the public signal, but it is decreasing in the public signal's precision for a large portion of precisions.

An intuition for these two comparative statics is as follows. Because agent n uses the public signal to filter out i_m from agent m 's drop-out time, the drop-out time of agent m becomes more informative to agent n . This implies that agent n can rely less on her own signal about c , which means that β decreases. This leads to an increase in the social surplus because the drop out time of agent n is less correlated with the noise term ε_n . On the other hand, agent n 's inference about c relies more on the drop-out time of agent m , so the winner's curse increases. This leads to a decrease in revenue. Since the social surplus and the winner's curse are both increasing, the net effect on revenue can be non-monotonic.

In the limit $\text{var}(\epsilon) \rightarrow 0$ the social surplus approaches the efficient outcome (i.e., the winner of the auction is the bidder with highest idiosyncratic shock) and revenue converges to zero (we prove this analytically in the Online Appendix). A simple intuition for this result can be given when $N = 2$. In the limit $\text{var}(\epsilon) \rightarrow 0$ the model approaches a model of common values with asymmetric agents. In other words, in this limit, the difference between both agents' valuation become common knowledge, so it is the same as a model with asymmetric agents where the only source of uncertainty is a common shock. The literature has observed that in a common values auction, small asymmetries between agents can lead to large differences in the equilibrium outcome (cf. Bulow et al. (1999); Bulow and Klemperer (2002)). In particular, the agent with the highest valuation always wins the object and the other bidders instantly drop out of the auction.

The literature has shown that, in a common values environment, a small departure from a symmetric environment can lead to large changes in the equilibrium outcome. Here we can also underscore the other direction. That is, adding an arbitrarily small common shock to a private values environment may reduce the revenues to essentially zero. As long as the uncertainty about the common shock is larger than the uncertainty about the realization of the average idiosyncratic shock, the revenues will be reduced to essentially zero.²⁹

²⁹ We can also interpret this result as showing that a natural perturbation of the private values environment can select an equilibrium different than the one in weakly dominant strategies. Weinstein and Yildiz (2007) proves the existence of perturbations that can select any equilibrium as a unique equilibrium in a broad class of games. In contrast to their general result, in our model, the "perturbation" is merely adding a common-value component to the valuations.

Our comparative statics analysis can be interpreted as a failure of the linkage principle.³⁰ Agent n 's expected valuation conditional on all signals

$$\mathbb{E}[v_n \mid \{i_m\}_{m \in N}, \{s_m\}_{m \in N}, \bar{s}]$$

is nondecreasing in all the conditioning variables, and all signals are positively correlated when $\text{var}(\epsilon)$ is not too small. In Figs. 2(a)-2(b), we identified the minimum variance for which the affiliation property holds.³¹ Thus, if $\text{var}(\epsilon)$ is larger than the lower bound indicated by the red dashed line, all assumptions made by Milgrom and Weber (1982) are satisfied *except* for the assumption that private signals are one-dimensional. Therefore, the linkage principle may fail for no other reason than the multi-dimensionality of the information structure.³²

Other public signals. Here we have focused on studying the effects of a public signal about the average idiosyncratic shock. It is also natural to study the effects of a public signal about the common shock. If the public signal is about the common shock, then the comparative statics on social surplus remain the same. However, the comparative statics on revenue may be different. For example, as the public signal about the common shock becomes arbitrarily precise, the revenue does not converge to zero.

When agents observe multi-dimensional signals, multiple natural public signals may have different effects. Goldstein and Yang (2019) study the impact of different public signals and stress the different effects these may have when agents observe multi-dimensional signals. They study a model where agents submit linear demands; our results show that the same tractability is preserved in an ascending auction and here too different types of public signals may have different effects.

7. Conclusions

The auction literature relies heavily on the assumption that agents observe one-dimensional signals. In this paper, we offer a tractable model of an ascending auction in which agents observe multi-dimensional signals. We show that there may be multiple symmetric equilibria in such environments and overturn some classic results on the effects of public signals. This paper provides a set of tools that can be used to increase our understanding of multi-dimensional environments and how they differ from their one-dimensional counterparts.

Appendix I. Proofs

Preamble. We first provide explicit expressions for the expectations with normal random variables. To do this, we use the definition of one-dimensional signal given in (5).

³⁰ The linkage principle states that public signals increase an auction's revenue.

³¹ The affiliation property can be checked by verifying that all the off-diagonal terms of the variance-covariance matrix's inverse are negative.

³² The linkage principle has been shown to fail in other environments. Perry and Reny (1999) establish that the linkage principle may fail in multi-unit auctions. The linkage principle has been shown to fail also in environments characterized by an asymmetric payoff structure (Krishna (2009)) or by private values (Foucault and Lovo (2003)). Axelson and Makarov (2016) demonstrate that the linkage principle fails in common values auctions when an agent must take an action after winning the good. As in our model, the bid of an agent in Axelson and Makarov (2016) does not fully reveal the signal observed by that agent; however, in their paper, the reason is that the good's final payoff is not strictly monotonic in the realization of signals observed by agents (see also Atakan and Ekmekci (2014)). Our paper identifies a new channel through which the linkage principle may fail.

Since $(v_1, \dots, v_N, s'_1, \dots, s'_N)$ are jointly Gaussian, it follows that the distribution of $(v_1, \dots, v_N)$ conditional on $(s'_1, \dots, s'_N)$ is also jointly Gaussian. Note that $\mathbb{E}[u(v_n) \mid s'_n, \dots, s'_n, s'_{n+1}, \dots, s'_N]$ is computed as if the realization of $(s'_1, \dots, s'_n)$ were equal to $(s'_n, \dots, s'_n)$. Because the conditional variance of normal random variables is independent of the realization of the random variables, we can write $\text{var}(v_n \mid s'_1, \dots, s'_N) = \text{var}(v_n \mid s'_n, \dots, s'_n, s'_{n+1}, \dots, s'_N)$. It follows that, for any strictly increasing function $u(\cdot)$:

$$\begin{aligned} u(\mathbb{E}[v_n \mid s'_1, \dots, s'_N]) &\geq u(\mathbb{E}[v_n \mid s'_n, \dots, s'_n, s'_{n+1}, \dots, s'_N]) \iff \mathbb{E}[v_n \mid s'_1, \dots, s'_N] \\ &\geq \mathbb{E}[v_n \mid s'_n, \dots, s'_n, s'_{n+1}, \dots, s'_N]. \end{aligned} \quad (22)$$

We now explicitly compute the coefficients of the Bayesian updating with the normal random variables. We have that

$$\mathbb{E}[v_n \mid s'_1, \dots, s'_N] = \kappa \cdot \left(s'_n + \lambda \sum_{m=1}^N s'_m \right), \quad (23)$$

where

$$\kappa \triangleq \frac{(1 - \rho_i) \sigma_i^2}{(1 - \rho_i) \sigma_i^2 + b^2 \cdot \sigma_\varepsilon^2}; \quad (24)$$

$$\lambda \triangleq \frac{1}{N} \left(\frac{((1 - \rho_i) + \rho_i \cdot N) \cdot \sigma_i^2 + b \cdot N \cdot \sigma_c^2}{((1 - \rho_i) + \rho_i \cdot N) \cdot \sigma_i^2 + b^2(N \cdot \sigma_c^2 + \sigma_\varepsilon^2)} \frac{(1 - \rho_i) \sigma_i^2 + b^2 \cdot \sigma_\varepsilon^2}{(1 - \rho_i) \sigma_i^2} - 1 \right). \quad (25)$$

To verify that the coefficients λ and κ are correctly computed, it is sufficient to check that

$$\forall m \in N, \quad \text{cov}(v_n - \mathbb{E}[v_n \mid s'_1, \dots, s'_N], s'_m) = 0; \quad (26)$$

for this we use (23) and the definitions of κ and λ .³³ Finally, note that $\kappa > 0$ and that, for all $n \in N$,

$$\begin{aligned} (1 + n \cdot \lambda) &= \frac{n}{N} \frac{((1 - \rho_i) + \rho_i \cdot N) \cdot \sigma_i^2 + b \cdot N \cdot \sigma_c^2}{((1 - \rho_i) + \rho_i \cdot N) \cdot \sigma_i^2 + b^2(N \cdot \sigma_c^2 + \sigma_\varepsilon^2)} \frac{(1 - \rho_i) \sigma_i^2 + b^2 \cdot \sigma_\varepsilon^2}{(1 - \rho_i) \sigma_i^2} \\ &\quad + \frac{N - n}{N} > 0. \end{aligned}$$

Proof of Proposition 1. The proof is standard in the literature (see e.g. Krishna (2009)). Nevertheless, we give the proof here for completeness and to check that all the conditions are satisfied. In particular, we check the following three conditions.

1. According to the equilibrium strategies (see (7)), agent $n + 1$ drops out of the auction before agent n does. This is a necessary condition for an equilibrium because, according to (7), agent

³³ That is, λ and κ solve the following system of equations:

$$\begin{aligned} \sigma_i^2 + b \sigma_c^2 &= \kappa \left(\sigma_i^2 + b^2(\sigma_c^2 + \sigma_\varepsilon^2) + \lambda \left(\sigma_i^2 + b^2(\sigma_c^2 + \sigma_\varepsilon^2) + (N - 1)(\rho_i \cdot \sigma_i^2 + b^2 \cdot \sigma_c^2) \right) \right); \\ \rho_i \cdot \sigma_i^2 + b \sigma_c^2 &= \kappa \left(\rho_i \sigma_i^2 + b^2 \cdot \sigma_c^2 + \lambda \left(\sigma_i^2 + b^2(\sigma_c^2 + \sigma_\varepsilon^2) + (N - 1)(\rho_i \cdot \sigma_i^2 + b^2 \cdot \sigma_c^2) \right) \right), \end{aligned}$$

which corresponds to (26) for $m = n$ and $m \neq n$, respectively.

n 's equilibrium strategy is conditioned on the signals $(s'_{n+1}, \dots, s'_N)$. Hence it is necessary to check that agents with higher signals drop out later in the auction.

By (23), the following expression holds:

$$\begin{aligned} \mathbb{E}[v_{n-1} | s'_{n-1}, \dots, s'_{n-1}, s'_n, \dots, s'_N] - \mathbb{E}[v_n | s'_n, \dots, s'_n, s'_{n+1}, \dots, s'_N] \\ = \kappa(1 + \lambda \cdot (n-1))(s'_{n-1} - s'_n) > 0. \end{aligned}$$

The equality is based on (23); the inequality reflects that $\kappa > 0$, $(1 + (n-1)\lambda) > 0$ (as previously shown), and $(s'_{n-1} - s'_n) > 0$ by construction. It now follows from (22) that, for all $n \in \{2, \dots, N\}$,

$$\mathbb{E}[u(v_{n-1}) | s'_{n-1}, \dots, s'_{n-1}, s'_n, \dots, s'_N] - \mathbb{E}[u(v_n) | s'_n, \dots, s'_n, s'_{n+1}, \dots, s'_N] > 0. \quad (27)$$

Therefore, agent n drops out of the auction before agent $n-1$.

2. We now check that agent 1 does not regret winning the auction (this is (8)).

Using (23), we note that

$$\mathbb{E}[v_1 | s'_1, \dots, s'_N] - \mathbb{E}[v_2 | s'_2, s'_2, \dots, s'_N] = \kappa(1 + \lambda)(s'_1 - s'_2) > 0.$$

It is clear that the inequality is satisfied also if we take the exponential of v_1 and v_2 , so (8) is satisfied. Hence agent 1 does not regret winning the auction.

3. Finally, we check that agent $m > 1$ does not regret waiting until agent 1 drops out of the auction (this is (9)).

By (23),

$$\mathbb{E}[v_m | s'_1, \dots, s'_N] - \mathbb{E}[v_1 | s'_1, s'_1, s'_2, \dots, s'_{m-1}, s'_{m+1}, \dots, s'_N] = \kappa(1 + \lambda)(s'_m - s'_1) < 0$$

holds. The inequality will also be satisfied if we take the exponential of v_m and v_1 , from which it follows that (9) is satisfied. Therefore, agent $m > 1$ does not regret waiting until agent 1 drops out of the auction.

We thus conclude that the equilibrium strategies constitute an ex post equilibrium. $\square$

Proof of Proposition 2. The proof proceeds in two steps.

Step 1. We first establish that $t_n = i_n + \beta \cdot s_n$ is an equilibrium statistic if and only if

$$\text{cov}(v_n - \mathbb{E}[v_n | t_1, \dots, t_N], i_n) = 0 \quad (28)$$

and $\beta \neq 0$.

“Only If”. Clearly, $\beta = 0$ is not an equilibrium statistic (simply note that $\mathbb{E}[v_n | i_1, \dots, i_N] \neq \mathbb{E}[v_n | s_n, i_1, \dots, i_N]$). By the construction of the expectation, we have

$$\text{cov}(v_n - \mathbb{E}[v_n | i_n, s_n, t_1, \dots, t_N], i_n) = 0. \quad (29)$$

So if $\mathbb{E}[v_n | i_n, s_n, t_1, \dots, t_N] = \mathbb{E}[v_n | t_1, \dots, t_N]$, then (28) must be satisfied.

“If”. Note that, by construction of the expectation,

$$\forall m \in N, \quad \text{cov}(v_n - \mathbb{E}[v_n | t_1, \dots, t_N], t_m) = 0. \quad (30)$$

Now, by the collinearity of the covariance, if (28) is satisfied and if (30) is satisfied with $\beta \neq 0$, then also

$$\text{cov}(v_n - \mathbb{E}[v_n | t_1, \dots, t_N], s_n) = 0. \quad (31)$$

This follows because s_n is a linear combination of t_n and i_n . Therefore, if (28) is satisfied then (30) and (31) are satisfied by construction. As a result, the covariance of $(i_n, s_n, t_1, \dots, t_N)$ and $(v_n - \mathbb{E}[v_n | t_1, \dots, t_N])$ is equal to 0. Hence we must have

$$\mathbb{E}[v_n | t_1, \dots, t_N] = \mathbb{E}[v_n | i_n, s_n, t_1, \dots, t_N]. \quad (32)$$

Thus $t_n = i_n + \beta \cdot s_n$ is an equilibrium statistic, which completes the proof's first step.

Step 2. We now prove that β satisfies (28) if and only if β solves the cubic polynomial (12). It is clear that $\text{cov}(v_n, i_n) = \sigma_i^2$. By (23), we have

$$\text{cov}(\mathbb{E}[v_n | t_1, \dots, t_N], i_n) = \kappa((\lambda + 1)\sigma_i^2 + \lambda(N - 1)\rho_i\sigma_i^2).$$

Hence we can rewrite (28) as

$$1 - \kappa((\lambda + 1) + \lambda(N - 1)\rho_i) = 0.$$

Multiplying both sides by:

$$-\frac{(\beta^2\sigma_\varepsilon^2 - (\rho_i - 1)\sigma_i^2)(\beta^2(N\sigma_c^2 + \sigma_\varepsilon^2) + \sigma_i^2((N - 1)\rho_i + 1))}{\beta(\rho_i - 1)\sigma_c^2\sigma_i^2\sigma_\varepsilon^2((N - 1)\rho_i + 1)}$$

yields the cubic polynomial (12), which proves the result. $\square$

Proof of Theorem 1. This theorem was proved in the main text. $\square$

Proof of Proposition 3. The proof uses similar arguments to those in Bikhchandani et al. (2002), with minor modifications to account for the multi-dimensional signals in our model. We fix a strictly increasing linear strategy $\{\hat{P}_k^*\}_{k \in N}$, where the linear combination of signals that determines $\{\hat{P}_k^*\}_{k \in N}$ is denoted by $i_n + b^*s_n$. We assume that this is a Nash equilibrium and then prove that the drop-out time of the last and second-to-last agent to drop out is the same as those described in Theorem 1. The case in which the strategy is decreasing (instead of increasing) can be proved in an analogous way.

We now fix an agent n and characterize agent n 's drop-out time when $N - 2$ agents have already dropped out of the auction and every agent different than n is using $\hat{P}^*$. We denote by $Y_1, \dots, Y_{N-1}$ the realization of the one-dimensional statistic $\{i_m + b^*s_m\}_{m \neq n}$ in decreasing order. For example, Y_1 (resp. Y_N) is the highest (resp. lowest) number in $\{i_m + b^*s_m\}_{m \neq n}$. We write shortly $\hat{P}_2^*(Y_1)$ for the drop-out time of the agent who observed Y_1 when every agent different than n uses strategy $\hat{P}^*$ and agent n has not dropped out of the auction (note that $\hat{P}_2^*(Y_1)$ may depend on $\{Y_m\}_{m \geq 2}$). We denote by $y_1(p)$ the inverse function of $\hat{P}_2^*(Y_1)$, which is monotonic, continuous, and differentiable for almost every $p \in \mathbb{R}$ (this follows from the strict monotonicity of $\hat{P}_2^*$). We denote by $\phi(y)$ the probability density function of Y_1 conditional on $(Y_2, \dots, Y_N, i_n, s_n)$. Finally, we note that the probability density function of the random variable $\hat{P}_2^*(Y_1)$ conditional on $(Y_2, \dots, Y_N, i_n, s_n)$ is given by $\frac{\partial y_1(p)}{\partial p} \phi(y_1(p))$.³⁴ In other words:

³⁴ Since $y_1(p)$ is continuous and monotonic, it is differentiable almost everywhere. For the points of non-differentiability, we define $\frac{\partial y_1(p)}{\partial p} \phi(y_1(p))$ as the right-limit of the derivative (this choice is inconsequential for the analysis).

$$\mathbb{P}\{\hat{P}_2^*(Y_1) \leq \tilde{p} \mid Y_2, \dots, Y_N, i_n, s_n\} = \int_{-\infty}^{y_1(\tilde{p})} \phi(y) dy = \int_{-\infty}^{\tilde{p}} \frac{\partial y_1(p)}{\partial p} \phi(y_1(p)) dp.$$

Agent n 's objective function when he observes signals (i_n, s_n) is given by:

$$p^*(i_n, s_n) \in \arg \max_{\tilde{p} \geq \hat{P}_3^*(Y_2)} \mathbb{E}[\mathbb{E}[u(v_n) \mid i_n, s_n, y_1(p), Y_2, \dots, Y_N] - p \mid p \leq \tilde{p}],$$

where the expectation is taken over p . We can write the expectation explicitly as an integral:

$$p^*(i_n, s_n) \in \arg \max_{\substack{\tilde{p} \geq \hat{P}_3^*(Y_2) \\ \hat{P}_3^*(Y_2)}} \int_{\hat{P}_3^*(Y_2)}^{\tilde{p}} \left(\mathbb{E}[u(v_n) \mid i_n, s_n, y_1(p), Y_2, \dots, Y_N] - p \right) \frac{\partial y_1(p)}{\partial p} \phi(y_1(p)) dp.$$

The term inside the integral is continuous, and so the first order condition implies that:

$$\left(\mathbb{E}[u(v_n) \mid i_n, s_n, y_1(p^*), Y_2, \dots, Y_N] - p^*(i_n, s_n) \right) \frac{\partial y_1(p^*(i_n, s_n))}{\partial p} = 0.$$

Since we conjecture this is a symmetric linear Nash equilibrium we have that $p^*(i_n, s_n) = \hat{P}_2^*(i_n + b^*s_n)$. Thus, we have that:

$$\left(\mathbb{E}[u(v_n) \mid i_n, s_n, y_1(\hat{P}_2^*(i_n + b^*s_n)), Y_2, \dots, Y_N] - \hat{P}_2^*(i_n + b^*s_n) \right) \times \frac{\partial y_1(\hat{P}_2^*(i_n + b^*s_n))}{\partial p} = 0.$$

We now note that,

$$\frac{\partial y_1(\hat{P}_2^*(i_n + b^*s_n))}{\partial p} \neq 0,$$

almost surely. This inequality follows from the fact that $\frac{\partial y_1(\hat{P}_2^*(i_n + b^*s_n))}{\partial p} = 0$ if and only if the agents never drops out at a price in a neighborhood around $\hat{P}_2^*(i_n + b^*s_n)$ according to strategy $\hat{P}^*$. Thus, by construction, $\frac{\partial y_1(\hat{P}_2^*(i_n + b^*s_n, Y_2, \dots, Y_N))}{\partial p} \neq 0$ almost surely. Also, it is clear that $y_1(\hat{P}_2^*(i_n + b^*s_n)) = i_n + b^*s_n$. Thus, the first-order condition implies that

$$\hat{P}_2^*(i_n + b^*s_n) = \mathbb{E}[u(v_n) \mid i_n, s_n, (i_n + b^*s_n), Y_2, \dots, Y_N]. \quad (33)$$

Here it is useful to recall that we have omitted the dependence of $\hat{P}_2^*(i_n + b^*s_n)$ on $Y_2, \dots, Y_N$ only to simplify the notation, but $\hat{P}_2^*(i_n + b^*s_n)$ depends on $Y_2, \dots, Y_N$. Thus, (33) implies that $\mathbb{E}[u(v_n) \mid i_n, s_n, (i_n + b^*s_n), Y_2, \dots, Y_N]$, is measurable with respect to $i_n + b^*s_n$ so we have that:

$$\mathbb{E}[u(v_n) \mid (i_n + b^*s_n), (i_n + b^*s_n), Y_2, \dots, Y_N] = \mathbb{E}[u(v_n) \mid i_n, s_n, (i_n + b^*s_n), Y_2, \dots, Y_N]. \quad (34)$$

We now recall that $Y_m = i_m + b^*s_m$ is equal to the one-dimensional statistic of the $N - m + 1$ bidder to drop out of the auction (for example, Y_N is the one-dimensional statistic of the first agent to drop out of the auction). It follows that b^* is an equilibrium statistic. Since we have

conjectured that $\hat{P}^*$ is monotonic, it follows that the agent who observed the highest equilibrium statistic wins the object. Moreover, the price at which the second-to-last agent drops out of the auction is given by $\mathbb{E}[u(v_n) \mid (i_n + b^*s_n), (i_n + b^*s_n), Y_2, \dots, Y_N]$, which is the same as in the equilibrium constructed in Theorem 1 (recall that here agent n is the agent who dropped out second-to-last of the auction). $\square$

Appendix II. Supplementary material

Supplementary material related to this article can be found online at <https://doi.org/10.1016/j.jet.2019.104938>.

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